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GENERALIZED (α, β)-RATIONAL CONTRACTIONS AND FIXED POINT THEOREMS IN ORDERED S_b-METRIC SPACES WITH APPLICATIONS

ABSTRACT. In this paper, we define generalized (α, β)-rational contraction. Based on this we have to prove some fixed point theorems. Applications related to integral equations and Homotopy theory are presented. Also we gave an example which supported our main results.

KEY WORDS: S_b-metric, generalized (α, β)-rational contraction, S_b-completeness, Homotopy theory, partial ordering and regular.

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1. Introduction

In 1922 S. Banach [3] formulated the contraction known is Banach contraction principle. Some results related with generalization of metric space can be found in ([1]-[12]).

In 1989, Bakhtin formulated the b-metric spaces [2], later several researchers work on this space and obtained so many results on this spaces can be found in ([5, 6, 7]).

In generalized contractions one is (α, β)-weak contraction. Using this contraction, researchers proved results (for detail see [13, 14, 15, 16, 17]). Currently the study of (α, β)-contractions gain the attractions of many researches. In this regards many fixed point results and their applications are studied (see [18, 19, 20] and the reference cited therein).

Mustafa et. al. defined the notion of G-metric space [4]. Sedghi et. al. gave the concept of an S-metric space [11]. Aghajani et.al. presented a new type of metric is called G_b-metric [1]. Recently Sedghi et al. [10] defined S_b-metric space by using the S-metric space [11].

The aim of present article is to prove applications to integral equations and Homotopy theory via generalized (α, β)-rational contraction, we can also gave related fixed point results and example.

First we recall some basic results.

2. Preliminaries

Definition 1 ([10])**.** *Let X be a non-empty set and $b \geq 1$ be given real number. Suppose that a mapping $S_b : X^3 \to [0, \infty)$ be a function satisfying the following properties:*

(S_b1) *$0 < S_b(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z \neq x$,*

(S_b2) *$S_b(x, y, z) = 0 \Leftrightarrow x = y = z$,*

(S_b3) *$S_b(x, y, z) \leq b(S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a))$ for all $x, y, z \in X$ and $a \in X$.*

Then the function S_b is called a S_b-metric on X and the pair (X, S_b) is called a S_b-metric space.

Remark 1 ([10])**.** It should be noted that, the class of S_b-metric spaces is effectively larger than that of S-metric spaces. Indeed each S-metric space is a S_b-metric space with $b = 1$.

Following example shows that a S_b-metric on X need not be a S-metric on X.

Example 1 ([10])**.** Let (X, S) be S-metric space and $S_*(x, y, z) = S(x, y, z)^p$, where $p > 1$ is a real number. Note that S_* is a S_b-metric with $b = 2^{2(p-1)}$. Also, (X, S_*) is not necessarily a S-metric space.

Definition 2 ([10])**.** *Let (X, S_b) be a S_b-metric space. Then, for $x \in X$, $r > 0$ we defined the open ball $B_{S_b}(x, r)$ and closed ball $B_{S_b}[x, r]$ with center x and radius r as follows respectively:*

$$B_{S_b}(x, r) = \{y \in X : S_b(y, y, x) < r\} \quad and$$
$$B_{S_b}[x, r] = \{y \in X : S_b(y, y, x) \leq r\}.$$

Lemma 1 ([10])**.** *In a S_b-metric space, we have*

$$S_b(u, u, w) \leq 2bS_b(u, u, v) + b^2 S_b(v, v, w).$$

Lemma 2 ([10])**.** *In a S_b-metric space, we have*

$$S_b(u, u, v) \leq bS_b(v, v, u) \quad and \quad S_b(v, v, u) \leq bS_b(u, u, v).$$

Definition 3 ([10])**.** *If (X, S_b) be a S_b-metric space. A sequence $\{x_n\}$ in X is said to be:*

(a) *S_b-Cauchy sequence if, for each $\epsilon > 0$, there exists $n_0 \in \mathcal{N}$ such that $S_b(x_n, x_n, x_m) < \epsilon$ for each $m, n \geq n_0$.*

(b) *S_b-convergent to a point $x \in X$ if, for each $\epsilon > 0$, there exists a positive integer n_0 such that $S_b(x_n, x_n, x) < \epsilon$ or $S_b(x, , x, x_n) < \epsilon$ for all $n \geq n_0$ and we denote by $\lim_{n \to \infty} x_n = x$.*

Definition 4 ([10]). *A S_b-metric space (X, S_b) is called complete if every S_b-Cauchy sequence is S_b-convergent in X.*

Lemma 3 ([10]). *If (X, S_b) be a S_b-metric space with $b \geq 1$ and suppose that $\{x_n\}$ is a S_b-convergent to x, then we have*

(i) $\quad \frac{1}{2b} S_b(y, y, x) \leq \lim_{n \to \infty} \inf \ S_b(y, y, x_n) \leq \lim_{n \to \infty} \sup S_b(y, y, x_n)$
$$\leq 2b S_b(y, y, x)$$

and

(ii) $\quad \frac{1}{b^2} S_b(x, x, y) \leq \lim_{n \to \infty} \inf S_b(x_n, x_n, y) \leq \lim_{n \to \infty} \sup S_b(x_n, x_n, y)$
$$\leq b^2 S_b(x, x, y)$$

for all $y \in X$.

In particular, if $x = y$, then we have $\lim_{n \to \infty} S_b(x_n, x_n, y) = 0$.

In the next section we gave our main results.

3. Main results

Definition 5. *Let (X, S_b) be S_b-metric space and let the mapping $E : X \to X$. We say that the mapping E satisfy generalized (α, β)-rational contraction if there exists continuous maps $\alpha, \beta : [0, \infty) \to [0, \infty)$ such that*

(5.1) $\frac{1}{4b^3} \min \{ S_b(x, x, Ex), S_b(y, y, Ey) \} \leq S_b(x, x, y) \Rightarrow$
$\alpha \left(4b^5 S_b \left(Ex, Ex, Ey \right) \right) \leq \alpha \left(N_E^i \left(x, y \right) \right) - \beta \left(N_E^i \left(x, y \right) \right),$
for all $x, y \in X$, x is comparable to y, $i = 3$ or 4 and

$$N_f^3 \left(x, y \right) = \max \left\{ \begin{array}{c} S_b(x, x, y), \frac{S_b(x, x, Ex) S_b(y, y, Ey)}{1 + S_b(x, x, y) + S_b(Ex, Ex, Ey)}, \\ \frac{S_b(x, x, Ey) S_b(y, y, Ex)}{1 + S_b(x, x, y) + S_b(Ex, Ex, Ey)} \end{array} \right\}.$$

$$N_E^4 \left(x, y \right) = \max \left\{ \begin{array}{c} S_b(x, x, y), S_b(x, x, Ex), S_b(y, y, Ey), \\ \frac{S_b(x, x, Ey) S_b(y, y, Ex)}{1 + S_b(x, x, y) + S_b(Ex, Ex, Ey)} \end{array} \right\},$$

(5.2) $\alpha(t)$ *and* $\beta(t)$ *vanish at* $t = 0$
(5.3) $\beta(t) > 0$ *for* $t > 0$

Definition 6. *Let $(X, S_b, \preceq)$ be a partially ordered complete S_b-metric space which is said to be regular if every two elements of X are comparable, i.e., if $x, y \in X$ implies either $x \preceq y$ or $y \preceq x$.*

Definition 7. *Suppose that $(X, \preceq)$ is a partially ordered set and E is a mapping of X into itself. We say that E is non-decreasing if for every $x, y \in X$, $x \preceq y$ implies that $Ex \preceq Ey$*

Theorem 1. *Let $(X, S_b, \preceq)$ be an partially ordered complete S_b-metric space, $E : X \to X$ satisfies generalized (α, β)-contraction with $i = 3$ and assume that the non decreasing function E is continuous or X is regular. If there exists $x_0 \in X$ with $x_0 \preceq Ex_0$. Then E has unique fixed point in X.*

Proof. Let $x_0 \in X$. Since E is self-map, there exists a sequence $\{x_n\}$ in X such that

$$x_{n+1} = E\, x_n, \quad n = 0, 1, 2, 3, \cdots.$$

Case (i): If $x_n = Ex_n = x_{n+1}$, then clearly proof is over.

Case (ii): Assume $x_n \neq Ex_n$, $\forall\, n$. Since $x_0 \preceq Ex_0 = x_1$ and by definition of E, we have

$$x_0 \preceq Ex_0 \preceq E^2 x_0 \preceq E^3 x_0 \preceq \cdots \preceq E^n x_0 \preceq E^{n+1} x_0 \preceq \cdots$$

Since $\frac{1}{4b^3} \min \{S_b(x_0, x_0, Ex_0), S_b(x_1, x_1, Ex_1)\} \leq S_b(x_0, x_0, x_1)$.
Now

$$
\begin{aligned}
\alpha \left(4b^5 S_b \left(Ex_0, Ex_0, E^2 x_0\right)\right) &= \alpha \left(4b^5 S_b \left(Ex_0, Ex_0, Ex_1\right)\right) \\
&\leq \alpha \left(N_f^3 (x_0, x_1)\right) - \beta \left(N_f^3 (x_0, x_1)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
N_f^3 (x_0, x_1) &= \max \left\{
\begin{array}{l}
S_b (x_0, x_0, x_1), \frac{S_b(x_0,x_0,Ex_0)S_b(x_1,x_1,Ex_1)}{1+S_b(x_0,x_0,x_1)+S_b(Ex_0,Ex_0,Ex_1)}, \\
\frac{S_b(x_0,x_0,Ex_1)S_b(x_1,x_1,Ex_0)}{1+S_b(x_0,x_0,Ex_0)+S_b(Ex_0,Ex_0,E^2x_0)}
\end{array}
\right\} \\
&= \max \left\{
\begin{array}{l}
S_b (x_0, x_0, Ex_0), \frac{S_b(x_0,x_0,Ex_0)S_b(Ex_0,Ex_0,E^2x_0)}{1+S_b(x_0,x_0,Ex_0)+S_b(Ex_0,Ex_0,E^2x_0)}, \\
\frac{S_b(x_0,x_0,E^2x_0)S_b(Ex_0,Ex_0,Ex_0)}{1+S_b(x_0,x_0,Ex_0)+S_b(Ex_0,Ex_0,E^2x_0)}
\end{array}
\right\} \\
&= S_b (x_0, x_0, Ex_0)
\end{aligned}
$$

Thus

$$\alpha \left(4b^5 S_b \left(Ex_0, Ex_0, E^2 x_0\right)\right) \leq \alpha \left(S_b (x_0, x_0, Ex_0)\right) - \beta \left(S_b (x_0, x_0, Ex_0)\right).$$

Also since $\frac{1}{4b^3} \min \{S_b(x_1, x_1, Ex_1), S_b(x_2, x_2, Ex_2)\} \leq S_b(x_1, x_1, x_2)$.
So that we have

$$
\begin{aligned}
\alpha \left(4b^5 S_b \left(E^2 x_0, E^2 x_0, E^3 x_0\right)\right) &\leq \alpha \left(S_b \left(Ex_0, Ex_0, E^2 x_0\right)\right) \\
&\quad - \beta \left(S_b \left(Ex_0, Ex_0, E^2 x_0\right)\right).
\end{aligned}
$$

Continuing this way we can conclude that

$$
\begin{aligned}
\alpha \left(4b^5 S_b \left(E^{n+1} x_0, E^{n+1} x_0, E^{n+2} x_0\right)\right) &\leq \alpha \left(S_b \left(E^n x_0, E^n x_0, E^{n+1} x_0\right)\right) \\
&\quad - \beta \left(S_b \left(E^n x_0, E^n x_0, E^{n+1} x_0\right)\right).
\end{aligned}
$$

Thus $\left\{S_b \left(E^n x_0, E^n x_0, E^{n+1} x_0\right)\right\}$ is non-increasing and must converges to a real number $\eta \geq 0$ (say). Also

$$
\begin{aligned}
\alpha \left(4b^5 S_b \left(E^{n+1} x_0, E^{n+1} x_0, E^{n+2} x_0\right)\right) &\leq \alpha \left(S_b \left(E^n x_0, E^n x_0, E^{n+1} x_0\right)\right) \\
&\quad - \beta \left(S_b \left(E^n x_0, E^n x_0, E^{n+1} x_0\right)\right).
\end{aligned}
$$

Letting $n \to \infty$, we have

$$\alpha(4b^5 \eta) \leq \alpha(\eta) - \beta(\eta).$$

It is clear that $\eta = 0$. that is

$$\lim_{n \to \infty} S_b \left(E^n x_0, E^n x_0, E^{n+1} x_0 \right) = 0.$$

Now we prove $\{E^n x_0\}$ is Cauchy sequence in (X, S_b). On contrary we suppose that $\{E^n x_0\}$ is not Cauchy. Then there exist $\epsilon > 0$ and monotonically increasing sequence of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$.

(1) $$S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k} x_0 \right) \geq \epsilon$$

and

(2) $$S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k - 1} x_0 \right) < \epsilon.$$

First we claim that

(3) $$\frac{1}{4b^3} \min \left\{ S_b \left(x_{m_k}, x_{m_k}, E x_{m_k} \right), S_b \left(x_{n_k - 1}, x_{n_k - 1}, E x_{n_k - 1} \right) \right\}$$
$$\leq S_b \left(x_{m_k}, x_{m_k}, x_{n_k - 1} \right).$$

On contrary suppose

$$\frac{1}{4b^3} \min \left\{ S_b \left(x_{m_k}, x_{m_k}, E x_{m_k} \right), S_b \left(x_{n_k - 1}, x_{n_k - 1}, E x_{n_k - 1} \right) \right\}$$
$$> S_b \left(x_{m_k}, x_{m_k}, x_{n_k - 1} \right).$$

Now consider

$$\begin{aligned}
\epsilon \leq{}& S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k} x_0 \right) \\
\leq{}& 2b S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k - 1} x_0 \right) + b^2 S_b \left(E^{n_k - 1} x_0, E^{n_k - 1} x_0, E^{n_k} x_0 \right) \\
<{}& \frac{1}{2b^2} \min \left\{ S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{m_k + 1} x_0 \right), S_b \left(x_{n_k - 1}, x_{n_k - 1}, x_{n_k} \right) \right\} \\
& + b^2 S_b \left(E^{n_k - 1} x_0, E^{n_k - 1} x_0, E^{n_k} x_0 \right).
\end{aligned}$$

Letting $k \to \infty$, it follows $\epsilon \leq 0$. It is a contradiction. Hence our claim (3) is holds. From (1) and (2), we have

$$\begin{aligned}
\epsilon \leq{}& S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k} x_0 \right) \\
\leq{}& 2b S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{m_k + 1} x_0 \right) + b^2 S_b \left(E^{m_k + 1} x_0, E^{m_k + 1} x_0, E^{n_k} x_0 \right).
\end{aligned}$$

Letting $k \to \infty$, we have $\frac{\epsilon}{b^2} \leq S_b \left(E^{m_k+1} x_0, E^{m_k+1} x_0, E^{n_k} x_0 \right)$. Then

$$\text{(4)} \quad \alpha \left(4b^3 \epsilon \right) \leq \lim_{k \to \infty} \alpha \left(4b^5 S_b \left(E^{m_k+1} x_0, E^{m_k+1} x_0, E^{n_k} x_0 \right) \right)$$

$$= \lim_{k \to \infty} \alpha \left(4b^5 S_b \left(E x_{m_k}, E x_{m_k}, E x_{n_k-1} \right) \right)$$

$$\leq \lim_{k \to \infty} \alpha \left(N_f^3 \left(x_{m_k}, x_{n_k-1} \right) \right) - \lim_{k \to \infty} \beta \left(N_f^3 \left(x_{m_k}, x_{n_k-1} \right) \right),$$

where

$$\lim_{k \to \infty} N_f^3 \left(x_{m_k}, x_{n_k-1} \right)$$

$$= \lim_{k \to \infty} \max \left\{ \begin{array}{c} S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k-1} x_0 \right), \\ \frac{S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{m_k+1} x_0 \right) S_b \left(E^{n_k-1} x_0, E^{n_k-1} x_0, E^{n_k} x_0 \right)}{1 + S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k-1} x_0 \right) + S_b \left(E^{m_k+1} x_0, E^{m_k+1} x_0, E^{n_k} x_0 \right)} \\ \frac{S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k} x_0 \right) S_b \left(E^{n_k-1} x_0, E^{n_k-1} x_0, E^{m_k+1} x_0 \right)}{1 + S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k-1} x_0 \right) + S_b \left(E^{m_k+1} x_0, E^{m_k+1} x_0, E^{n_k} x_0 \right)} \end{array} \right\}.$$

But

$$\lim_{k \to \infty} \frac{S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k} x_0 \right) S_b \left(E^{n_k-1} x_0, E^{n_k-1} x_0, E^{m_k+1} x_0 \right)}{1 + S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k-1} x_0 \right) + S_b \left(E^{m_k+1} x_0, E^{m_k+1} x_0, E^{n_k} x_0 \right)}$$

$$= \lim_{k \to \infty} \frac{\left(\begin{array}{c} \left[2b S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k-1} x_0 \right) \right. \\ \left. + b^2 S_b \left(E^{n_k-1} x_0, E^{n_k-1} x_0, E^{n_k} x_0 \right) \right] \\ \times \left[2b S_b \left(E^{n_k-1} x_0, E^{n_k-1} x_0, E^{m_k} x_0 \right) \right. \\ \left. + b^2 S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{m_k} x_0 \right) \right] \end{array} \right)}{1 + S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k-1} x_0 \right) + S_b \left(E^{m_k+1} x_0, E^{m_k+1} x_0, E^{n_k} x_0 \right)}$$

$$\leq 4b^3 \epsilon.$$

and

$$\lim_{k \to \infty} \frac{S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{m_k+1} x_0 \right) S_b \left(E^{n_k-1} x_0, E^{n_k-1} x_0, E^{n_k} x_0 \right)}{1 + S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k-1} x_0 \right) + S_b \left(E^{m_k+1} x_0, E^{m_k+1} x_0, E^{n_k} x_0 \right)}$$

$$= \lim_{k \to \infty} \frac{\left(\begin{array}{c} \left[2b S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k-1} x_0 \right) \right. \\ \left. + b^2 S_b \left(E^{n_k-1} x_0, E^{n_k-1} x_0, E^{m_k+1} x_0 \right) \right] \\ \times \left[2b S_b \left(E^{n_k-1} x_0, E^{n_k-1} x_0, E^{m_k} x_0 \right) \right. \\ \left. + b^2 S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{m_k} x_0 \right) \right] \end{array} \right)}{1 + S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k-1} x_0 \right) + S_b \left(E^{m_k+1} x_0, E^{m_k+1} x_0, E^{n_k} x_0 \right)}$$

$$\leq 4b^3 \epsilon.$$

Now from (4), we have

$$\alpha \left(4b^3 \epsilon \right) \leq \alpha \left(4 \ b^3 \ \epsilon \right) - \lim_{k \to \infty} \beta \left(N_E^4 \left(x_{m_k}, x_{n_k-1} \right) \right) < \alpha \left(4b^3 \epsilon \right).$$

Is a contradiction.

Hence $\{E^n x_0\}$ is Cauchy sequence in (X, S_b). By completeness of (X, S_b), it follows the sequence $\{E^n x_0\} \to \vartheta \in (X, S_b)$. That is

$$\lim_{k \to \infty} E^n x_0 = \vartheta = \lim_{k \to \infty} E^{n+1} x_0.$$

First we claim that for each $n \geq 1$, at least one of the following assertion is holds.

$$\frac{1}{4b^3} S_b(x_{n+1}, x_{n+1}, x_n) \leq S_b(\vartheta, \vartheta, x_n) \quad \text{or}$$

$$\frac{1}{4b^3} S_b(x_n, x_n, x_{n-1}) \leq S_b(\vartheta, \vartheta, x_{n-1}).$$

On contrary suppose that

$$\frac{1}{4b^3} S_b(x_{n+1}, x_{n+1}, x_n) > S_b(\vartheta, \vartheta, x_n) \quad \text{and}$$

$$\frac{1}{4b^3} S_b(x_n, x_n, x_{n-1}) > S_b(\vartheta, \vartheta, x_{n-1}).$$

Now consider

$$
\begin{aligned}
S_b(x_{n-1}, x_{n-1}, x_n) &\leq 2b S_b(x_{n-1}, x_{n-1}, \vartheta) + b^2 S_b(\vartheta, \vartheta, x_n) \\
&< 2b^2 S_b(\vartheta, \vartheta, x_{n-1}) + b^2 \frac{1}{4b^3} S_b(x_{n+1}, x_{n+1}, x_n) \\
&< 2b^2 \frac{1}{4b^3} S_b(x_n, x_n, x_{n-1}) + \frac{1}{4b} S_b(x_{n+1}, x_{n+1}, x_n) \\
&= \frac{1}{2b} b\, S_b(x_{n-1}, x_{n-1}, x_n) + \frac{1}{4b} b\, S_b(x_n, x_n, x_{n+1}) \\
&\leq \frac{1}{2} S_b(x_{n-1}, x_{n-1}, x_n) + \frac{1}{4b^3} S_b(x_{n-1}, x_{n-1}, x_n) \\
&= \frac{2b^3 + 1}{4b^3} S_b(x_{n-1}, x_{n-1}, x_n) \\
&\leq \frac{3}{4} S_b(x_{n-1}, x_{n-1}, x_n).
\end{aligned}
$$

It is a contradiction. Hence our claim is holds. Since $E x_n \to \vartheta$ and (X, S_b) is regular, it follows x_n is comparable to ϑ. Suppose $E\vartheta \neq \vartheta$. From (5.1) and by the definition of α, Lemma (3), we have

$$
\begin{aligned}
\text{(5)} \quad \alpha\left(2b^4 S_b(E\vartheta, E\vartheta, \vartheta)\right) &\leq \liminf_{n \to \infty} \alpha\left(4b^5 S_b\left(E\vartheta, E\vartheta, E^{n+1} x_0\right)\right) \\
&\leq \liminf_{n \to \infty} \alpha\left(N_f^3(\vartheta, x_n)\right) \\
&\quad - \liminf_{n \to \infty} \beta\left(N_f^3(\vartheta, x_n)\right)
\end{aligned}
$$

Here

$$\lim_{n \to \infty} N_f^3(\vartheta, x_n) = \lim_{n \to \infty} \max \left\{ \begin{array}{c} S_b(\vartheta, \vartheta, E\vartheta), \frac{S_b(\vartheta, \vartheta, E\vartheta), S_b(x_n, x_n, Ex_n)}{1 + S_b(\vartheta, \vartheta, x_n) S_b(E\vartheta, E\vartheta, x_n)}, \\ \frac{S_b(\vartheta, \vartheta, Ex_n) S_b(x_n, x_n, E\vartheta)}{1 + S_b(\vartheta, \vartheta, x_n) + S_b(E\vartheta, E\vartheta, Ex_n)} \end{array} \right\}$$

$$= S_b(\vartheta, \vartheta, E\vartheta).$$

Hence from (5), we have

$$\alpha \left(2b^2 \ S_b(E\vartheta, E\vartheta, \vartheta) \right) \leq \alpha \left(S_b(\vartheta, \vartheta, E\vartheta) \right) - \beta \left(S_b(\vartheta, \vartheta, E\vartheta) \right)$$
$$\leq \alpha \left(S_b(\vartheta, \vartheta, E\vartheta) \right).$$

Clearly ϑ is fixed point of E. Assume ϑ^* is also fixed point of $E \ni \vartheta \neq \vartheta^*$. Since $\frac{1}{4b^3} \min \{ S_b(\vartheta, \vartheta, E\vartheta), S_b(\vartheta^*, \vartheta^*, E\vartheta^*) \} \leq S_b(\vartheta, \vartheta, \vartheta^*)$.

Consider

$$\alpha \left(4b^5 S_b(\vartheta, \vartheta, \vartheta^*) \right) \leq \alpha \left(N_E^4(\vartheta, \vartheta^*) \right) - \beta \left(N_E^4(\vartheta, \vartheta^*) \right)$$
$$= \alpha \left(S_b(\vartheta, \vartheta, \vartheta^*) \right) - \beta \left(S_b(\vartheta, \vartheta, \vartheta^*) \right)$$
$$< \alpha \left(S_b(\vartheta, \vartheta, \vartheta^*) \right).$$

Clearly ϑ is unique fixed point of E in (X, S_b).   ∎

Example 2. Let us define $S_b : X \times X \times X \to \mathbb{R}^+$ by $S_b(u, v, w) = (|v + w - 2u| + |v - w|)^2$ where $X = [0, 1]$ and $\preceq$ by $u \preceq w \iff u \leq w$. So clearly $(X, S_b, \preceq)$ is complete ordered S_b - metric space with $b = 4$. Define $E : X \to X$ by $E(u) = \frac{u}{16\sqrt{2}}$, also $\alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ by $\alpha(t) = t$ and $\beta(t) = \frac{(2b^2-1)t}{2b^2}$.

$$\alpha \left(4b^5 S_b(Eu, Eu, Ev) \right) = 4b^5(|Eu + Ev - 2Eu| + |Eu - Ev|)^2$$
$$= 4b^5 \left(2 \left| \frac{u}{16\sqrt{2}} - \frac{v}{16\sqrt{2}} \right| \right)^2$$
$$= \frac{1}{2b^2} S_b(u, u, v)$$
$$\leq \alpha \left(N_f^3(u, v) \right) - \beta \left(N_f^3(u, v) \right).$$

Hence from Theorem 1, 0 is unique fixed point of E.

Theorem 2. *Let $(X, S_b, \preceq)$ be an partially ordered complete S_b metric space and let $E : X \to X$ be satisfies generalized (α, β)-contraction with $i = 4$ and assume that X is regular or the non decreasing function E is continuous. If there exists $x_0 \in X$ with $x_0 \preceq Ex_0$. Then E has unique fixed point in X.*

Proof. If we replace $N_E^4(x, y)$ in place of $N_E^3(x, y)$, the rest of proof follows from Theorem 1. ∎

Theorem 3. *Let $(X, S_b, \preceq)$ be an partially ordered complete S_b metric space and let $E : X \to X$ be satisfies (3.1) $\frac{1}{4b^3} \min \{S_b(x, x, Ex), S_b(y, y, Ey)\} \leq S_b(x, x, y)$ implies that*

$$4b^5 S_b(Ex, Ex, Ey) \leq N_E^i(x, y) - \beta(N_E^i(x, y)),$$

where $\beta : [0, \infty) \to [0, \infty)$ continuous with $\beta(t) > 0$ for $t > 0$ and $i = 3$ and X is regular or the non decreasing function E is continuous. If there exists $x_0 \in X$ with $x_0 \preceq Ex_0$. Then E has unique fixed point in X.

Proof. Let $x_0 \in X$. Since E is self-map, there exists a sequence $\{x_n\}$ in X such that

$$x_{n+1} = E\ x_n, n = 0, 1, 2, 3, \cdots.$$

Case (i)**:** If $x_n = Ex_n = x_{n+1}$, then clearly proof is over.

Case (ii)**:** Assume $x_n \neq Ex_n, \forall n$. Since $x_0 \preceq Ex_0 = x_1$ and by definition of E, we have

$$x_0 \preceq Ex_0 \preceq E^2 x_0 \preceq E^3 x_0 \preceq \cdots \preceq E^n x_0 \preceq E^{n+1} x_0 \preceq \cdots$$

Since $\frac{1}{4b^3} \min \{S_b(x_0, x_0, Ex_0), S_b(x_1, x_1, Ex_1)\} \leq S_b(x_0, x_0, x_1)$. Now

$$
\begin{aligned}
4b^5 S_b(Ex_0, Ex_0, E^2 x_0) &= 4b^5 S_b(Ex_0, Ex_0, Ex_1) \\
&\leq N_f^3(x_0, x_1) - \beta(N_f^3(x_0, x_1)),
\end{aligned}
$$

where

$$
N_f^3(x_0, x_1) = \max \left\{
\begin{array}{l}
S_b(x_0, x_0, x_1), \frac{S_b(x_0, x_0, Ex_0) S_b(x_1, x_1, Ex_1)}{1 + S_b(x_0, x_0, x_1) + S_b(Ex_0, Ex_0, Ex_1)}, \\
\frac{S_b(x_0, x_0, Ex_1) S_b(x_1, x_1, Ex_0)}{1 + S_b(x_0, x_0, Ex_0) + S_b(Ex_0, Ex_0, E^2 x_0)}
\end{array}
\right\}
$$

$$
= \max \left\{
\begin{array}{l}
S_b(x_0, x_0, Ex_0), \frac{S_b(x_0, x_0, Ex_0) S_b(Ex_0, Ex_0, E^2 x_0)}{1 + S_b(x_0, x_0, Ex_0) + S_b(Ex_0, Ex_0, E^2 x_0)}, \\
\frac{S_b(x_0, x_0, E^2 x_0) S_b(Ex_0, Ex_0, Ex_0)}{1 + S_b(x_0, x_0, Ex_0) + S_b(Ex_0, Ex_0, E^2 x_0)}
\end{array}
\right\}
$$

$$
= S_b(x_0, x_0, Ex_0)
$$

Thus

$$4b^5 S_b(Ex_0, Ex_0, E^2 x_0) \leq S_b(x_0, x_0, Ex_0) - \beta(S_b(x_0, x_0, Ex_0)).$$

Also since $\frac{1}{4b^3} \min \{S_b(x_1, x_1, Ex_1), S_b(x_2, x_2, Ex_2)\} \leq S_b(x_1, x_1, x_2)$.

So that we have

$$4b^5 S_b \left(E^2 x_0, E^2 x_0, E^3 x_0\right) \leq S_b \left(E x_0, E x_0, E^2 x_0\right)$$
$$- \beta \left(S_b \left(E x_0, E x_0, E^2 x_0\right)\right).$$

Continuing this way we can conclude that

$$4b^5 S_b \left(E^{n+1} x_0, E^{n+1} x_0, E^{n+2} x_0\right) \leq S_b \left(E^n x_0, E^n x_0, E^{n+1} x_0\right)$$
$$- \beta \left(S_b \left(E^n x_0, E^n x_0, E^{n+1} x_0\right)\right).$$

Thus $\left\{S_b \left(E^n x_0, E^n x_0, E^{n+1} x_0\right)\right\}$ is non-increasing and must converges to a real number $\eta \geq 0$ (say). Also

$$4b^5 S_b \left(E^{n+1} x_0, E^{n+1} x_0, E^{n+2} x_0\right) \leq S_b \left(E^n x_0, E^n x_0, E^{n+1} x_0\right)$$
$$- \beta \left(S_b \left(E^n x_0, E^n x_0, E^{n+1} x_0\right)\right).$$

Letting $n \to \infty$, we have
$$4b^5 \eta \leq \eta - \beta(\eta).$$

It is clear that $\eta = 0$. that is

$$\lim_{n \to \infty} S_b \left(E^n x_0, E^n x_0, E^{n+1} x_0\right) = 0.$$

Now we prove $\{E^n x_0\}$ is Cauchy sequence in (X, S_b). On contrary suppose that $\{E^n x_0\}$ is not Cauchy. Then there exist $\epsilon > 0$ and monotonically increasing sequence of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$.

(6) $$S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k} x_0\right) \geq \epsilon$$

and

(7) $$S_b \left(E^{m_k} x_0, E^{m_k} x_0, E^{n_k - 1} x_0\right) < \epsilon.$$

First we claim that

(8) $$\frac{1}{4b^3} \min \left\{S_b \left(x_{m_k}, x_{m_k}, E x_{m_k}\right), S_b \left(x_{n_k - 1}, x_{n_k - 1}, E x_{n_k - 1}\right)\right\}$$
$$\leq S_b \left(x_{m_k}, x_{m_k}, x_{n_k - 1}\right).$$

On contrary suppose that

$$\frac{1}{4b^3} \min \left\{S_b \left(x_{m_k}, x_{m_k}, E x_{m_k}\right), S_b \left(x_{n_k - 1}, x_{n_k - 1}, E x_{n_k - 1}\right)\right\}$$
$$> S_b \left(x_{m_k}, x_{m_k}, x_{n_k - 1}\right).$$

Now consider

$$\begin{aligned}
\epsilon \ &\leq \ S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{n_k}x_0\right) \\
&\leq \ 2bS_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0\right) + b^2 S_b\left(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{n_k}x_0\right) \\
&< \ \frac{1}{2b^2}\min\left\{S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{m_k+1}x_0\right), S_b\left(x_{n_k-1}, x_{n_k-1}, x_{n_k}\right)\right\} \\
&\quad + b^2 S_b\left(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{n_k}x_0\right).
\end{aligned}$$

Letting $k \to \infty$, it follows that $\epsilon \leq 0$. It is a contradiction. Hence our claim (8) is holds. From (6) and (7), we have

$$\begin{aligned}
\epsilon \ &\leq \ S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{n_k}x_0\right) \\
&\leq \ 2bS_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{m_k+1}x_0\right) + b^2 S_b\left(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0\right).
\end{aligned}$$

Letting $k \to \infty$, we have $\frac{\epsilon}{b^2} \leq S_b\left(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0\right)$. Then

$$\begin{aligned}
(9) \qquad 4b^3\epsilon \ &\leq \ \lim_{k\to\infty} 4b^5 S_b\left(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0\right) \\
&= \ \lim_{k\to\infty} 4b^5 S_b\left(Ex_{m_k}, Ex_{m_k}, Ex_{n_k-1}\right) \\
&\leq \ \lim_{k\to\infty} N_f^3\left(x_{m_k}, x_{n_k-1}\right) - \lim_{k\to\infty} \beta\left(N_f^3\left(x_{m_k}, x_{n_k-1}\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
&\lim_{k\to\infty} N_f^3\left(x_{m_k}, x_{n_k-1}\right) \\
&= \lim_{k\to\infty} \max\left\{ \begin{array}{c}
S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0\right), \\
\frac{S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{m_k+1}x_0\right)S_b\left(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{n_k}x_0\right)}{1+S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0\right)+S_b\left(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0\right)} \\
\frac{S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{n_k}x_0\right)S_b\left(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{m_k+1}x_0\right)}{1+S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0\right)+S_b\left(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0\right)}
\end{array}\right\}.
\end{aligned}$$

But

$$\begin{aligned}
&\lim_{k\to\infty} \frac{S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{n_k}x_0\right) S_b\left(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{m_k+1}x_0\right)}{1 + S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0\right) + S_b\left(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0\right)} \\
&= \lim_{k\to\infty} \frac{\left(\begin{array}{c}
\left[2bS_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0\right)\right. \\
\left.+ b^2 S_b\left(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{n_k}x_0\right)\right] \\
\times \left[2bS_b\left(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{m_k}x_0\right)\right. \\
\left.+ b^2 S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{m_k}x_0\right)\right]
\end{array}\right)}{1 + S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0\right) + S_b\left(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0\right)} \\
&\leq 4b^3\epsilon.
\end{aligned}$$

and

$$\lim_{k\to\infty} \frac{S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{m_{k+1}}x_0\right) S_b\left(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{n_k}x_0\right)}{1 + S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0\right) + S_b\left(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0\right)}$$

$$= \lim_{k\to\infty} \frac{\left(\begin{array}{c}\left[2bS_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0\right)\right. \\ \left.+ b^2 S_b\left(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{m_k+1}x_0\right)\right] \\ \times \left[2bS_b\left(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{m_k}x_0\right)\right. \\ \left.+ b^2 S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{m_k}x_0\right)\right]\end{array}\right)}{1 + S_b\left(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0\right) + S_b\left(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0\right)}$$

$$\leq 4b^3\epsilon.$$

Now from (9), we have

$$4b^3\epsilon \leq 4b^3\epsilon - \lim_{k\to\infty} \beta\left(N_E^4\left(x_{m_k}, x_{n_k-1}\right)\right) < 4b^3\epsilon.$$

Is a contradiction. Hence $\{E^n x_0\}$ is Cauchy sequence in (X, S_b). By completeness of (X, S_b), it follows that the sequence $\{E^n x_0\} \to \vartheta \in (X, S_b)$. That is

$$\lim_{k\to\infty} E^n x_0 = \vartheta = \lim_{k\to\infty} E^{n+1}x_0.$$

First we claim that for each $n \geq 1$, at least one of the following assertion is holds.

$$\frac{1}{4b^3}S_b\left(x_{n+1}, x_{n+1}, x_n\right) \leq S_b(\vartheta, \vartheta, x_n) \quad \text{or}$$

$$\frac{1}{4b^3}S_b\left(x_n, x_n, x_{n-1}\right) \leq S_b(\vartheta, \vartheta, x_{n-1}).$$

On contrary suppose that

$$\frac{1}{4b^3}S_b\left(x_{n+1}, x_{n+1}, x_n\right) > S_b(\vartheta, \vartheta, x_n) \quad \text{and}$$

$$\frac{1}{4b^3}S_b\left(x_n, x_n, x_{n-1}\right) > S_b(\vartheta, \vartheta, x_{n-1}).$$

Now consider

$$S_b\left(x_{n-1}, x_{n-1}, x_n\right) \leq 2bS_b\left(x_{n-1}, x_{n-1}, \vartheta\right) + b^2 S_b\left(\vartheta, \vartheta, x_n\right)$$

$$\leq \frac{1}{2}S_b\left(x_{n-1}, x_{n-1}, x_n\right) + \frac{1}{4b^3}S_b\left(x_{n-1}, x_{n-1}, x_n\right)$$

$$= \frac{2b^3 + 1}{4b^3}S_b\left(x_{n-1}, x_{n-1}, x_n\right) \leq \frac{3}{4}S_b\left(x_{n-1}, x_{n-1}, x_n\right).$$

It is a contradiction. Hence our claim is holds. Since $Ex_n \to \vartheta$ and (X, S_b) is regular, it follows x_n is comparable to ϑ. Suppose $E\vartheta \neq \vartheta$. From (3) and Lemma (3), we have

$$(10) \quad 2b^4 S_b(E\vartheta, E\vartheta, \vartheta) \leq \lim_{n\to\infty} \inf\left(4b^5 S_b\left(E\vartheta, E\vartheta, E^{n+1}x_0\right)\right)$$

$$\leq \lim_{n\to\infty} \inf\left(N_f^3\left(\vartheta, x_n\right)\right) - \lim_{n\to\infty} \inf \beta\left(N_f^3\left(\vartheta, x_n\right)\right)$$

Here

$$\lim_{n\to\infty} N_f^3 (\vartheta, x_n) = \lim_{n\to\infty} \max \left\{ \begin{array}{c} S_b (\vartheta, \vartheta, E\vartheta) , \frac{S_b(\vartheta,\vartheta,E\vartheta),S_b(x_n,x_n,Ex_n)}{1+S_b(\vartheta,\vartheta,x_n)S_b(E\vartheta,E\vartheta,x_n)}, \\ \frac{S_b(\vartheta,\vartheta,Ex_n)S_b(x_n,x_n,E\vartheta)}{1+S_b(\vartheta,\vartheta,x_n)+S_b(E\vartheta,E\vartheta,Ex_n)} \end{array} \right\}$$
$$= S_b (\vartheta, \vartheta, E\vartheta) .$$

Hence from (10), we have

$$2b^2 S_b(E\vartheta, E\vartheta, \vartheta) \leq S_b (\vartheta, \vartheta, E\vartheta) - \beta (S_b (\vartheta, \vartheta, E\vartheta))$$
$$\leq S_b (\vartheta, \vartheta, E\vartheta) .$$

Clearly ϑ is fixed point of E. Assume ϑ^* is also fixed point of $E \ni \vartheta \neq \vartheta^*$. Since $\frac{1}{4b^3} \min \{S_b (\vartheta, \vartheta, E\vartheta) , S_b (\vartheta^*, \vartheta^*, E\vartheta^*)\} \leq S_b (\vartheta, \vartheta, \vartheta^*)$.

Consider

$$4b^5 S_b (\vartheta, \vartheta, \vartheta^*) \leq N_E^4 (\vartheta, \vartheta^*) - \beta \left(N_E^4 (\vartheta, \vartheta^*) \right)$$
$$= S_b (\vartheta, \vartheta, \vartheta^*) - \beta (S_b (\vartheta, \vartheta, \vartheta^*))$$
$$< S_b (\vartheta, \vartheta, \vartheta^*) .$$

Clearly ϑ is unique fixed point of E in (X, S_b). ∎

Theorem 4. *Let $(X, S_b, \preceq)$ be an partially ordered complete S_b metric space and let $E : X \to X$ be satisfies generalized (α, β)-contraction with $i = 4$ and assume that X is regular or the non decreasing function E is continuous. If there exists $x_0 \in X$ with $x_0 \preceq Ex_0$. Then E has unique fixed point in X.*

Proof. If we replace $N_E^4 (x, y)$ in place of $N_E^3 (x, y)$, the rest of proof follows from Theorem 3. ∎

Theorem 5. *Let $(X, S_b, \preceq)$ be an partially ordered complete S_b metric space and let $E : X \to X$ be satisfies (5.1) $\frac{1}{4b^3} \min \{S_b(x, x, Ex), S_b(y, y, Ey)\} \leq S_b(x, x, y)$ implies that*

$$S_b (Ex, Ex, Ey) \leq \lambda N_E^i (x, y) ,$$

where $\lambda \in \left[0, \frac{1}{4b^5}\right)$ and $i = 3, 4$ and the non decreasing function E is continuous or X is regular. If there exists $x_0 \in X$ with $x_0 \preceq Ex_0$. Then E has unique fixed point in X.

4. Application to integral equations

In this section, we study the existence of a unique solution to an initial value problem as an application to Theorem 3.

Theorem 6. *Consider the I. V. P.*

(11) $u'(x) = P(x, u(x)), \quad x \in I = [0, 1], \quad u(0) = u_0$

where $P : I \times \left[\frac{u_0}{4}, \infty\right) \to \left[\frac{u_0}{4}, \infty\right)$ and $u_0 \in \mathbb{R}$. Then (11) has unique solution
.

Proof. The integral equation of I. V. P. (11) is

$$u(x) = u_0 + 5b^4 \int_0^x P(t, u(t))dt.$$

Let $X = C\left(I, \left[\frac{u_0}{4}, \infty\right)\right)$ and $S_b(u, v, w) = (|v + w - 2v| + |v - w|)^2$ for $u, v, w \in X$. Define $\beta : [0, \infty) \to [0, \infty)$ by $\beta(x) = \frac{(25b^3 - 4)x}{25b^3}$. Define $E : X \to X$ by

(12) $E(u)(x) = \dfrac{u_0}{5b^4} + \displaystyle\int_0^x P(t, u(t))dt.$

Now

$$
\begin{aligned}
4b^5 & S_b(Eu(x), Eu(x), Ev(x)) \\
&= 4b^5 \left\{ |Eu(x) + Ev(x) - 2Eu(x)| + |Eu(x) - Ev(x)| \right\}^2 \\
&= 16b^5 |Eu(x) - Ev(x)|^2 \\
&= \frac{16b^5}{25b^8} \left| u_0 + 5b^4 \int_0^x P(t, u(t))dt - v_0 - 5b^4 \int_0^x P(x, v(x))dx \right|^2 \\
&= \frac{16}{25b^3} |u(x) - v(x)|^2 = \frac{4}{25b^3} S(u, u, v) \\
&\leq N_f^3(u, v) - \beta\left(N_f^3(u, v) \right)
\end{aligned}
$$

It follows from Theorem 3, E has a unique fixed point in X.   ∎

5. Application to homotopy

In this section, we study the existence of a unique solution to homotopy theory.

Theorem 7. *Let (X, S_b) be complete S_b-metric space, U be an open subset of X and $\overline{U}$ be closed subset of X such that $U \subseteq \overline{U}$. Suppose $H_b : \overline{U} \times [0, 1] \to X$ be an operator with following conditions are satisfying, (7.1)*

$u \neq H_b(u, \kappa)$ *for each* $u \in \partial U$ *and* $\kappa \in [0, 1]$ *(Here* $\partial U =$ *boundary of* U *in* X), (7.2) $\frac{1}{4b^3} \min \{S_b(u, u, H_b(u, \kappa)), S_b(v, v, H_b(v, \kappa))\} \leq S_b(u, u, v)$ *implies that*

$$4b^5 S_b(H_b(u, \kappa), H_b(u, \kappa), H_b(v, \kappa)) \leq S_b(u, u, v) - \beta(S_b(u, u, v))$$

for all $u, v \in \overline{U}$ *and* $\kappa \in [0, 1]$, *where* β *defined in Theorem(3),* (7.3) $\exists$ $M_b \geq 0 \ni S_b(H_b(u, \kappa), H_b(u, \kappa), H_b(u, \zeta)) \leq M_b|\kappa - \zeta|$ *for every* $u \in \overline{U}$ *and* $\kappa, \zeta \in [0, 1]$. *Then* $H_b(\cdot, 0)$ *has a fixed point* $\iff$ $H_b(\cdot, 1)$ *has a fixed point.*

Proof. Let the set

$$B = \{\kappa \in [0, 1] : u = H_b(u, \kappa) \text{ for some } u \in U\}.$$

Since $H_b(., 0)$ has a fixed point in U, so $0 \in B$.

Now we show that B is both closed and open in $[0, 1]$ and hence by the connectedness $B = [0, 1]$. Let $\{\kappa_n\}_{n=1}^{\infty} \subseteq B$ with $\kappa_n \to \kappa \in [0, 1]$ as $n \to \infty$. We must show $\kappa \in B$. Since $\kappa_n \in B$ for $n = 1, 2, 3, \cdots$, there exists $u_n \in U$ with $u_n = H_b(u_n, \kappa_n)$.

Consider

$$
\begin{aligned}
S_b(u_n, u_n, u_{n+1}) &= S_b(H_b(u_n, \kappa_n), H_b(u_n, \kappa_n), H_b(u_{n+1}, \kappa_{n+1})) \\
&\leq 2b S_b(H_b(u_n, \kappa_n), H_b(u_n, \kappa_n), H_b(u_{n+1}, \kappa_n)) \\
&\quad + b^2 S_b(H_b(u_{n+1}, \kappa_n), H_b(u_{n+1}, \kappa_n), H_b(u_{n+1}, \kappa_{n+1})) \\
&\leq 2b S_b(H_b(u_n, \kappa_n), H_b(u_n, \kappa_n), H_b(u_{n+1}, \kappa_n)) \\
&\quad + b^2 M|\kappa_n - \kappa_{n+1}|.
\end{aligned}
$$

Letting $n \to \infty$, we get

$$
\lim_{n \to \infty} S_b(u_n, u_n, u_{n+1})
$$
$$
\leq \lim_{n \to \infty} 2b S_b(H_b(u_n, \kappa_n), H_b(u_n, \kappa_n), H_b(u_{n+1}, \kappa_n)) + 0.
$$

Since

$$
\frac{1}{4b^3} \min \{S_b(u_n, u_n, H_b(u_n, \kappa)), S_b(u_{n+1}, u_{n+1}, H_b(u_{n+1}, \kappa))\}
$$
$$
\leq S_b(u_n, u_n, u_{n+1}).
$$

Therefore, from 7, we have

$$
\lim_{n \to \infty} 2b^4 S_b(u_n, u_n, u_{n+1}) \leq \lim_{n \to \infty} 4b^5 S_b(H_b(u_n, \kappa_n), H_b(u_n, \kappa_n), H_b(u_{n+1}, \kappa_n))
$$
$$
\leq \lim_{n \to \infty} [S_b(u_n, u_n, u_{n+1}) - \beta(S_b(u_n, u_n, u_{n+1}))].
$$

It follows that

$$(13) \qquad \lim_{n \to \infty} S_b(u_n, u_n, u_{n+1}) = 0.$$

Now we prove $\{u_n\}$ is a S_b−Cauchy sequence in (X, S_b). On contrary suppose $\{u_n\}$ is not a S_b−Cauchy. There exists an $\epsilon > 0$ and monotone increasing sequence of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$,

$$(14) \qquad S_b(u_{m_k}, u_{m_k}, u_{n_k}) \geq \epsilon$$

and

$$(15) \qquad S_b(u_{m_k}, u_{m_k}, u_{n_k-1}) < \epsilon.$$

Therefore from (14) and (15), we have

$$\epsilon \leq S_b(u_{m_k}, u_{m_k}, u_{n_k})$$
$$\leq 2b S_b(u_{m_k}, u_{m_k}, u_{m_k+1}) + b^2 S_b(u_{m_k+1}, u_{m_k+1}, u_{n_k}).$$

Letting $k \to \infty$, we have

$$\frac{\epsilon}{b^2} \leq \lim_{n \to \infty} S_b\left(u_{m_k+1}, u_{m_k+1}, u_{n_k}\right).$$

But

$$\lim_{n \to \infty} S_b\left(u_{m_k+1}, u_{m_k+1}, u_{n_k}\right) \leq \lim_{n \to \infty} 9b^4 S_b\left(H_b(u_{m_k+1}, \kappa_{m_k+1}),\right.$$
$$\left. H_b(u_{m_k+1}, \kappa_{m_k+1}), H_b(u_{n_k}, \kappa_{n_k})\right)$$
$$\leq \lim_{n \to \infty}\left[S_b(u_{m_k+1}, u_{m_k+1}, u_{n_k})\right.$$
$$\left. - \beta(S_b(u_{m_k+1}, u_{m_k+1}, u_{n_k}))\right].$$

It follows

$$\lim_{n \to \infty} S_b\left(u_{m_k+1}, u_{m_k+1}, u_{n_k})\right) \leq 0.$$

So that

$$\epsilon \leq 0,$$

it is a contradiction.

Hence $\{u_n\}$ is a S_b−Cauchy sequence in (X, S_b). By completeness $\exists \eta \in U \ni$

$$(16) \qquad \lim_{n \to \infty} u_n = \eta = \lim_{n \to \infty} u_{n+1}.$$

Since

$$\frac{1}{4b^3} \min\left\{S_b\left(\eta, \eta, H_b(\eta, \kappa)\right), S_b\left(u_n, u_n, H_b(u_n, \kappa)\right)\right\} \leq S_b\left(\eta, \eta, u_n\right).$$

$$\frac{1}{2b} S_b\left(H_b(\eta, \kappa), H_b(\eta, \kappa), \eta\right) \leq \lim_{n \to \infty} \inf S_b\left(H_b(\eta, \kappa), H_b(\eta, \kappa), H_b(u_n, \kappa)\right)$$
$$\leq \lim_{n \to \infty} \inf 4b^5 S_b\left(H_b(\eta, \kappa), H_b(\eta, \kappa), H_b(u_n, \kappa)\right)$$
$$\leq \lim_{n \to \infty} \inf[S_b\left(\eta, \eta, u_n\right) - \beta(S_b(\eta, \eta, u_n))]$$
$$= 0.$$

It follows that $\eta = H_b(\eta, \kappa)$. Thus $\kappa \in B$. Clearly B is closed in $[0, 1]$. Let $\kappa_0 \in B$, then $\exists u_0 \in U \ni u_0 = H_b(u_0, \kappa_0)$. Since U is open, then there exists $r > 0$ such that $B_{S_b}(u_0, r) \subseteq U$. Choose $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$ such that $|\kappa - \kappa_0| \leq \frac{1}{M^n} < \epsilon$. Then for $u \in \overline{B_p(u_0, r)} = \{u \in X/S_b(u, u, u_0) \leq r + b^2 S_b(u_0, u_0, u_0)\}$. Also

$$\frac{1}{4b^3} \min \{S_b(u, u, H_b(u, \kappa), S_b(u_0, u_0, H_b(u_0, \kappa)))\} \leq S_b(u, u, u_0).$$

$$
\begin{aligned}
S_b(H_b(u, \kappa), H_b(u, \kappa), x_0) &= S_b(H_b(u, \kappa), H_b(u, \kappa), H_b(u_0, \kappa_0)) \\
&\leq 2b S_b(H_b(u, \kappa), H_b(u, \kappa), H_b(u, \kappa_0)) \\
&\quad + b^2 S_b(H_b(u, \kappa_0), H_b(u, \kappa_0), H_b(u_0, \kappa_0)) \\
&\leq 2bM|\kappa - \kappa_0| \\
&\quad + b^2 S_b(H_b(u, \kappa_0), H_b(u, \kappa_0), H_b(u_0, \kappa_0)) \\
&\leq \tfrac{2b}{M^{n-1}} + b^2 S_b(H_b(u, \kappa_0), H_b(u, \kappa_0), H_b(u_0, \kappa_0)).
\end{aligned}
$$

Letting $n \to \infty$, we obtain

$$
\begin{aligned}
S_b(H_b(u, \kappa), H_b(u, \kappa), x_0) &\leq b^2 S_b(H_b(u, \kappa_0), H_b(u, \kappa_0), H_b(u_0, \kappa_0)) \\
&\leq 4b^5 S_b(H_b(u, \kappa_0), H_b(u, \kappa_0), H_b(u_0, \kappa_0)) \\
&\leq S_b(u, u, u_0) - \beta(S_b(u, u, x_0)) \\
&\leq S_b(u, u, u_0).
\end{aligned}
$$

$$
\begin{aligned}
S_b(H_b(u, \kappa), H_b(u, \kappa), u_0) &\leq S_b(u, u, u_0) \\
&\leq r + b^2 S_b(u_0, u_0, u_0).
\end{aligned}
$$

Thus for each fixed $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$, $H_b(., \kappa) : \overline{B_p(u_0, r)} \to \overline{B_p(u_0, r)}$. Then all conditions of Theorem (7) are satisfied. Thus we conclude that $H_b(., \kappa)$ has a fixed point in $\overline{U}$. But this must be in U. Therefore, $\kappa \in B$ for $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$. Hence $(\kappa_0 - \epsilon, \kappa_0 + \epsilon) \subseteq B$. Clearly B is open in $[0, 1]$. To prove the reverse, we can use the similar process. ∎

6. Conclusions

In this paper we conclude some applications to integral equations and homotopy theory by using (α, β)-rational contraction fixed point theorems in partially ordered S_b-metric spaces

7. Declaration

— **Author contributions :** All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.
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