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UNIFIED (p, q)-BERNOULLI-HERMITE POLYNOMIALS

Abstract. The Concepts of p-Bernoulli numbers $B_{n,p}$ and p-Bernoulli polynomials $B_{n,p}(x)$ are generalized to (p, q)-Bernoulli numbers $B_{n,p,q}$ and (p, q)-Bernoulli polynomials $B_{n,p,q}(x)$, respectively. Some properties, generating functions and Laplace hypergeometric integral representations of (p, q)-Bernoulli numbers $B_{n,p,q}$ and (p, q)-Bernoulli polynomials $B_{n,p,q}(x)$, are established. Unified (p, q)-Bernoulli-Hermite polynomials are defined by a generating function which aid in proving the generalizations of the results of Khan et al [8], Kargin and Rahmani [7], Dattoli [4] and Pathan [9]. Some explicit summation formulas and some relationships between Appell's function F_1, Gauss hypergeomtric function, Hurwitz zeta function and Euler's polynomials are also given.

Key words: Bernoulli polynomials and Bernoulli numbers, generating functions, Gauss hypergeomtric function, Hurwitz zeta function and Euler's polynomials.

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1. Introduction

The Pochhammer's symbol or Appell's symbol or shifted factorial or rising factorial or generalized factorial function [13] is defined by

$$(b,\ k) = (b)_k = \frac{\Gamma(b+k)}{\Gamma(b)}$$

where b is neither zero nor negative integer and the notation Γ stands for Gamma function. Generalized Gaussian hypergeometric function ${}_AF_B$ [15,p.42(1)] of one variable is defined by

$$(1)\quad {}_AF_B\left[\begin{array}{l} a_1, a_2, \cdots, a_A\ ; \\ b_1, b_2, \cdots, b_B\ ; \end{array} z\right] = {}_AF_B(a_1, a_2, \cdots, a_A; b_1, b_2, \cdots, b_B; z)$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}$$

where denominator parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and A, B are non-negative integers and we assume that variable z takes on complex values. The Appell function F_1 of two variables is defined by [14]

$$F_1 = F_1(a, b_1, b_2; c; z_1, z_2) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n z_1^m z_2^n}{(c)_{m+n} m! n!} \tag{2}$$

$$F_1 = \sum_{m=0}^{\infty} \frac{(a)_m (b_1)_m}{(c)_m} {}_2F_1(a+m, b_2; c+m; z_2) \frac{z_1^m}{m!}, \tag{3}$$

$\max\{|z_1|, |z_2|\} < 1$, and is represented by a single integral of Euler's type (see [14] and [15])

$$F_1 = \frac{1}{B(a, c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} (1-tz_1)^{-b_1} (1-tz_2)^{-b_2} dt, \tag{4}$$

where $\Re(c) > \Re(a) > 0$ and $B(a, c-a)$ is beta function [13].

The generalized Hermite polynomials (known as Gould-Hopper polynomials) $H_n^r(x, y)$ defined by

$$e^{xt+yt^r} = \sum_{n=0}^{\infty} H_n^r(x, y) \frac{t^n}{n!} \tag{5}$$

are 2-variable Kampe· de Fe·riet generalization of the Hermite polynomials [1] and [4]

$$H_n(x, y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!} \tag{6}$$

These polynomials usually defined by the generating function

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}, H_n(x, 0) = x^n \tag{7}$$

reduce to the ordinary Hermite polynomials $H_n(x)$ (see [13]) when $y = -1$ and x is replaced by $2x$

We recall that the Hermite numbers H_n are the values of the Hermite polynomials $H_n(x)$ at zero argument, that is, $H_n(0)$.
A closed formula for H_n is given by

$$H_n = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \frac{(-1)^{n/2} n!}{(\frac{n}{2})!}, & \text{if } n \text{ is even}. \end{cases} \tag{8}$$

2. Preliminaries

Observe that there are three independent linear transformations of Appell's function F_1.The number of z variables is diminished by one if (i) one of the b parameters equal to zero, (ii) one of the z variables equals zero, (iii) one of the z variables equals unity, or (iv) two z variables are equal. One of the reduction formula out of these four cases is considered by Rahmani [12] in the form of p-Bernoulli numbers $B_{n,p}$ by constructing an infinite matrix.The situation $z_1 = z_2 = 1 - e^t$ in (2) leads to a generating function

$${}_2F_1(1,1;p+2;1-e^t) = \sum_{n=0}^{\infty} B_{n,p} \frac{t^n}{n!}, \tag{9}$$

where ${}_2F_1$ denotes the Gauss hypergeometric function given by (1), for the p-Bernoulli numbers $B_{n,p}$ which are closely related to Bernoulli numbers B_n by the formula

$$B_{n,p} = \frac{p+1}{p} \sum_{j=0}^{p} (-1)^j \left[\begin{matrix} p \\ j \end{matrix} \right] B_{n+j}, \tag{10}$$

where $\left[\begin{smallmatrix} p \\ j \end{smallmatrix} \right]$ is the Stirling number of the first kind [6].

Utilizing the fact that for $z_1 = z_2 = 1 - e^t$ in (2),we have a reduction formula

$$F_1(1,1,q;p+2;1-e^t,1-e^t) = {}_2F_1(1,q+1;p+2;1-e^t)$$

which is an immediate consequence of the familiar formula [15, p. 54(14)]

$$F_1(a,a,b;c;x,x) = {}_2F_1(a,b+1;c;x).$$

In particular,we have the following definition of the generalized p-Bernoulli numbers.

Definition 1. *(p,q)-Bernoulli numbers $B_{n,p,q}$ are given by the generating function*

$$\sum_{n=0}^{\infty} B_{n,p,q}\frac{t^n}{n!} = F_1(1,1,q;p+2;1-e^t,1-e^t) = {}_2F_1(1,q+1;p+2;1-e^t) \tag{11}$$

for every integer $p \geq -1$.

Equation (11) provides the equivalent definition to the Rahmani's original one given by (9). For $q=0$, (11) reduces to (9).Thus we have $B_{n,p,0} = B_{n,p}$ and $B_{n,0,0} = B_n$. (p,q)-Bernoulli numbers $B_{n,p,q}$ have a series representation involving Stirling numbers of second kind [6]

$$B_{n,p,q} = \sum_{k=0}^{n}(-1)^k \frac{(q+1)_k}{(p+2)_k}\left\{ \begin{array}{c} n \\ k \end{array} \right\} k!\,, \tag{12}$$

where Stirling numbers of second kind are defined by the generating function

$$\sum_{n=0}^{\infty}\left\{ \begin{array}{c} n \\ k \end{array} \right\}\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!}\,.$$

For $q=0$,(12) reduces to Kargin and Rahmani[7, p. 2(1.1)]

$$B_{n,p} = \sum_{k=0}^{n}(-1)^k \left\{ \begin{array}{c} n \\ k \end{array} \right\}\left(\begin{array}{c} k+p+1 \\ k \end{array} \right)^{-1} k!\,.$$

The first generating function for (p,q)-Bernoulli numbers $B_{n,p,q}$ for $p=1$, $q=0$ is

$$\sum_{n=0}^{\infty} B_{n,1,0}\frac{t^n}{n!} = \frac{2[(t-1)e^t+1]}{(e^t-1)^2}$$

and for $p=q=1$, we have

$$\sum_{n=0}^{\infty} B_{n,1,1}\frac{t^n}{n!} = 2\frac{(e^t-1-t)}{(1-e^t)^2} = 2\sum_{k=0}^{\infty}\frac{(1-e^t)^k}{2+k}\,,$$

where $(1-e^t)^k$ is given by

$$(1-e^t)^k = k!(-1)^k\sum_{n=0}^{\infty}\left\{ \begin{array}{c} n \\ k \end{array} \right\}\frac{t^n}{n!}\,.$$

Furthermore

$$\sum_{n=0}^{\infty} B_{n,1,2}\frac{t^n}{n!} = \sum_{n=0}^{\infty} B_{n,2,1}\frac{t^n}{n!} = \frac{3}{(1-e^t)^3}[(1-e^t)(1+e^t)+2te^t].$$

Since [5, eqn. (2.11.34)]

$$ {}_2F_1(a,b+1;a-b;z) = (1+z)^{-a} {}_2F_1(a/2,a/2+1/2;a-b;\frac{4z}{(1+z)^2}), $$

the special case of (11) when $a=1$, $p=-q-1$ and $z=1-e^t$ yields

$$ \sum_{n=0}^{\infty} B_{n,-q-1,q}\frac{t^n}{n!} = (2-e^t)^{-1} {}_2F_1(1/2,1;1-q;\frac{4(1-e^t)}{(2-e^t)^2}) . \tag{13} $$

From (4), we obtain an integral representation of Euler's type

$$ \sum_{n=0}^{\infty} B_{n,p,q}\frac{t^n}{n!} = \frac{1}{B(1+q,p-q+1)}\int_0^1 \frac{t^q(1-t)^{p-q}}{(1-t(1-e^t))}dt \tag{14} $$

where $\Re(p-q) > -1$.

Integral representations of Laplace type stem from the well-known results of Appell's F_1 [see, for example, 14, p. 282(26) and(27)]

$$ \sum_{n=0}^{\infty} B_{n,p,q}\frac{t^n}{n!} = \frac{1}{\Gamma(q)}\int_0^{\infty} e^{-t}t^{q-1}\Phi_1(1,1;p+2;1-e^t,(1-e^t)t)dt, \tag{15} $$

$Re(q) > 0$

$$ \sum_{n=0}^{\infty} B_{n,p,q}\frac{t^n}{n!} = \int_0^{\infty} e^{-t}\Phi_2(1,q;p+2;(1-e^t)t,(1-e^t)t)dt\,, \tag{16} $$

where Φ_1 and Φ_2 are confluent forms of Appell's series defined by Humbert [14, p. 25].

One can notice an interesting reduction of (p,q)-Bernoulli polynomials

$$ {}_2F_1(1,b;b+1;1-e^t) = b\zeta(1-e^t,1,b), \quad b \neq 0,-1,-2,\dots, \tag{17} $$

where Hurwitz zeta function ζ is defined as [11]

$$ \zeta(z,s,b) = \sum_{m=0}^{\infty} \frac{z^k}{(b+k)^s} . $$

3. A generating function for (p,q)-Bernoulli numbers

Kargin and Rahmani [7] derived a generating function for the p-Bernoulli numbers $B_{n,p}$. To this end they used the generating function of the geometric polynomials. Their result is

$$ \sum_{0}^{\infty} B_{n,p}\frac{t^n}{n!} = (p+1)[\frac{(t-H_p)e^{pt}}{(e^t-1)^{p+1}} + \sum_{k=1}^{p}\frac{p!}{k!(p-k)}\frac{H_k}{(e^t-1)^{k+1}}] , \tag{18} $$

where H_n is the $n-th$ harmonic number defined by [6, p. 258]

$$H_n = \sum_{j=1}^{n} \frac{1}{j}.$$

It may be remarked that the above result of Kargin and Rahmani is a direct consequence of the following known result [11, p. 462(128)]

$$ {}_2F_1(1,n;m;z) = \frac{(m-1)!}{(m-n-1)!z}\{\sum_{k=1}^{m-n-1} \frac{(m-n-k-1)!}{(m-k-1)!}(\frac{z-1}{z})^{k-1} - \frac{z}{(n-1)!}(\frac{z-1}{z})^{m-n-1}[\sum_{k=1}^{n-1} \frac{z^{-k}}{n-k} + z^{-n}\ln(1-z)]\}, \quad m>n.$$

Substitution of $z=1-e^t$, $n=q+1$ and $m=p+2$ in the above result provide us a more general result belonging to (p,q)-Bernoulli numbers considered in the preceding section.Thus we can write after a certain amount of algebra that

$$\sum_{n=0}^{\infty} B_{n,p,q}\frac{t^n}{n!} = \frac{(p+1)!}{(1-e^t)(p-q)!}[\sum_{k=1}^{p-q} \frac{(p+q-k)!}{(p+1-k)!}(\frac{e^t}{e^t-1})^{k-1} + \frac{(e^t-1)}{q!}(\frac{e^t}{e^t-1})^{p-q}(\sum_{k=1}^{q} \frac{(1-e^t)^{-k}}{(q-k+1)} + \frac{t}{(1-e^t)^{q+1}})], \tag{19}$$

$p > q-1$. For $q=0$, (19) reduces to

$$\sum_{n=0}^{\infty} B_{n,p}\frac{t^n}{n!} = (p+1)[\frac{te^{pt}}{(e^t-1)^{p+1}} + \sum_{k=1}^{p} \frac{e^{(k-1)t}}{(p-k)(e^t-1)^k}] \tag{20}$$

and for $q=0$ and $p=m-2$ where $m=2,3,4,\ldots$, we have

$$\sum_{n=0}^{\infty} B_{n,m-2}\frac{t^n}{n!} = (m-1)e^{-2t}[\sum_{k=2}^{m-1} \frac{1}{m-k}(\frac{e^t}{e^t-1})^k - t(\frac{e^t}{e^t-1})^m] \tag{21}$$

Also a calculation shows that,for $q=p=m-1$

$$\sum_{n=0}^{\infty} B_{n,m-1,m-1}\frac{t^n}{n!} = -\frac{m}{(1-e^t)^m}[t + \sum_{k=1}^{m-1} \frac{(1-e^t)^k}{k}], \quad m=1,2,3,\ldots.$$

4. Unified (p,q)-Bernoulli polynomials

This section is devoted to (p,q)-Bernoulli polynomials and their representations involving Euler polynomials $E_n(x)$ [11].

Definition 2 ((p,q)-Bernoulli polynomials)**.** *An explicit formula for (p,q)-Bernoulli polynomials is given by the generating function*

$$e^{xt}{}_2F_1(1,q+1;p+2;1-e^t)=\sum_{n=0}^{\infty}B_{n,p,q}(x)\frac{t^n}{n!} \tag{22}$$

for every integer $p\geq -1$.

For $x=0$ in (22),we get

$${}_2F_1(1,q+1;p+2;1-e^t)=\sum_{n=0}^{\infty}B_{n,p,q}\frac{t^n}{n!}\,. \tag{23}$$

Since this reduces to Bernoulli number B_n for special values of p and q, we shall call $B_{n,p,q}$, a (p,q)-Bernoulli number represented by the generating function (23). Since we have

$$\begin{aligned}{}_2F_1(1,q+1;p+2;1-e^t) &= e^{-xt}\sum_{n=0}^{\infty}B_{n,p,q}(x)\frac{t^n}{n!}\\ &= \sum_{m=0}^{\infty}\frac{(-x)^m}{m!}\sum_{n=0}^{\infty}B_{n,p,q}(x)\frac{t^{n+m}}{n!}.\end{aligned} \tag{24}$$

Therefore comparing (25) and (23),we have

$$B_{n,p,q}=n!\sum_{m=0}^{n}\frac{(-x)^m}{m!(n-m)!}B_{n-m,p,q}(x).$$

Similarly, when $x=-1$, (22) implies

$$B_{n,p,q}(-1)=n!\sum_{m=0}^{n}\frac{(-1)^mB_{n-m,p,q}}{m!(n-m)!}. \tag{25}$$

Theorem 1. *Let $p\geq 1$. Then following representation for (p,q)- Bernoulli polynomials $B_{n,p,q}(x)$ involving Euler polynomials $E_n(x)$ holds true:*

$$B_{n,p,q}(x)=\frac{n!}{2}\Big[\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{B_{m-k,p,q}E_{n-m}(x)}{(m-k)!k!(n-m)!}+\sum_{m=0}^{n}\frac{B_{m,p,q}E_{n-m}(x)}{m!(n-m)!}\Big] \tag{26}$$

Proof. The generating function for the Euler polynomials $E_n(x)$ gives

$$e^{xt} = \frac{e^t+1}{2}\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{27}$$

Substituting this value of e^{xt} in (22) gives

$$\frac{e^t+1}{2}\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \;\; {}_2F_1(1,q+1;p+2;1-e^t) = \sum_{n=0}^{\infty} B_{n,p,q}(x)\frac{t^n}{n!}. \tag{28}$$

Now using

$$e^t{}_2F_1(1,q+1;p+2;1-e^t) = \sum_{m=0}^{\infty} t^m \sum_{k=0}^{m} \frac{B_{m-k,p,q}}{(m-k)!k!}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \;\; &{}_2F_1(1,q+1;p+2;1-e^t) \\ &= \sum_{n=0}^{\infty} E_n(x)\sum_{m=0}^{\infty} B_{m,p,q}\frac{t^m}{m!} = \sum_{n=0}^{\infty} t^n \sum_{m=0}^{n} \frac{E_{n-m}(x)B_{m,p,q}}{(n-m)!m!}\end{aligned} \tag{29}$$

in the left hand side of (28),we get

$$\begin{aligned}\frac{1}{2}[&\sum_{m=0}^{\infty} t^m \sum_{k=0}^{m} \frac{B_{m-k,p,q}}{(m-k)!k!}\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \\ &+ \sum_{n=0}^{\infty} t^n \sum_{m=0}^{n} \frac{E_{n-m}(x)B_{m,p,q}}{(n-m)!m!}] = \sum_{n=0}^{\infty} B_{n,p,q}(x)\frac{t^n}{n!}.\end{aligned} \tag{30}$$

Finally, replacing n by $n-m$ and comparing the coefficients of t^n, we get the required result. ∎

Using the following result [11, p. 766]

$$E_n(x) = \frac{2}{n+1}(B_{n+1}(x) - 2^{n+1}B_{n+1}(x/2)), \tag{31}$$

in (26), we have

Corollary 1.

$$\begin{aligned}B_{n,p,q}(x) = n!\sum_{m=0}^{n}\{&\sum_{k=0}^{m} \frac{B_{m-k,p,q}}{(m-k)!k!(n-m)!} \\ &+ \frac{B_{m,p,q}}{m!(n-m)!}\}\frac{(B_{n-m+1}(x) - 2^{n-m+1}B_{n-m+1}(x/2))}{n-m+1}\end{aligned} \tag{32}$$

Setting $x=0$ in (32), we have the following result

Corollary 2.

$$B_{n,p,q} = n!\sum_{m=0}^{n}\{\sum_{k=0}^{m}\frac{B_{m-k,p,q}}{(m-k)!k!(n-m)!}+\frac{B_{m,p,q}}{m!(n-m)!}\}\frac{(1-2^{n-m+1})B_{n-m+1}}{n-m+1}. \tag{33}$$

5. Unified (p,q)-Bernoulli-Hermite polynomials

In this section, we define (p,q)-Bernoulli-Hermite polynomials and give some explicit formulas for these generalized polynomials.

Definition 3 ((p,q)-Bernoulli-Hermite polynomials)**.** *Given integer $p \geq -1$ and integer q, a (p,q)-Bernoulli-Hermite polynomials, denoted by ${}_HB^r_{n,p,q}(x,y)$ are defined by the generating function*

$$e^{xt+yt^r}{}_2F_1(1,q+1;p+2;1-e^t)=\sum_{n=0}^{\infty}{}_HB^r_{n,p,q}(x,y)\frac{t^n}{n!}. \tag{34}$$

Using (5) and (11) in (34),we can write

$$\sum_{n=0}^{\infty}{}_HB^r_{n,p,q}(x,y)\frac{t^n}{n!}=\sum_{n=0}^{\infty}H^r_n(x,y)\frac{t^n}{n!}\sum_{m=0}^{\infty}B_{m,p,q}\frac{t^m}{m!}$$

which on replacing n by $n-m$ and comparing the coefficients of t^n yields the representation

$${}_HB^r_{n,p,q}(x,y)=n!\sum_{m=0}^{n}\frac{H^r_{n-m}(x,y)B_{m,p,q}}{(n-m)!m!}.$$

With x replaced by $2x, r=2$ and $y=-1$ it turns to the equality

$${}_HB_{n,p,q}(2x,-1)=n!\sum_{m=0}^{n}\frac{H_{n-m}(x)B_{m,p,q}}{(n-m)!m!}. \tag{35}$$

To obtain (p,q)-Bernoulli-Hermite numbers denoted by ${}_HB_{n,p,q}$, we take in (34), $r=2$, $y=-1$ and replace x by $2x$. Once this has been done then we take $x=0$ and use (8) to get

$$\begin{aligned}{}_HB_{n,p,q}(0,-1)={}_HB_{n,p,q}&=n!\sum_{m=0}^{n}\frac{H_{n-m}B_{m,p,q}}{(n-m)!m!}\\ &=n!\sum_{m=0}^{n}\frac{B_{m,p,q}}{(n-m)!m!}\begin{cases}0, & \text{if } n-m \text{ is odd}\\ \frac{(-1)^{(n-m)/2}}{(n-m)!}(\frac{n-m}{2})!, & \text{if } n-m \text{ is even,}\end{cases}\end{aligned} \tag{36}$$

where Hermite number H_n is given by (8).

Theorem 2. *The following summation formulae for (p,q)-Bernoulli-Hermite polynomials ${}_HB_{n,p,q}$ holds true:*

$$(37)\qquad {}_HB^r_{n,p,q}(z,u) = n!\sum_{m=0}^{n}\frac{{}_HB^r_{n-m,p,q}(x-\alpha,y-\beta)H^r_m(\alpha-x+z,\beta-y+u)}{(n-m)!m!},$$

or, in its equivalent form

$$(38)\qquad {}_HB^r_{n,p,q}(z-\alpha+x,u-\beta+y) = n!\sum_{m=0}^{n}\frac{{}_HB^r_{n-m,p,q}(x-\alpha,y-\beta)H^r_m(z,u)}{(n-m)!m!}.$$

Proof. By exploiting the generating function (34), we can write

$$\begin{aligned} fty{}_HB^r_{n,p,q}(z,u)\frac{t^n}{n!} &= e^{zt+ut^r}{}_2F_1(1,q+1;p+2;1-e^t)\\ &= e^{-(x-z-\alpha)t-(y-u-\beta)t^r}e^{(x-\alpha)t+(y-\beta)t^r}{}_2F_1(1,q+1;p+2;1-e^t)\\ &= e^{-(x-z-\alpha)t-(y-u-\beta)t^r}\sum_{n=0}^{\infty}{}_HB^r_{n,p,q}(x-\alpha,y-\beta)\frac{t^n}{n!}\end{aligned}$$

which readily yields

$$\sum_{n=0}^{\infty}{}_HB^r_{n,p,q}(z,u)\frac{t^n}{n!} = \sum_{m=0}^{\infty}H^r_m(\alpha-x+z,\beta-y+u)\frac{t^m}{m!}\sum_{n=0}^{\infty}{}_HB^r_{n,p,q}(x-\alpha,y-\beta)\frac{t^n}{n!},$$

and thus by replacing n by $n-m$ and comparing the coefficients of t^n, we get (37). On replacing z by $z-\alpha-x$ and u by $u-\beta+y$ in (37), we get (38).

Setting $z=u=0$ in (37) and noting that ${}_HB^r_{n,p,q}(0,0) = {}_HB^r_{n,p,q}$, we have the following result for (p,q)-Bernoulli polynomials $B_{n,p,q}$ ■

Corollary 3.

$$B^r_{n,p,q} = n!\sum_{m=0}^{n}\frac{{}_HB^r_{n-m,p,q}(x-\alpha,y-\beta)H^r_m(\alpha-x,\beta-y)}{(n-m)!m!}.$$

Setting $z=u=q=0$ in (37) and noting that ${}_HB^r_{n,p,0}(0,0) = B^r_{n,p}$, we have the following result for p-Bernoulli polynomials.

Corollary 4.

$$B^r_{n,p} = n!\sum_{m=0}^{n}\frac{{}_HB^r_{n-m,p}(x-\alpha,y-\beta)H^r_m(\alpha-x,\beta-y)}{(n-m)!m!}.$$

Theorem 3. *The following summation formulae connecting Gauss hypergeometric functions ${}_2F_1$ and (p,q)-Bernoulli-Hermite polynomials ${}_HB^r_{m,p,q}$ holds true:*

$$\begin{aligned}
{}_2F_1(1,p+1-q;p+2;1-e^{-t}) &= \sum_{n=0}^{\infty}P_{n,p.q}(x,y)\frac{t^n}{n!}\\
{}_2F_1(p+1,q+1;p+2;1-e^{-t}) &= \sum_{n=0}^{\infty}Q_{n,p,q}(x,y)\frac{t^n}{n!}\\
{}_2F_1(p+1,p+1-q;p+2;1-e^{t}) &= \sum_{n=0}^{\infty}R_{n,p,q}(x,y)\frac{t^n}{n!},
\end{aligned} \tag{39}$$

where $P_{n,p,q}$, $Q_{n,p,q}$ and $R_{n,p.q}$ are given by

$$\begin{aligned}
P_{n,p,q}(x,y) &= n!\sum_{m=0}^{n}\frac{H^r_{n-m}(1-x,-y){}_HB^r_{m,p,q}(x,y)}{(n-m)!m!}\\
Q_{n,p,q}(x,y) &= n!\sum_{m=0}^{n}\frac{H^r_{n-m}(1-x+q,-y){}_HB^r_{m,p,q}(x,y)}{(n-m)!m!}\\
R_{n,p,q}(x,y) &= n!\sum_{m=0}^{n}\frac{H^r_{n-m}(q-x-p,-y){}_HB^r_{m,p,q}(x,y)}{(n-m)!m!}.
\end{aligned} \tag{40}$$

Proof. It is easy to use the linear transformations (also known as Euler's transformations) [see, S and M p. 33(19), (20), (21)]of Gauss hypergeometric functions to prove that

$$\begin{aligned}
{}_2F_1(1,q+1;p+2;1-e^{t}) &= e^{-t}{}_2F_1(1,p+1-q;p+2;1-e^{-t})\\
&= e^{-qt-t}{}_2F_1(p+1,q+1;p+2;1-e^{-t})\\
&= e^{pt-qt}{}_2F_1(p+1,p+1-q;p+2;1-e^{t}).
\end{aligned}$$

We further rewrite these relations in the form

$$\begin{aligned}
&{}_2F_1(1,p+1-q;p+2;1-e^{-t})\\
&\qquad = e^{t-xt-yt^r}[e^{xt+yt^r}{}_2F_1(1,q+1;p+2;1-e^{t})]\\
&{}_2F_1(p+1,q+1;p+2;1-e^{-t})\\
&\qquad = e^{qt+t-xt-yt^r}[e^{xt+yt^r}{}_2F_1(1,q+1;p+2;1-e^{t})]\\
&{}_2F_1(p+1,p+1-q;p+2;1-e^{t})\\
&\qquad = e^{-pt+qt-xt-yt*r}[e^{xt+yt^r}{}_2F_1(1,q+1;p+2;1-e^{t})]
\end{aligned}$$

We may split right hand sides of these relations into product of the generating functions of the polynomials $H_n^r(x,y)$ and (p,q)-Bernoulli-Hermite polynomials ${}_HB^r_{m,p,q}(x,y)$. Finally, replacement of n by $n-m$ and use of series arrangement technique prove the result (39). ■

Remark 1. An important application comes from the reduction of (39) for $q=0$. Within such a context, we use (17) and the first two equations of (39) to get a series representation of Hurwitz zeta function

$$(p+1)\zeta(1-e^{-t},1,p+1)=\sum_{n=0}^{\infty}P_{n,p}(x,y)\frac{t^n}{n!}=\sum_{n=0}^{\infty}Q_{n,p}(x,y)\frac{t^n}{n!},$$

where

$$P_{n,p}(x,y)=Q_{n,p}(x,y)=n!\sum_{m=0}^{n}\frac{H^r_{n-m}(1-x,-y)\,{}_HB^r_{m,p}(x,y)}{(n-m)!m!}. \tag{41}$$

On the other hand,we use (11) with $q=p$ in the last equation of (39). Then we have

$$B_{n,p,p}=R_{n,p,p}(x,y)=n!\sum_{m=0}^{n}\frac{H^r_{n-m}(-x,-y)\,{}_HB^r_{m,p,p}(x,y)}{(n-m)!m!}, \tag{42}$$

where

$$B_{n,p,p}=(p+1)\sum_{m=0}^{n}\frac{(-1)^m m!}{p+1+m}\left\{\begin{matrix} n\\ m\end{matrix}\right\}.$$

6. Implicit formulae involving (p,q)-Bernoulli-Hermite polynomials

This section of the paper is devoted to employing the definition of the Hermite-Bernoulli polynomials ${}_HB_n^{[\alpha,m-1]}(x,y)$ in proving the generalizations of the results of Khan et al [8], Dattoli [4, p. 386(1.7)] and Pathan [9] (see also [10]). For the derivation of implicit formulae involving the Hermite-Bernoulli polynomials ${}_HB_n^{[\alpha,m-1]}(x,y)$, the same considerations as developed for the ordinary Hermite and related polynomials in Khan et al [8]and Dattoli et al [2] and [3] hold as well. First we prove the following results involving Hermite-Bernoulli polynomials ${}_HB_n^{[\alpha,m-1]}(x,y)$.

Theorem 4. *The following summation formulae for (p,q) Bernoulli-Hermite polynomials ${}_HB^r_{n,p,q}(x,y)$ holds true:*

$${}_HB^r_{n,p,q}(x,y)=n!\sum_{k=0}^{n}\frac{B_{n-k,p}(x-z)H^r_k(z,y)}{(n-k)!k!}. \tag{43}$$

Proof. By exploiting the generating function (22), we can write equation (34) as

$$\sum_{n=0}^{\infty} {}_HB^r_{n,p,q}(x,y)\frac{t^n}{n!} = e^{(x-z)t}{}_2F_1(1,q+1;p+2;1-e^t)e^{zt+yt^r}, \tag{44}$$

$$\sum_{n=0}^{\infty} {}_HB^r_{n,p,q}(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty} B_{n,p}(x-z)\frac{t^n}{n!}\sum_{k=0}^{\infty} H^r_k(z,y)\frac{t^k}{k!}. \tag{45}$$

Now replacing n by $n-k$ using the Lemma 3 [15, p. 101(1)] in the right hand side of equation (46), we get

$$\sum_{n=0}^{\infty} {}_HB^r_{n,p,q}(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B_{n-k,p}(x-z)H^r_k(z,y)\frac{t^n}{(n-k)!k!} \tag{46}$$

On equating the coefficients of the like powers of t, we get (43). ■

Remark 2. Letting $z=x$ in (43) gives an equivalent form of (5.3). For $p=q=0$ in (43), we get a known result of Pathan [10] which further for $r=2$ gives a known result of Dattoli [4, p. 386(1.7)]. Again taking $y=0$ in formula (43), we obtain

$$ {}_HB^r_{n,p,q}(x) = n!\sum_{k=0}^{n}\frac{z^n B_{n-k,p}(x-z)}{(n-k)!k!}. \tag{47}$$

Theorem 5. *The following summation formulae for (p,q) Bernoulli-Hermite polynomials ${}_HB^r_{n,p,q}(x,y)$ and (p,q) Bernoulli polynomials $B^r_{n,p,q}$ holds true:*

$$\sum_{m=0}^{[\frac{n}{r}]}\frac{B^r_{n-mr,p,q}(x)B_{m,p,q}(y)}{(n-mr)!m!} = \sum_{m=0}^{[\frac{n}{r}]}\frac{{}_HB^r_{n-mr,p,q}(x,y)B_{m,p,q}}{(n-mr)!m!}, \tag{48}$$

$$\sum_{m=0}^{[\frac{n}{r}]}\frac{B^r_{n-mr,p,q}(x)B_{m,p,q}(y)}{(n-mr)!m!} = \sum_{k=0}^{n}\sum_{m=0}^{n-k}\frac{H^r_{n-k}(x,y)B_{n-m-k,p,q}B_{m,p,q}}{k!(n-m-k)!m!}. \tag{49}$$

Proof. Consider the definition of (p,q) Bernoulli-polynomials ${}_HB^r_{n,p,q}$

$$\sum_{m=0}^{\infty} B^r_{m,p,q}(y)\frac{t^{mr}}{m!} = e^{yt^r}{}_2F_1(1,q+1;p+2;1-e^{t^r}), \tag{50}$$

where x is replaced by y and t is replaced by t^r in (22). On multiplying (22) and (49),we have

$$\begin{aligned}(51)\quad &\sum_{m=0}^{\infty} B^r_{m,p,q}(y)\frac{t^{mr}}{m!}\sum_{n=0}^{\infty} B_{n,p,q}(x)\frac{t^n}{n!}\\ &= e^{xt+yt^r}{}_2F_1(1,q+1;p+2;1-e^t){}_2F_1(1,q+1;p+2;1-e^{t^r})\\ &= \sum_{n=0}^{\infty} {}_HB^r_{n,p,q}(x,y)\frac{t^n}{n!}\sum_{m=0}^{\infty} B_{m,p,q}\frac{t^{mr}}{m!}.\end{aligned}$$

Now replacing n by $n-rm$, using the Lemma 3 [15, p. 101(1)] in the right hand side of equation (46) and then equating the coefficients of the like powers of t, we get (48). Another way of defining the right hand side of (51) is suggested by replacing e^{xt+yt^r} by its series representation

$$\begin{aligned}&\sum_{m=0}^{\infty} B^r_{m,p,q}(y)\frac{t^{mr}}{m!}\sum_{n=0}^{\infty} B_{n,p,q}(x)\frac{t^n}{n!}\\ &\quad= \sum_{k=0}^{\infty} H^r_n(x,y)\sum_{n=0}^{\infty} B_{n,p,q}\frac{t^n}{n!}\sum_{m=0}^{\infty} B_{m,p,q}\frac{t^{mr}}{m!}.\end{aligned}$$

Next,we rearrange the order of summation and then equating the coefficients of the like powers of t, we get (49). ■

Setting $r=2$ and $y=0$ in the above theorem,we have the following result for (p,q)-Bernoulli polynomials

Corollary 5.

$$(52)\quad \sum_{m=0}^{[\frac{n}{2}]}\frac{B_{n-2r,p,q}(x)B_{m,p,q}}{(n-2m)!m!} = \sum_{k=0}^{n}\sum_{m=0}^{n-k}\frac{H_n(x)B_{n-m-k,p,q}B_{m,p,q}}{k!(n-m-k)!m!}.$$

Theorem 6. *The following summation formulae for (p,q)-Bernoulli-Hermite polynomials ${}_HB^k_{n,p,q}(x,y)$ and (p,q)-Bernoulli polynomials $B^k_{n,p,q}$ holds true:*

$$\begin{aligned}(53)\quad &\sum_{m=0}^{n}\sum_{r=0}^{[\frac{n-m}{k}]}\left(\frac{x}{y^k}-\frac{y}{x^k}\right)^r\frac{{}_HB^k_{n-kr-m,p,q}(x,y)B_{m,p,q}}{r!m!(n-m-kr)!y^m x^{n-m-kr}}\\ &\quad= \sum_{m=0}^{n}\frac{{}_HB^k_{n-m,p,q}(y,x)B_{m,p,q}}{(n-m)!m!x^m y^{n-m}}.\end{aligned}$$

Proof. On replacing t by t/x and r by k, we can write equation (34) as

$$(54)\quad \sum_{n=0}^{\infty} {}_HB^k_{n,p,q}(x,y)\frac{t^n}{x^n n!} = e^{t+y\frac{t^k}{x^k}}{}_2F_1(1,q+1;p+2;1-e^{t/x}).$$

Now interchanging x and y, we have

$$\sum_{n=0}^{\infty} {}_H B^k_{n,p,q}(y,x)\frac{t^n}{y^n n!} = e^{t+x\frac{t^k}{y^k}} {}_2F_1(1,q+1;p+2;1-e^{t/y}) \tag{55}$$

Comparison of (54) and (55) yields

$$\begin{aligned} e^{x\frac{t^k}{y^k}-y\frac{t^k}{x^k}} {}_2F_1(1,q+1;p+2;1-e^{t/y}) \sum_{n=0}^{\infty} {}_H B^k_{n,p,q}(x,y)\frac{t^n}{x^n n!} \\ = {}_2F_1(1,q+1;p+2;1-e^{t/x}) \sum_{n=0}^{\infty} {}_H B^k_{n,p,q}(y,x)\frac{t^n}{y^n n!} \end{aligned} \tag{56}$$

Using (36), we get

$$\begin{aligned} \sum_{r=0}^{\infty} \frac{(\frac{x}{y^k}-\frac{y}{x^k})^r t^{kr}}{r!} \sum_{m=0}^{\infty} B_{m,p,q}\frac{t^m}{y^m m!} \sum_{n=0}^{\infty} {}_H B^k_{n,p,q}(x,y)\frac{t^n}{x^n n!} \\ = \sum_{m=0}^{\infty} B_{m,p,q}\frac{t^m}{x^m m!} \sum_{n=0}^{\infty} {}_H B^k_{n,p,q}(y,x)\frac{t^n}{y^n n!}. \end{aligned} \tag{57}$$

■

Next, we rearrange the order of summation and then equating the coefficients of the like powers of t, we get (53).

Setting $p = 0$ and $q = 0$ in the above theorem,we have the following result for Bernoulli polynomials

Corollary 6.

$$\begin{aligned} \sum_{m=0}^{n} \sum_{r=0}^{[\frac{n-m}{k}]} (\frac{x}{y^k}-\frac{y}{x^k})^r \frac{{}_H B^k_{n-kr-m}(x,y)B_m}{r!m!(n-m-kr)!y^m x^{n-m-kr}} \\ = \sum_{m=0}^{n} \frac{{}_H B^k_{n-m}(y,x)B_m}{(n-m)!m!x^m y^{n-m}} \end{aligned} \tag{58}$$

7. Concluding remarks

The formulae we have provided in previous sections illustrate how properties of Appell's function F_1 can sometimes be established or understood more easily by using generalized hypergeometric series of several variables.Gauss's function ${}_2F_1$ and Appell's function F_1 are both special cases of Lauricella hypergeometric function F_D^n [15, p. 60] defined by the power series

$$F_D^n[a, b_1, \ldots, b_n; c; x_1, \ldots, x_n] = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}} \frac{z_1^m}{m_1!} \cdots \frac{z_n^{m_n}}{m_n!}, \tag{59}$$

where for convergence $|z_m| < 1$, $m = 1, 2, \ldots$. The following reduction of F_D^n will be useful

$$F_D^n[a, b_1, \ldots, b_n; c; x, \ldots, x] = {}_2F_1(a, b_1 + \ldots + b_n; c; x), \tag{60}$$

because it allows one to conclude that for (p, q)-Bernoulli numbers given by (11) for every $p \geq -1$, we can write

$$\begin{aligned} &F_D^n[1, q, 1, 0, 0, \ldots, 0; p+2; 1-e^t, \ldots, 1-e^t] \\ &\quad = F_D^n[1, q, b_2, \ldots, b_n; c; 1-e^t, \ldots, 1-e^t] \\ &\quad = {}_2F_1(1, q+1; p+2; 1-e^t) \end{aligned}$$

where $b_2 + b_3 + \ldots + b_n = 1$.

Our speculations for Lauricella hypergeometric function F_D^n [15,p.60] and their reduction cases will yield perhaps a further feeling on the usefulness of the series. This process of using different analytical means on their respective generating functions can be extended to drive new relations for conventional and generalized Bernoulli numbers and polynomials.

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