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# A GENERAL FIXED POINT THEOREM FOR HYBRID PAIRS OF MAPPINGS SATISFYING A NEW TYPE OF LIMIT RANGE PROPERTY

Abstract. In this paper a new type of common limit range property is introduced, which generalizes the notion of strongly tangential property [9] and joint common limit range property [16]. A general fixed point theorem for two pairs of hybrid mappings involving altering distance and satisfying an implicit relation is proved, generalizing the results from [9], [16] and other papers. As application, we obtain new results for contractive mappings satisfying a contractive condition of integral type, for mappings satisfying $\phi$ - contractive conditions and satisfying $(\psi, \phi)$ - contractive conditions.

Key words: Coincidence point, fixed point, hybrid pair, multi - valued mappings, common limit range property, strict tangential property, altering distance, implicit relation.

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## 1. Introduction

In 1969, Nadler [21] proved an analog Banach principle with set valued mappings employing Hausdorff - Pompeiu metric.

In 2004, Pathak and Shahzad [23], introduced the notion of pointwise tangential mappings and used the same to prove a common fixed point theorem of Greguš type for two pairs of mappings satisfying a strict general contractive condition of integral type.

Sintunavarat and Kumam [42] explained the notion of tangential property to hybrid mappings and extended an integral type common fixed point theorem of Greguš type for four mappings under strict contractive condition.

In a noted review of [42], M. Balaj pointed out that the results contained in [42] (and also [43]) are not valid under given conditions without closedness of suitable image subspaces.

Quite recently, S. Chauhan et al. [9] and other authors show that the results [42], [43] can be recovered without the closedness requirement to the subspace by slightly restricting the tangential property which is accomplished by introducing the strong tangential property for non - self hybrid mappings besides utilizing the same to prove a metric fixed point theorem satisfying a general contractive condition of integral type.

In 2011, Sintunavarat and Kumam [41] introduced the notion of common limit range property for single valued mappings.

Imdad et al. [15] established common limit range property for a hybrid pair of mappings and proved some fixed point results in symmetric spaces.

Quite recently, Imdad et al. [16] introduced the notion of joint common limit range property for two pairs of hybrid mappings.

The study of fixed points for mappings satisfying a contractive condition of integral type is initiated by Branciari [8]. It is proved in [29] that the study of fixed points for single valued mappings satisfying a contractive condition of integral type is reduced to the study of fixed points in symmetric spaces. Also, it is proved in [30] that the study of fixed points of single valued mappings and multi - valued mappings satisfying integral conditions is reduced to the study of fixed points for mappings involving altering distances.

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation [24], [25] and other papers. Recently, this method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, $b$ - metric spaces, ultra - metric spaces, Hilbert spaces, reflexive metric spaces, compact metric spaces, in two and three metric spaces, for single - valued mappings, hybrid pairs of mappings and set - valued mappings.

Quite recently, the method is used in the study of fixed points for mappings satisfying contractive / extensive conditions of integral type, in fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces, $G$ - metric spaces and partial metric spaces.

With this method the proofs of existence of fixed points are more simple.

Also, the method allows the study of local and global properties of fixed point structures.

In [26] - [28] and in other papers, the study of fixed points for hybrid pairs of mappings satisfying implicit relations is initiated. A general fixed point theorem for a pair of hybrid mappings with common limit range property satisfying implicit relations is proved in [10].

## 2. Preliminaries

Let $(X, d)$ be a metric space. We denote by $CL(X)$ (respectively, $CB\,(X)$) the set of all nonempty closed (respectively, nonempty closed bounded) subsets of $(X, d)$ and by $H$ the Hausdorff - Pompeiu metric, i.e.

$$H(A, B) = \max\{\sup_{x \in A}\{d(x, B), \sup_{x \in B} d(x, A)\},$$

where $A, B \in CL(X)$ and

$$d(x, A) = \inf_{y \in A}\{d(x, y)\}.$$

**Definition 1.** *Let $f : X \to X$ and $F : X \to CL(X)$ be.*
1) *A point $x \in X$ is said to be a coincidence point of $f$ and $F$ if $fx \in Fx$. The set of all coincidence points of $f$ and $F$ is denoted by $\mathcal{C}(f, F)$.*
2) *A point $x \in X$ is a common fixed point of $f$ and $F$ if $x = fx \in Fx$.*

**Definition 2** ([39])**.** *Let $f : X \to X$ and $F : X \to CL(X)$ be. Then, $f$ and $F$ are said to be compatible if $\lim_{n\to\infty} H(Ffx_n.fFx_n) = 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $Fx_n \to A \in CL(X)$ and $fx_n \to t$ as $n \to \infty$.*

**Definition 3** ([40])**.** *Let $f : X \to X$ and $F : X \to CL(X)$ be. Then, $f$ and $F$ are said to be weakly commuting if $ffx \in Ffx$.*

**Definition 4** ([17], [22])**.** *$f : X \to X$ and $F : X \to CL(X)$ are said to be weakly compatible if $fFx = Ffx$ for each $x \in \mathcal{C}(f, F)$.*

**Definition 5** ([1], [38])**.** *Let $f : X \to X$ and $F : X \to CL(X)$ be. Then, $f$ and $F$ are said to be compatible of type $N$ if $x \in \mathcal{C}(f, F)$ implies $ffx \in Ffx$.*

**Definition 6.** *Let $(X, d)$ be a metric space and $Y$ be an arbitrary nonempty subset of $X$ with $f : Y \to X$, $F : Y \to 2^X$. The pair of hybrid mappings $(f, F)$ is said to be quasi - coincidentally commuting if $fx \in Fx$, for $x \in X$ with $fx \in Fx \subseteq Y$ implies $fFx \subset Ffx$.*

**Remark 1.** If $(f, F)$ is quasi - coincidentally commuting then $(f, F)$ is compatible of type $N$, because if $fx \in Fx$, then $ffx \in fFx \subset Ffx$. The converse is not true.

**Example 1** ([32])**.** Let $X = [0, 1]$ be a metric space with the usual metric, $fx = 1 - x$, $Fx = \left[0, \frac{1}{2}\right]$. Then $\mathcal{C}(f, F) = \left\{\frac{1}{2}\right\}$, $fF\left(\frac{1}{2}\right) = \left[\frac{1}{2}, 1\right] \not\subset Ff\left(\frac{1}{2}\right) = \left[0, \frac{1}{2}\right)$ and $ff\frac{1}{2} = \frac{1}{2} \in Ff\frac{1}{2}$.

**Definition 7** ([13])**.** *Let $f : Y \subset X \to X$ be a single valued mapping and $F : X \to CL(X)$. The mapping $f$ is said to be coincidentally idempotent with respect to $F$ if $fx \in Fx$, with $fx \in Y$ imply $fx = ffx$, that is $f$ is idempotent at coincidence points of the pair $(f, F)$.*

**Definition 8** ([15]). *Let $f : Y \subset X \to X$ be a single valued mapping and $F : Y \subset X \to CL(X)$ be a multivalued mapping. Then, $(f, F)$ has common limit range property if there exists a sequence $\{x_n\}$ in $Y$ such that $\lim_{n\to\infty} fx_n = fu \in A = \lim_{n\to\infty} Fx_n$, for some $u \in Y$ and $A \in CL(X)$.*

**Definition 9** ([43]). *Let $f, g : Y \subset X \to X$ be two single valued mappings and $F, G : Y \subset X \to CL(X)$ be two multivalued mappings. The pair $(f, g)$ is called strongly tangential with respect to the pair $(F, G)$ if*

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gy_n = D \in CL(X),$$

*whenever $\{x_n\}$ and $\{y_n\}$ are in $Y$ such that*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = z$$

*and $z \in f(Y) \cap g(Y)$.*

**Definition 10** ([16]). *Let $(X, d)$ be a metric space whereas $Y$ is an arbitrary nonempty set of $X$ with $F, G : Y \to CL(X)$ and $f, g : Y \to X$. Then the pairs of hybrid mappings $(F, f)$ and $(G, g)$ are said to have joint common limit range property $(JCLR)$ - property there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $Y$ and $A, B \in CL(X)$ such that*

$$\lim_{n\to\infty} Fx_n = A, \qquad \lim_{n\to\infty} Gy_n = B,$$

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = t \in A \cap B \cap f(Y) \cap g(Y),$$

*i.e. there exist $u$ and $v$ in $Y$ such that $t = fu = gv \in A \cap B$.*

The following results are obtained in [16].

**Theorem 1** ([16]). *Let $(X, d)$ be a metric space whereas $Y$ is an arbitrary nonempty set of $X$ with $F, G : Y \to CL(X)$ and $f, g : Y \to X$. Suppose that*

1) *the hybrid pairs $(f, F)$ and $(g, G)$ share the $(JCLR)$ - property,*
2) *for all $x \neq y$ in $Y$ and $k \in (0, 2)$*

$$H(Fx, Gy) < \max\{d(fx, gy), \frac{k}{2}[d(fx, Fx) + d(gy, Gy)], \frac{k}{2}[d(fx, Gy) + d(gy, Fx)]\}. \tag{1}$$

*Then the pairs $(F, f)$ and $(G, g)$ have a coincidence point each.*

*In particular if $Y \subset X$ and the pairs are quasi - coincidentally commuting and coincidentally idempotent, then the pairs $(F, f)$ and $(G, g)$ have a common fixed point in $X$.*

**Remark 2** ([16])**.** The conclusion of Theorem 1 remains true if inequality (1) is replaced by one of the following inequalities:

$$(2)\quad H(Fx,Gy) < \max\{d(fx,gy), k[d(fx,Fx)+d(gy,Gy)], k[d(gy,Fx)+d(fx,Gy)]\}, \quad \text{where } k\in(0,1),$$

$$(3)\quad H(Fx,Gy) \le k\max\{d(fx,gy), d(fx,Fx), d(gy,Gy), \frac{d(fx,Gy)+d(gy,Fx)}{2}\}, \quad \text{where } k\in(0,1),$$

$$(4)\quad \begin{aligned} H(Fx,Gy) \le\ & ad(fx,gy)+b\max\{d(fx,Fx), d(gy,Gy)\} \\ & + c\max\{d(fx,Fx)+d(gy,Gy), \\ & \qquad d(fx,Gy)+d(gy,Fx)\}, \end{aligned}$$

where $a,b,c\ge 0$ and $a+b+2c<1$.

Now we introduce a new type of common limit range property.

**Definition 11.** *Let $(X,d)$ be a metric space, $A: X\to CL(X)$ and $S,T: X\to X$. Then $(A,S)$ and $T$ satisfy common limit range property with respect to $T$, denoted $CLR_{(A,S)T}$ - property, if there exists a convergent sequence $\{x_n\}$ in $X$ such that*

$$\lim_{n\to\infty} Sx_n = t \in \lim_{n\to\infty} Ax_n \quad \textit{and} \quad t\in S(X)\cap T(X).$$

**Example 2.** Let $X=[0,\infty)$ be a metric space with the usual metric. $A(X)=\left[\frac{1}{4},1\right]$, $Sx=\frac{x^2+1}{2}$, $Tx=x+\frac{1}{4}$. Then $S(X)=\left[\frac{1}{2},\infty\right)$, $T(X)=\left[\frac{1}{4},\infty\right)$, $S(X)\cap T(X)=\left[\frac{1}{2},\infty\right)$. Let $\{x_n\}$ be a sequence such that $x_n\to 0$. Then

$$\lim_{n\to\infty} Sx_n = t = \frac{1}{2} \in \left[\frac{1}{4},\frac{1}{2}\right] = \lim_{n\to\infty} Ax_n.$$

Hence $t\in S(X)\cap T(X)$.

**Remark 3.** *a*) Let $(X,d)$ be a metric space, $A,B: Y\to CL(X)$, where $Y\subset X$ and $S,T: Y\to X$. If $(A,S)$ and $(B,T)$ satisfy $(JCLR)$ - property, then $(A,S)$ and $T$ satisfy $CLR_{(A,S)T}$ - property.

*b*) Let $(X,d)$ be a metric space, $A,B: Y\to CL(X)$, where $Y\subset X$ and $S,T: Y\to X$. If $(S,T)$ is strongly tangential with respect to $(A,B)$, then $(A,S)$ and $T$ satisfy $CLR_{(A,S)T}$ - property.

**Definition 12** ([19])**.** *An altering distance is a mapping $\psi: [0,\infty)\to[0,\infty)$ such that*

$(\psi_1)$ : *$\psi$ is increasing and continuous,*

$(\psi_2)$ : *$\psi(t)=0$ if and only if $t=0$.*

Fixed point problems involving altering distances have been studied in [30], [36], [37] and in other papers.

## 3. Implicit relations

The following class of implicit relations is introduced in [4].

**Definition 13.** *Let $\mathcal{F}$ be the set of all lower semi - continuous functions $F : \mathbb{R}_{+}^{6} \to \mathbb{R}$ satisfying the following conditions:*
$(F_1)$ : *$F$ is nondecreasing in variable $t_1$,*
$(F_2)$ : $F(t, 0, 0, t, t, 0) > 0, \forall t > 0$,
$(F_3)$ : $F(t, 0, t, 0, 0, t) > 0, \forall t > 0$.

**Example 3.** $F(t_1, ..., t_6) = t_1 - \max\left\{t_2, \frac{k}{2}(t_3 + t_4), \frac{k}{2}(t_5 + t_6)\right\}$, where $k \in [0, 2)$.

**Example 4.** $F(t_1, ..., t_6) = t_1 - \max\{t_2, k(t_3 + t_4), k(t_5 + t_6)\}$, where $k \in [0, 1)$.

**Example 5.** $F(t_1, ..., t_6) = t_1 - k\max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\right\}$, where $k \in [0, 1)$.

**Example 6.** $F(t_1, ..., t_6) = t_1 - at_2 - b\max\{t_3, t_4\} - c\max\{t_3 + t_4, t_5 + t_6\}$, where $a, b, c \geq 0$ and $a + b + 2c < 1$.

**Example 7.** $F(t_1, ..., t_6) = t_1 - \alpha\max\{t_2, t_3, t_4\} - (1 - \alpha)(at_5 + bt_6)$, where $\alpha \in (0, 1)$, $a, b \geq 0$ and $a + b < 1$.

**Example 8.** $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, where $a, b, c, d, e \geq 0$, $c + d < 1$ and $b + e < 1$.

**Example 9.** $F(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c\min\{t_5, t_6\}$, where $a, b, c \geq 0$ and $a + b + c < 1$.

**Example 10.** $F(t_1, ..., t_6) = t_1 - at_2 - \frac{b(t_5+t_6)}{1+t_3+t_4}$, where $a$, $b \geq 0$ and $a + 2b < 1$.

**Example 11.** $F(t_1, ..., t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, where $c \in (0, 1)$, $a, b \geq 0$ and $a + b + c \leq 1$.

**Example 12.** $F(t_1, ..., t_6) = t_1 - at_2 - b\max\{t_3, t_4\} - c\max\{t_2, t_5, t_6\}$, where $a, b, c > 0$ and $a + b + c < 1$.

Other examples are in [28] and [4].

## 4. Main results

**Theorem 2.** *Let $(X,d)$ be a metric space, whereas $Y$ is an arbitrary nonempty set of $X$ with $A,B:Y\to CL(X)$ and $S,T:Y\to X$. Suppose that*

$$F\left(\begin{array}{c}\psi\left(H\left(Ax,By\right)\right),\psi\left(d\left(Sx,Ty\right)\right),\psi\left(d\left(Sx,Ax\right)\right),\\ \psi\left(d\left(Ty,By\right)\right),\psi\left(d\left(Sx,By\right)\right),\psi\left(d\left(Ty,Ax\right)\right)\end{array}\right)\leq 0 \tag{5}$$

*for all $x\neq y\in Y$, where $F\in\mathcal{F}$ and $\psi$ is a altering distance. If $A$, $S$ and $T$ satisfy $CLR_{(A,S)T}$ - property, then*

1) $\mathcal{C}(A,S)\neq\emptyset$,

2) $\mathcal{C}(B,T)\neq\emptyset$.

*Moreover, if $(A,S)$ and $(B,T)$ are compatible of type $N$ and*

*a) $(A,S)$ is coincidentally idempotent, then $A$ and $S$ have a common fixed point;*

*b) $(B,T)$ is coincidentally idempotent, then $B$ and $T$ have a common fixed point;*

*c) $(A,S)$ and $(B,T)$ are coincidentally idempotent each, then $A,B,S$ and $T$ have a common fixed point.*

**Proof.** Since $(A,S)$ and $T$ satisfy $CLR_{(A,S)T}$ - property, then there exists a sequence $\{x_n\}$ in $Y$ such that

$$\lim_{n\to\infty}Sx_n=t\in D=\lim_{n\to\infty}Ax_n\quad\text{and}\quad t\in S(Y)\cap T(Y).$$

Since $t\in T(Y)$, there exists $u\in Y$ such that $t=Tu$.

By (5) we have

$$F\left(\begin{array}{c}\psi\left(H\left(Ax_n,Bu\right)\right),\psi\left(d\left(Sx_n,Tu\right)\right),\psi\left(d\left(Sx_n,Ax_n\right)\right),\\ \psi\left(d\left(Tu,Bu\right)\right),\psi\left(d\left(Sx_n,Bu\right)\right),\psi\left(d\left(Tu,Ax_n\right)\right)\end{array}\right)\leq 0. \tag{6}$$

Letting $n$ tends to infinity, by (6) we obtain

$$F\left(\psi\left(H\left(D,Bu\right)\right),0,0,\psi\left(d\left(t,Bu\right)\right),\psi\left(d\left(t,Bu\right)\right),0\right)\leq 0.$$

Since $t\in D$, $d(t,Bu)\leq H(D,Bu)$.

By $(F_1)$ and (6) we obtain

$$F\left(\psi\left(d\left(t,Bu\right)\right),0,0,\psi\left(d\left(t,Bu\right)\right),\psi\left(d\left(t,Bu\right)\right),0\right)\leq 0,$$

a contradiction of $(F_2)$ if $\psi(d(t,Bu))>0$. Hence $\psi(d(t,Bu))=0$, which implies $d(t,Bu)=0$. Then $t=Tu\in Bu$ and $\mathcal{C}(T,B)\neq\emptyset$.

On the other hand, $t\in S(Y)$. Hence, there exists $v\in Y$ such that $t=Sv$.

By (5) we obtain

$$F\left(\begin{array}{c}\psi\left(H\left(Av,Bu\right)\right),\psi\left(d\left(Sv,Tu\right)\right),\psi\left(d\left(Sv,Av\right)\right),\\ \psi\left(d\left(Tu,Bu\right)\right),\psi\left(d\left(Sv,Bu\right)\right),\psi\left(d\left(Tu,Av\right)\right)\end{array}\right)\leq 0. \tag{7}$$

Since $t\in Bu$, then $d\left(t,Av\right)\leq H\left(Av,Bu\right)$.
By $(F_1)$ and (7) we obtain

$$F\left(\psi\left(d\left(t,Av\right)\right),0,\psi\left(d\left(t,Av\right)\right),0,0,\psi\left(d\left(t,Av\right)\right)\right)\leq 0,$$

a contradiction of $(F_3)$ if $\psi\left(d\left(t,Av\right)\right)>0$. Hence $\psi\left(d\left(t,Av\right)\right)=0$, which implies $d\left(t,Av\right)=0$. Then $t=Sv\in Av$. Therefore, $\mathcal{C}\left(A,S\right)\neq\emptyset$ and $t$ is a common point of coincidence of $(A,S)$ and $(B,T)$.

Moreover,

*a*) if $(A,S)$ is compatible of type $N$ and $S$ is coincidentally idempotent, then $t=Sv=SSv=St$ and $t$ is a fixed point of $S$. Since $(A,S)$ is compatible of type $N$, then $t=Sv=SSv\in ASv=At$ and $t$ is a fixed point of $A$. Hence, $t$ is a common fixed point of $S$ and $A$;

*b*) if $(B,T)$ is compatible of type $N$ and $T$ is coincidentally idempotent, then as in the proof of *a*) follows that $t$ is a common fixed point of $B$ and $T$;

*c*) if $(A,S)$ and $(B,T)$ are compatible of type $N$ and $S$ and $T$ are coincidentally idempotent, then by a) and b) it follows that $t$ is a common fixed point of $A$, $B$, $S$ and $T$. ■

If $\psi(t)=t$ by Theorem 2 we obtain

**Theorem 3.** *Let $(X,d)$ be a metric space, whereas $Y$ is an arbitrary nonempty set of $X$ with $A,B:Y\rightarrow CL\left(X\right)$ and $S,T:Y\rightarrow X$. Suppose that*

$$F\left(\begin{array}{c}H\left(Ax,By\right),d\left(Sx,Ty\right),d\left(Sx,Ax\right),\\ d\left(Ty,By\right),d\left(Sx,By\right),d\left(Ty,Ax\right)\end{array}\right)\leq 0 \tag{8}$$

*for all $x\neq y\in Y$. If $A,S$ and $T$ satisfy $CLR_{(A,S)T}$ - property, then*

1) $\mathcal{C}\left(A,S\right)\neq\emptyset$,

2) $\mathcal{C}\left(B,T\right)\neq\emptyset$.

*Moreover, if $(A,S)$ and $(B,T)$ are compatible of type $N$ and*

*a*) *$(A,S)$ is coincidentally idempotent, then $A$ and $S$ have a common fixed point;*

*b*) *$(B,T)$ is coincidentally idempotent, then $B$ and $T$ have a common fixed point;*

*c*) *$(A,S)$ and $(B,T)$ are coincidentally idempotent each, then $A,B,S$ and $T$ have a common fixed point.*

By Theorem 3 and Example 3 we obtain

**Theorem 4.** *Let* $(X, d)$ *be a metric space, whereas* $Y$ *is an arbitrary nonempty subset of* $X$ *with* $A, B : Y \to CL(X)$ *and* $S, T : Y \to X$. *Suppose that inequality (1) holds for all* $x \neq y \in Y$. *If* $A, S$ *and* $T$ *satisfy* $CLR_{(A,S)T}$ *- property, then*

1) $\mathcal{C}(A, S) \neq \emptyset$,
2) $\mathcal{C}(B, T) \neq \emptyset$.

*Moreover, if* $(A, S)$ *and* $(B, T)$ *are compatible of type* $N$ *and*

*a)* $(A, S)$ *is coincidentally idempotent, then* $A$ *and* $S$ *have a common fixed point;*

*b)* $(B, T)$ *is coincidentally idempotent, then* $B$ *and* $T$ *have a common fixed point;*

*c)* $(A, S)$ *and* $(B, T)$ *are coincidentally idempotent each, then* $A$, $B$, $S$ *and* $T$ *have a common fixed point.*

**Remark 4.** 1) Similarly, by Theorem 3 and Examples 4 - 6 we obtain some results as in Remark 2.

2) Theorem 4 is a generalization of Theorem 1.

3) By Examples 7 - 12 we obtain new particular results.

## 5. Applications

### 5.1. Coincidence and common fixed points for mappings satisfying a relation of integral type

In [8], Branciari established the following theorem, which opened the way to the study of fixed points for mappings satisfying a contractive condition of integral type.

**Theorem 5** ([8]). *Let* $(X, d)$ *be a complete metric space,* $c \in (0, 1)$ *and* $f : X \to X$ *be a mapping such that for each* $x, y \in X$

$$\int_0^{d(fx,fy)} h(t)dt \leq c \int_0^{d(x,y)} h(t)dt, \tag{9}$$

*where* $h : [0, \infty) \to [0, \infty)$ *is a Lebesgue measurable mapping which is summable (i.e. with finite integral) on each compact subset of* $[0, \infty)$ *such that for* $\varepsilon > 0$, $\int_0^{\varepsilon} h(t)dt > 0$.

*Then,* $f$ *has a unique fixed point* $z \in X$ *such that for all* $x \in X$, $z = \lim_{n\to\infty} f^n x$.

Some fixed point results for mappings satisfying contractive conditions of integral type are obtained in [20], [30], [35] and in other papers.

**Lemma 1** ([30]). *Let $h : [0, \infty) \to [0, \infty)$ be as in Theorem 5. Then $\psi(t) = \int_0^t h(x)dx$ is an altering distance.*

In [9] the following result is obtained.

**Theorem 6** ([9]). *Let $f, g$ be two mappings from a subset $Y$ of a metric space $(X, d)$ into $X$ and $F, G$ be two mappings from $Y$ into $CB(X)$ satisfying the inequality*

$$\int_0^{H(Fx,Gy)} h(t)dt \le \alpha \int_0^{\max\{d(fx,gy),d(fx,Fx),d(gy,Gy)\}} h(t)dt + (1-\alpha)\left[a\int_0^{\frac{d(fx,Gy)}{2}} h(t)dt + b\int_0^{\frac{d(gy,Fx)}{2}} h(t)dt\right], \tag{10}$$

*for all $x \neq y \in Y$, where $\alpha \in (0, 1)$, $a, b \ge 0$, $a + b < 1$ and $h(t)$ is as in Theorem 5*

*Suppose that the pair $(f, g)$ is strongly tangential with respect to $(F, G)$. Then,*

*a) $f$ and $F$ have a coincidence point in $Y$,*

*b) $g$ and $G$ have a coincidence point in $Y$.*

*Moreover, the mapping $f$, $g$, $F$ and $G$ have a common fixed point providing that $(f, F)$ and $(g, G)$ are quasi - coincidentally commuting and coincidentally idempotent.*

Let $\Phi$ be the set of all upper semi - continuous functions $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ satisfying the conditions

$(\phi_1)$ : $\phi$ is nondecreasing in each coordinate variable,

$(\phi_2)$ : for any $t > 0$,

$$\phi(t) = \max\{\phi(0, t, 0, 0, t), \phi(0, 0, t, t, 0), \phi(t, 0, 0, t, t)\} < t.$$

Quite recently, the following theorem is "proved" in [6].

**Theorem 7** ([6]). *Let $(X, d)$ be a metric space, $f, g$ be two single valued mappings and $S, T : X \to CB(X)$ be two multi - valued maps such for all $x, y \in X$ we have*

$$\int_0^{H(Sx,Ty)} h(t)dt \le \phi\Big(\int_0^{d(fx,gy)} h(t)dt, \int_0^{d(fx,Sx)} h(t)dt, \int_0^{d(gy,Ty)} h(t)dt, \int_0^{d(fx,Ty)} h(t)dt, \int_0^{d(gy,Sx)} h(t)dt\Big), \tag{11}$$

*where $\phi \in$ and $h(t)$ is as in Theorem 5.*

*Suppose that $(f, S)$ and $(g, T)$ are weakly compatible and $(f, g)$ is strongly tangential with respect to $(S, T)$.*

*Then $f$, $g$, $S$ and $T$ have a unique common fixed point.*

**Remark 5.** Let $A, B \in CB(X)$ and $a \in A$, $b \in B$. It is known that the inequality $d(a,b) \leq H(A,B)$ is not correct.

In [6], page 69, line 4 (from top), is used the inequality $d(z,fz) \leq H(M,Sz)$, because $z \in M$ and $fz \in Sz$.

In line 7 (from bottom) is used the inequality $d(z,fz) \leq H(M,Tz)$, because $z \in M$ and $gz \in Tz$.

Other mistake is in the proof of the uniqueness of fixed point at page 70, line 2 (from top).

**Theorem 8.** *Let $(X,d)$ be a metric space, whereas $Y$ is an arbitrary nonempty subset of $X$ with $A, B : Y \to CL(X)$ and $S, T : Y \to X$. Suppose that*

$$F\left(\begin{array}{c} \int_0^{H(Ax,By)} h(t)\,dt, \int_0^{d(Sx,Ty)} h(t)\,dt, \int_0^{d(Sx,Ax)} h(t)\,dt, \\ \int_0^{d(Ty,By)} h(t)\,dt, \int_0^{d(Sx,By)} h(t)\,dt, \int_0^{d(Ty,Ax)} h(t)\,dt \end{array}\right) < 0 \tag{12}$$

*for all $x \neq y$, $h(t)$ is as in Theorem 5 and $F \in \mathcal{F}$.*

*If $A, S$ and $T$ satisfy $CLR_{(A,S)T}$ - property, then*

*1) $\mathcal{C}(A,S) \neq \emptyset$,*

*2) $\mathcal{C}(B,T) \neq \emptyset$.*

*Moreover, if $(A,S)$ and $(B,T)$ are compatible of type $N$ and*

*a) $(A,S)$ is coincidentally idempotent, then $A$ and $S$ have a common fixed point;*

*b) $(B,T)$ is coincidentally idempotent, then $B$ and $T$ have a common fixed point;*

*c) $(A,S)$ and $(B,T)$ are coincidentally idempotent each, then $A, B, S$ and $T$ have a common fixed point.*

**Proof.** Let $\psi(t)$ as in Lemma 1. Then

$$\begin{aligned} \psi(H(Ax,By)) &= \int_0^{H(Ax,By)} h(t)\,dt, \psi(d(Sx,Ty)) = \int_0^{d(Sx,Ty)} h(t)\,dt, \\ \psi(d(Sx,Ax)) &= \int_0^{d(Sx,Ax)} h(t)\,dt, \psi(d(Ty,By)) = \int_0^{d(Ty,By)} h(t)\,dt, \\ \psi(d(Sx,By)) &= \int_0^{d(Sx,By)} h(t)\,dt, \psi(d(Ty,Ax)) = \int_0^{d(Ty,Ax)} h(t)\,dt. \end{aligned}$$

Then, by (12) we obtain

$$F\left(\begin{array}{c} \psi(H(Ax,By)), \psi(d(Sx,Ty)), \psi(d(Sx,Ax)), \\ \psi(d(Ty,By)), \psi(d(Sx,By)), \psi(d(Ty,Ax)) \end{array}\right) < 0.$$

Hence, the conditions of Theorem 2 are satisfied and the proof of Theorem 8 it follows by Theorem 2. ∎

By Theorem 8 and Example 7 we obtain

**Theorem 9.** *Let $(X,d)$ be a metric space, whereas $Y$ is an arbitrary nonempty subset of $X$ with $A,B:Y\to CL(X)$ and $S,T:Y\to X$. Suppose that*

$$\begin{aligned}(13)\qquad \int_0^{H(Ax,By)} h(t)\,dt \le \alpha \max\{&\int_0^{d(Sx,Ty)} h(t)\,dt,\\ &\int_0^{d(Sx,Ax)} h(t)\,dt, \int_0^{d(Ty,By)} h(t)\,dt\}\\ &+ (1-\alpha)\left[a\int_0^{d(Sx,By)} h(t)\,dt + b\int_0^{d(Ty,Ax)} h(t)\,dt\right]\end{aligned}$$

*for all $x\neq y\in Y$, $\alpha\in(0,1)$, $a,b\ge 0$, $a+b<1$ and $h(t)$ is as in Theorem 5.*

*If $A,S$ and $T$ satisfy $CLR_{(A,S)T}$ - property, then*

1) $\mathcal{C}(A,S)\neq\emptyset$,

2) $\mathcal{C}(B,T)\neq\emptyset$.

*Moreover, if $(A,S)$ and $(B,T)$ are compatible of type $N$ and*

*a) $(A,S)$ is coincidentally idempotent, then $A$ and $S$ have a common fixed point;*

*b) $(B,T)$ is coincidentally idempotent, then $B$ and $T$ have a common fixed point;*

*c) $(A,S)$ and $(B,T)$ are coincidentally idempotent each, then $A,B,S$ and $T$ have a common fixed point.*

**Corollary 1.** *Let $(X,d)$ be a metric space, whereas $Y$ is an arbitrary nonempty subset of $X$ with $A,B:Y\to CL(X)$ and $S,T:Y\to X$ such that*

$$\begin{aligned}(14)\qquad \int_0^{H(Ax,By)} h(t)\,dt \le \alpha &\int_0^{\max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By)\}} h(t)\,dt\\ &+ (1-\alpha)\left[a\int_0^{\frac{d(Sx,By)}{2}} h(t)\,dt + b\int_0^{\frac{d(Ty,Ax)}{2}} h(t)\,dt\right]\end{aligned}$$

*for all $x\neq y\in Y$, $\alpha\in(0,1)$, $a,b\ge 0$, $a+b<1$ and $h(t)$ is as in Theorem 5.*

*If $A,S$ and $T$ satisfy $CLR_{(A,S)T}$ - property, then*

1) $\mathcal{C}(A,S)\neq\emptyset$,

2) $\mathcal{C}(B,T)\neq\emptyset$.

*Moreover, if $(A,S)$ and $(B,T)$ are compatible of type $N$ and*

*a) $(A,S)$ is coincidentally idempotent, then $A$ and $S$ have a common fixed point;*

*b) $(B,T)$ is coincidentally idempotent, then $B$ and $T$ have a common fixed point;*

*c) $(A,S)$ and $(B,T)$ are coincidentally idempotent each, then $A,B,S$ and $T$ have a common fixed point.*

**Proof.** The proof it follows by Theorem 9 and

$$\begin{aligned}\int_0^{H(Ax,By)} h(t)\,dt &\leq \alpha \int_0^{\max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By)\}} h(t)\,dt \\ &+ (1-\alpha)\,[a \int_0^{\frac{d(Sx,By)}{2}} h(t)\,dt + b \int_0^{\frac{d(Ty,Ax)}{2}} h(t)\,dt] \\ &\leq \alpha \max\{\int_0^{d(Sx,Ty)} h(t)\,dt, \int_0^{d(Sx,Ax)} h(t)\,dt, \int_0^{d(Ty,By)} h(t)\,dt\} \\ &+ (1-\alpha)\,[a \int_0^{d(Sx,By)} h(t)\,dt + b \int_0^{d(Ty,Ax)} h(t)\,dt].\end{aligned}$$

■

**Remark 6.** Corollary 1 is a generalization of Theorem 6.

Let $\Phi'$ be the set of all upper semi - continuous functions $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ satisfying the conditions

$(\phi_1')$ : $\phi$ is nondecreasing in each coordinate variable,

$(\phi_2')$ : for any $t > 0$,

$$\phi(t) = \max\{\phi(0,t,0,0,t), \phi(0,0,t,t,0)\} < t.$$

If $\phi \in \Phi'$ then $F(t_1, ..., t_6) = t_1 - \phi(\max\{t_2, t_3, t_4, t_5, t_6\})$ satisfies conditions $(F_1), (F_2), (F_3)$.

**Theorem 10.** *Let $(X, d)$ be a metric space, whereas $Y$ is an arbitrary nonempty subset of $X$ with $A, B : Y \to CL(X)$ and $S, T : Y \to X$ such that*

$$\begin{aligned}\int_0^{H(Ax,By)} h(t)\,dt &< \phi(\max\{\int_0^{d(Sx,Ty)} h(t)\,dt, \int_0^{d(Sx,Ax)} h(t)\,dt, \\ &\int_0^{d(Ty,By)} h(t)\,dt, \int_0^{d(Sx,By)} h(t)\,dt + b \int_0^{d(Ty,Ax)} h(t)\,dt\}\end{aligned} \tag{15}$$

*for all $x \neq y \in Y$, $\phi \in \Phi'$ and $h(t)$ is as in Theorem 5.*

*If $A$, $S$ and $T$ satisfy $CLR_{(A,S)T}$ - property, then*

*1) $\mathcal{C}(A, S) \neq \emptyset$,*

*2) $\mathcal{C}(B, T) \neq \emptyset$.*

*Moreover, if $(A, S)$ and $(B, T)$ are compatible of type $N$ and*

*a) $(A, S)$ is coincidentally idempotent, then $A$ and $S$ have a common fixed point;*

*b) $(B, T)$ is coincidentally idempotent, then $B$ and $T$ have a common fixed point;*

*c) $(A, S)$ and $(B, T)$ are coincidentally idempotent each, then $A, B, S$ and $T$ have a common fixed point.*

**Remark 7.** This is a generalization and the correct form of Theorem 7.

### 5.2. Coincidence and common fixed points for $\varphi$ - contractive hybrid mappings

As in [16], let $\Phi$ be the set of all nondecreasing upper semi - continuous functions $\varphi : [0,\infty) \to [0,\infty)$ such that

1) $\varphi(t) < t$ for all $t \in (0,\infty)$,
2) $\varphi(0) = 0$.

The following functions $F : \mathbb{R}_+^6 \to \mathbb{R}_+$ satisfy conditions $(F_1), (F_2), (F_3)$.

**Example 13.** $F(t_1, ..., t_6) = t_1 - \varphi(\max\{t_2, t_3, t_4, t_5, t_6\})$.

**Example 14.** $F(t_1, ..., t_6) = t_1 - \varphi\left(\max\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\}\right)$.

**Example 15.** $F(t_1, ..., t_6) = t_1 - \varphi\left(\max\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\}\right)$.

**Example 16.** $F(t_1, ..., t_6) = t_1 - \varphi\left(\max\{t_2, \sqrt{t_3t_4}, \sqrt{t_3t_5}, \sqrt{t_4t_6}, \sqrt{t_5t_6}\}\right)$.

**Example 17.** $F(t_1, ..., t_6) = t_1 - \varphi(at_2 + bt_3 + ct_4 + dt_5 + et_6)$, where $a, b, c, d, e \geq 0$ and $a + b + c + d + e \leq 1$.

**Example 18.** $F(t_1, ..., t_6) = t_1 - \varphi\left(at_2 + \frac{b\sqrt{t_5t_6}}{1+t_3+t_4}\right)$, where $a, b \geq 0$ and $a + b < 1$.

**Example 19.** $F(t_1, ..., t_6) = t_1 - \varphi\left(at_2 + b\max\{t_3, t_4\} + c\max\{\frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\}\right)$, where $a, b, c \geq 0$ and $a + b + c \leq 1$.

**Example 20.** $F(t_1, ..., t_6) = t_1 - \varphi\left(at_2 + b\max\{\frac{2t_4+t_5}{3}, \frac{2t_4+t_6}{3}\}\right)$, where $a, b \geq 0$ and $a + b \leq 1$.

The following theorem is proved in [16].

**Theorem 11.** *Let $(X, d)$ be a metric space, whereas $Y$ is an arbitrary nonempty subset of $X$ with $F, G : Y \to CL(X)$ and $f, g : Y \to X$. Suppose that*

1) *the hybrid pairs $(F, f)$ and $(G, g)$ share $(JCLR)$ - property,*
2) *for $x \neq y$,*

$$H(Fx, Gy) \leq \varphi(m(x, y)),$$

*where*

$$\begin{aligned} m(x,y) \ = \ &\max\{d(fx, gy), d(fx, Fx), d(gy, Gy), \\ &d(fx, Gy), d(gy, Fx)\}. \end{aligned}$$

*Then, the pairs $(f, F)$ and $(g, G)$ have, each, a coincidence point.*

*In particular, if the pairs $(f, F)$ and $(g, G)$ are coincidentally commuting and coincidentally idempotent, then $f$, $F$, $g$ and $G$ have a common fixed point.*

By Theorem 8 and Example 13 we obtain

**Theorem 12.** *Let $(X,d)$ be a metric space, whereas $Y$ is an arbitrary nonempty subset of $X$ with $A,B : Y \to CL(X)$ and $S,T : Y \to X$. Suppose that*

$$\begin{aligned}(16)\quad \psi(H(Ax,By)) \le \varphi(\max\{&\psi(d(Sx,Ty)),\psi(d(Sx,Ax)),\\ &\psi(d(Ty,By)),\psi(d(Sx,By)),\psi(d(Ty,Ax))\})\end{aligned}$$

*for all $x \neq y$, $\psi(t)$ is an altering distance and $\varphi \in \Phi$.*

*If $A$, $S$ and $T$ satisfy $CLR_{(A,S)T}$ - property, then*

*1) $\mathcal{C}(A,S) \neq \emptyset$,*

*2) $\mathcal{C}(B,T) \neq \emptyset$.*

*Moreover, if $(A,S)$ and $(B,T)$ are compatible of type $N$ and*

*a) $(A,S)$ is coincidentally idempotent, then $A$ and $S$ have a common fixed point;*

*b) $(B,T)$ is coincidentally idempotent, then $B$ and $T$ have a common fixed point;*

*c) $(A,S)$ and $(B,T)$ are coincidentally idempotent each, then $A$, $B$, $S$ and $T$ have a common fixed point.*

By Theorem 8 and Example 20 we obtain

**Theorem 13.** *Let $(X,d)$ be a metric space, whereas $Y$ is an arbitrary nonempty subset of $X$ with $A,B : Y \to CL(X)$ and $S,T : Y \to X$. Suppose that*

$$\begin{aligned}(17)\quad \int_0^{H(Ax,By)} h(t)\,dt \le \varphi(\max\{&\int_0^{d(Sx,Ty)} h(t)\,dt, \int_0^{d(Sx,Ax)} h(t)\,dt,\\ &\int_0^{d(Ty,By)} h(t)\,dt, \int_0^{d(Sx,By)} h(t)\,dt, \int_0^{d(Ty,Ax)} h(t)\,dt\})\end{aligned}$$

*for all $x \neq y$ and $\varphi \in \Phi$.*

*If $A, S$ and $T$ satisfy $CLR_{(A,S)T}$ - property, then*

*1) $\mathcal{C}(A,S) \neq \emptyset$,*

*2) $\mathcal{C}(B,T) \neq \emptyset$.*

*Moreover, if $(A,S)$ and $(B,T)$ are compatible of type $N$ and*

*a) $(A,S)$ is coincidentally idempotent, then $A$ and $S$ have a common fixed point;*

*b) $(B,T)$ is coincidentally idempotent, then $B$ and $T$ have a common fixed point;*

*c) $(A,S)$ and $(B,T)$ are coincidentally idempotent each, then $A, B, S$ and $T$ have a common fixed point.*

If $h(t) = 1$ we obtain

**Theorem 14.** *Let $(X,d)$ be a metric space, whereas $Y$ is an arbitrary nonempty subset of $X$ with $A,B:Y\to CL(X)$ and $S,T:Y\to X$. Suppose that*

$$H(Ax,By)) \leq \varphi(\{d(Sx,Ty),d(Sx,Ax),\\ d(Ty,By),d(Sx,By),d(Ty,Ax)\}) \tag{18}$$

*for all $x\neq y$ and $\varphi\in\Phi$.*

*If $A,S$ and $T$ satisfy $CLR_{(A,S)T}$ - property, then*

*1) $\mathcal{C}(A,S)\neq\emptyset$,*

*2) $\mathcal{C}(B,T)\neq\emptyset$.*

*Moreover, if $(A,S)$ and $(B,T)$ are compatible of type $N$ and*

*a) $(A,S)$ is coincidentally idempotent, then $A$ and $S$ have a common fixed point;*

*b) $(B,T)$ is coincidentally idempotent, then $B$ and $T$ have a common fixed point;*

*c) $(A,S)$ and $(B,T)$ are coincidentally idempotent each, then $A,B,S$ and $T$ have a common fixed point.*

**Remark 8.** 1) Theorem 14 is a generalization of Theorem 11.

2) By Examples 14 - 20 we obtain new particular results.

### 5.3. Coincidence and common fixed points for $(\psi,\varphi)$ - weakly compatible hybrid pairs of mappings

In 1997, Alber and Guerre-Delabrierre [3] defined the concept of weak contraction as a generalization of contraction and established the existence of fixed points for self mappings in Hilbert spaces. Rhoades [34] extended this concept in metric spaces. In [5] the authors studied the existence of fixed points for a pair of $(\psi,\varphi)$ - weakly compatible mappings.

New results are obtained in [2], [7], [11], [12], [31] and in other papers. Some fixed point theorems for mappings with common limit range property satisfying weakly contractive conditions are proved in [14] and in other papers.

**Definition 14.** 1) *Let $\Psi$ be the set of all functions $\psi:[0,\infty)\to[0,\infty)$ satisfying*

*a) $\psi$ is continuous,*

*b) $\psi(0)=0$ and $\psi(t)>0$, $\forall t>0$.*

2) *Let $\Phi$ be the set of all functions $\phi:[0,\infty)\to[0,\infty)$ satisfying*

*a) $\phi$ is lower semi - continuous,*

*b) $\phi(0)=0$ and $\phi(t)>0$, $\forall t>0$.*

The following functions $F:\mathbb{R}_+^6\to\mathbb{R}_+$ satisfy conditions $(F_1),(F_2),(F_3)$.

**Example 21.** $F(t_1,...,t_6) = \psi(t_1) - \varphi\left(\max\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\}\right) + \phi(\max\{t_3, t_4,t_5,t_6\})$.

**Example 22.** $F(t_1,...,t_6) = \psi(t_1) - \varphi\left(\max\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\}\right) + \phi(\max\{t_2, t_3,t_4,t_5,t_6\})$.

**Example 23.** $F(t_1,...,t_6) = \psi(t_1) - \varphi(\max\{t_2,t_3,t_4,t_5,t_6\}) + \phi\left(\max\{t_2, t_3,t_4,\frac{t_5+t_6}{2}\}\right)$.

**Example 24.** $F(t_1,...,t_6) = \psi(t_1) - \varphi\left(\max\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\}\right) + \phi\left(\max\{t_2, \frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\}\right)$.

**Example 25.** $F(t_1,...,t_6) = \psi(t_1) - \varphi\left(\max\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\}\right) + \phi\left(\max \{\sqrt{t_2t_5},\sqrt{t_3t_6},\sqrt{t_4t_6}\}\right)$.

**Example 26.** $F(t_1,...,t_6) = \psi(t_1) - \varphi\left(\max\{\sqrt{t_3t_6},\sqrt{t_2t_5},\sqrt{t_4t_6}\}\right) + \phi(\max\{t_2,t_3,t_4,t_5,t_6\})$.

**Example 27.** $F(t_1,...,t_6) = \psi(t_1) - \varphi\left(\frac{\sqrt{t_3t_6}+\sqrt{t_4t_6}+\sqrt{t_2t_6}}{1+\sqrt{t_3t_4}+\sqrt{t_4t_6}+\sqrt{t_2t_3}}\right) + \phi(\max \{t_2,t_3,t_4,t_5,t_6\})$.

By Theorem 4 and Example 21 we obtain

**Theorem 15.** *Let $(X,d)$ be a metric space, whereas $Y$ is an arbitrary nonempty subset of $X$ with $A,B:Y\to CL(X)$ and $S,T:Y\to X$. Suppose that*

$$(19)\qquad \begin{aligned} H(Ax,By) \leq{} & \psi\left(\max\left\{\begin{array}{c} d(Sx,Ty), d(Sx,Ax), \\ d(Ty,By), \frac{d(Sx,By)+d(Ty,Ax)}{2} \end{array}\right\}\right) \\ & + \phi(\max\{d(Sx,Ax), d(Ty,By) \\ & \quad + d(Sx,By), d(Ty,By)\}), \end{aligned}$$

*for all $x\neq y\in Y$, $\psi\in\Psi$ and $\phi\in\Phi$. If $A$, $S$ and $T$ satisfy $CLR_{(A,S)T}$ - property, then*

*1) $\mathcal{C}(A,S)\neq\emptyset$,*

*2) $\mathcal{C}(B,T)\neq\emptyset$.*

*Moreover, if $(A,S)$ and $(B,T)$ are compatible of type $N$ and*

*a) $(A,S)$ is coincidentally idempotent, then $A$ and $S$ have a common fixed point;*

*b) $(B,T)$ is coincidentally idempotent, then $B$ and $T$ have a common fixed point;*

*c) $(A,S)$ and $(B,T)$ are coincidentally idempotent each, then $A$, $B$, $S$ and $T$ have a common fixed point.*

**Remark 9.** By Examples 22 - 27 we obtain new particular results.

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