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Y. Tabet Zatla and N. Daoudi-Merzagui

POSITIVE SOLUTION FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION WITH INTEGRAL BOUNDARY VALUE CONDITION

Abstract. In this paper, we consider a fractional differential equation, with integral boundary conditions, when the nonlinearities are sign changing. Our approach is based on the Krasnoselskii theorem in double cones. We generalize some recent results.

Key words: fractional differential equation, integral boundary value condition, positive solution, fixed point theorem, cones.

AMS Mathematics Subject Classification: 34A08, 34B05, 34B15.

1. Introduction

Differential equations of fractional order have recently proved to be a valuable tools in the modelling of many phenomena in various fields of science and engineering. These models have been applied successfully, e.g., in physics [22], mechanics (theory of viscoelasticity and viscoplasticity) [15], (bio-)chemistry (modelling of polymers and proteins) [20], [26], electrical engineering (transmission of ultrasound waves), bio-engineering [17], control theory, movement through porous media [28], electromagnetics, and electrochemistry [31].

The history, definitions, theory, and applications of fractional calculus are well laid out in the books by Miller and Ross [29], Oldham and Spanier [32] , Samko, Kilbas, and Marichev [33].

In recent years, the study of positive solutions for fractional differential equation boundary value problems (FBVPs for short) has attracted considerable attention, and abundance of papers treating this subject attest on that. For a small example of such work, we refer the reader to [2], [3], [4], [5], [6], [9], [11], [12], [16], [18], [13], [30], [34], [40], [42], [43] and the references therein.

Many researchers have investigated FBVPs where the nonlinear term is positive with nonlocal boundary conditions. This kind of conditions appears for example in the study of population dynamics [8] and cellular systems [1].

Their results are based on different methods: the application of fixed point theorems [3] (alternative of Leray-Schauder), [13] (Leggett-Williams), [9], [10], [11], [43] (Krasnoselskii), the theory of fixed point index [7], the method of upper-lower solutions [24], variational methods [35], [36] and so on.

When the nonlinearity is allowed to change sign, a number of papers have been carried out whether concerning integer or fractional order differential equations. For ordinary differential equations, by using the fixed point theorem in double cones [19], Guo in [21] showed the existence of positive solutions for second-order three point BVP and Chen [14] considered an m-point BVP associated to the second order differential equation. In [27], we showed the existence of at least two positive solutions for BVP with integral conditions. For fractional differential equation, some authors establish the same result by investigating the properties of the associated Green function and utilizing a topological approach, we refer the reader to [38], [39].

Motivated by the works mentioned above, we shall improve the results in [27], [11], and obtain a new one.

In this paper, using a fixed point theorem in double cones we prove the existence of multiple positive solutions of FBVP, when the nonlinear term is allowed to change sign.

$$ {}^cD_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, \qquad 0 < t < 1, \quad 1 < \alpha \leq 2 \tag{1} $$

$$ u'(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)ds, \tag{2} $$

where $0 \leq \lambda < 1$, ${}^cD_{0+}^{\alpha}$ is the Caputo's differential operator of order α and the nonlinear term f satisfies

$(H1)$ (i) $f : [0,1] \times [0,+\infty) \to \mathbb{R}$ is continuous, and can changing sign,

(ii) $f(t,0) \geq 0 (\not\equiv 0)$ for all t in $[0,1]$ (i.e; there exists an interval J_0 of $[0,1]$ such that $f(t,0) > 0$ for all t in J_0).

The paper is organized as follows: Section 2 contains the basic preliminaries. The main result is given in Section 3. An example is given in Section 4.

2. Preliminaries

We present the necessary definitions and some basic results from fractional calculs theory.

Definition 1 ([23]). *The Riemann-Liouville fractional integral operator of order $Re(\alpha) > 0$ of a continuous function $h : [0,1] \to \mathbb{R}$ is defined as*

$$I_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds.$$

Lemma 1 ([23]). *The relation*

$$I_{0^+}^{\alpha} I_{0^+}^{\beta} h(t) = I_{0^+}^{\alpha+\beta} h(t)$$

is valid in the following case

$$Re\beta > 0, \quad Re(\alpha+\beta) > 0, \quad h \in C^0[0,1].$$

Definition 2 ([23]). *The Caputo's fractional derivative of order $Re(\alpha > 0)$ for a function $h \in C^n[0,1]$ $(n \geq 1)$ is defined as*

$${}^c D_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^n(s) ds,$$

where $n-1 < \alpha \leq n$.

Lemma 2 ([23]). *Let $n-1 < \alpha \leq n$, $h \in C^n[0,1]$. Then*

$$I_{0^+}^{\alpha} {}^c D_{0^+}^{\alpha} h(t) = h(t) - c_1 - c_2 t - \cdots - c_n t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$.

Definition 3. *Let X be a Banach space, and K be a closed nonempty subset of X. K is said to be a cone of X if it satisfies the following conditions:*

(i) *$x \in K$ and $\lambda \geq 0$ implies $\lambda x \in K$,*
(ii) *$x \in K$ and $-x \in K$ implies $x = 0$.*

Now, we introduce some notations.

1. For some cone K in a Banach space $(X, ||.||)$ and a constant $r > 0$, we define the following sets:

$$K_r = \{x \in K : ||x|| < r\},$$

$$\partial K_r = \{x \in K : ||x|| = r\},$$

and if $\theta : K \to \mathbb{R}^+$ is a continuous functional such that $\theta(\lambda x) \leq \theta(x)$ for $\lambda \in (0,1)$, we define:

$$K(b) = \{x \in K : \theta(x) < b\},$$

$$\partial K(b) = \{x \in K : \theta(x) = b\},$$

and

$$K_a(b) = \{x \in K : a < ||x||, \theta(x) < b\},$$

where a, b are two positive constants.

2. For $u \in X$, we define $\psi : X \to K$ such that $\psi(u) = u^+ = \max\{u, 0\}$.

The following well-known result, Krasnoselskii fixed point theorem, is crucial in our arguments.

Theorem 1. *Let X be a real Banach space with norm $||.||$ and $K, K' \subset X$ two cones with $K' \subset K$. Suppose $T : K \to K$ and $T^* : K' \to K'$ are two completely continuous operators and $\theta : K' \to \mathbb{R}^+$ is a continuous functional satisfying $\theta(x) \le ||x|| \le M\theta(x)$ for all $x \in K'$, where M is a constant such that $M \ge 1$. If there are constants $b > a > 0$ such that*
$(C1)$ $||Tx|| < a$ for $x \in \partial K_a$;
$(C2)$ $||T^*x|| < a$ for $x \in \partial K'_a$ and $\theta(T^*x) > b$ for $x \in \partial K'(b)$;
$(C3)$ $Tx = T^*x$, for $x \in K'_a(b) \cap \{u : T^*u = u\}$.
Then T has at least two fixed points y_1 and y_2 in K, such that

$$0 \le ||y_1|| < a < ||y_2||, \quad \theta(y_2) < b.$$

Proof. For the proof of this result, we refer the reader to [41]. ■

Now, we present some lemmas. Let I be the interval $[0, 1]$, and $\|u\| = \sup\{|u(t)|,\ t \in I\}$ denote the norm of $u \in C(I)$, ($C(I)$ is the space of real-valued continuous functions on I). We put $X = C(I)$.

Lemma 3. *Let $0 < \lambda < 1$, $y \in X$. Then the boundary value problem*

$$ {}^cD^{\alpha}_{0+}u(t) + y(t) = 0, \qquad 0 < t < 1, \quad 1 < \alpha \le 2 \tag{3}$$

$$u'(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)ds, \tag{4}$$

is equivalent to the following integral equation

$$u(t) = \int_0^1 G(t, s)y(s)ds,$$

where

$$G(t,s) = \begin{cases} \frac{(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)-(1-\lambda)\alpha(t-s)^{\alpha-1}}{(1-\lambda)\Gamma(\alpha+1)} & \text{if } \ 0 \le s \le t \le 1, \\ \frac{(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(1-\lambda)\Gamma(\alpha+1)} & \text{if } \ 0 \le t \le s \le 1. \end{cases}$$

Here G is called the Green's function boundary value problem.

Proof. Applying the operator I^{α}_{0+} to the equation (3) and using Lemma 2, we obtain the following integral equation

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + c_1 + c_2 t.$$

Now, (4) imply that $c_2 = 0$, and,

$$c_1 = u(1) + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds.$$

Therefore, we obtain,

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + u(1) + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds.$$

Let put $\eta = \int_0^1 u(t)dt$, then, from the previous equality, using that $u(1) = \lambda\eta$, we deduce that

$$\begin{aligned} \eta \; &= \; -\int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)dsdt + \int_0^1 u(1)dt \\ &\quad + \int_0^1 \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)dsdt, \end{aligned}$$

by changing the order of integration, we obtain

$$\begin{aligned} \eta \; &= \; -\int_0^1 \int_s^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)dtds + \int_0^1 u(1)dt \\ &\quad + \int_0^1 \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)dsdt \\ &= \; -\int_0^1 \frac{(1-s)^{\alpha}}{\alpha\Gamma(\alpha)} y(s)ds + \lambda\eta + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds. \end{aligned}$$

Thus,

$$\eta = \frac{1}{1-\lambda} \int_0^1 \frac{(1-s)^{\alpha-1}(\alpha-1+s)}{\alpha\Gamma(\alpha)} y(s)ds,$$

which implies that

$$c_1 = \frac{\lambda}{1-\lambda} \int_0^1 \frac{(1-s)^{\alpha-1}(\alpha-1+s)}{\alpha\Gamma(\alpha)} y(s)ds + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds.$$

Finally,

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \int_0^1 \frac{(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(1-\lambda)\alpha\Gamma(\alpha)} y(s)ds.$$

So

$$\begin{aligned} u(t) \; &= \; \int_0^t \frac{(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (1-\lambda)\alpha(t-s)^{\alpha-1}}{(1-\lambda)\Gamma(\alpha+1)} y(s)ds \\ &\quad + \int_t^1 \frac{(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(1-\lambda)\Gamma(\alpha+1)} y(s)ds. \end{aligned}$$

Thus

$$u(t) = \int_0^t G(t,s)y(s)ds.$$

■

Lemma 4. *Assume that $0 < \lambda < 1$. Then the Green's function G satisfies the following properties:*
1) $G(t,s) \geq 0$ *for all* $s,t \in [0,1]$*;*
2) $G(1,s) \leq G(t,s) \leq \frac{\alpha}{\lambda(\alpha-1)} G(1,s)$ *for all* $s,t \in [0,1]$*;*
3) $G(t,s)$ *is continuous function for all* $s,t \in [0,1]$*;*
4) $\max_{t,s\in[0,1]} G(t,s) \leq \frac{1}{(1-\lambda)\Gamma(\alpha)}$

Proof.

1. It is not difficult to verify that for $0 \leq s \leq t$ the following inequalities hold

$$\begin{aligned} G(t,s) &= \frac{(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (1-\lambda)\alpha(t-s)^{\alpha-1}}{(1-\lambda)\Gamma(\alpha+1)} \\ &\geq \frac{(1-s)^{\alpha-1}\lambda(\alpha+s-1)}{(1-\lambda)\Gamma(\alpha+1)} \geq 0, \end{aligned}$$

using $1 < \alpha \leq 2$ and $s \geq 0$, we get

$$G(t,s) \geq 0.$$

Also, for $t < s$ we have

$$G(t,s) = \frac{(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(1-\lambda)\Gamma(\alpha+1)} \geq 0.$$

2. We consider two cases:

— If $0 \leq s \leq t \leq 1$, then we have

$$G(t,s) \geq (1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (1-\lambda)\alpha(1-s)^{\alpha-1} = G(1,s)$$

and

$$\frac{G(t,s)}{G(1,s)} \leq \frac{\alpha-\lambda+\lambda s}{\lambda(s+\alpha-1)} \leq \frac{\alpha}{\lambda(\alpha-1)}.$$

— If $0 \leq t \leq s \leq 1$, the following inequalities hold

$$1 \leq \frac{s-1+\frac{\alpha}{\lambda}}{s-1+\alpha} \leq \frac{G(t,s)}{G(1,s)} \leq \frac{\alpha}{\lambda(\alpha-1)}.$$

3. It is obvious from the definition of the function G.

4. For all $t, s \in [0,1]$ we have that

$$G(t,s) \leq \frac{(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(1-\lambda)\Gamma(\alpha+1)} \leq \frac{1}{(1-\lambda)\Gamma(\alpha)}.$$

■

Consider the subsets K, K' of X, defined by,

$$K = \{u \in X : u(t) \geq 0, t \in I\},$$

and

$$K' = \left\{u \in X : \min_{0\leq t\leq 1} u(t) \geq \gamma ||u||\right\},$$

where

$$0 < \gamma = \frac{\lambda(\alpha-1)}{\alpha} < 1.$$

Clearly, $K, K' \subset X$ are cones with $K' \subset K$.

For all $u \in K$, we define

$$\theta(u) = \min_{0\leq t\leq 1} u(t),$$

and the operators T, A, T^* by: $T : K \to K$, $A : K \to X$ and $T^* : K' \to K'$, such that:

$$\begin{aligned}
Tu(t) &= \left[\int_0^1 G(t,s)f(s,u(s))ds\right]^+, \quad \text{for all } t \in I,\\
Au(t) &= \int_0^1 G(t,s)f(s,u(s))ds, \quad \text{for all } t \in I,\\
T^*u(t) &= \int_0^1 G(t,s)f^+(s,u(s))ds, \quad \text{for all } t \in I.
\end{aligned}$$

By the above notation we have that:

$$T = \psi \circ A.$$

Lemma 5. *$T^* : K' \to K'$ is completely continuous.*

Proof. Let $u \in K'$. For all $t \in [0,1]$, we have

$$\begin{aligned}
T^*u(t) &\geq \int_0^1 G(1,s)f(s,u(s))ds\\
&\geq \frac{\lambda(\alpha-1)}{\alpha}\int_0^1 \left\{\max_{0\leq t\leq 1} G(t,s)\right\} f(s,u(s))ds.\\
&\geq \frac{\lambda(\alpha-1)}{\alpha}\max_{0\leq t\leq 1}\left\{\int_0^1 G(t,s)f(s,u(s))ds\right\}\\
&= \frac{\lambda(\alpha-1)}{\alpha}\|T^*u\| = \gamma\|T^*u\|.
\end{aligned}$$

Hence, $T^*u : K' \to K'$. By using the Arzela-Ascoli Theorem, we can prove that T^* is completely continuous operator (for the proof see [10]). ■

Lemma 6. *A function $u(t)$ is a solution of BVP* (1)-(2) *if and only if $u(t)$ is a fixed point of the operator A.*

Lemma 7. *If $A : K \to X$ is completely continuous, then $T = \psi \circ A : K \to K$ is also completely continuous.*

Proof. The complete continuity of A implies that A is continous and applies each bounded subset of K on a relatively compact set of X. Given a function $h \in X$, for each $\varepsilon > 0$ there is $\sigma > 0$ such that

$$||Ah - Ak|| < \varepsilon \;\text{ for }\; k \in X, \;\; ||h - k|| < \sigma.$$

Since

$$\begin{aligned} |(\psi Ah)(t) - (\psi Ak)(t)| &= |\max\{(Ah)(t), 0\} - \max\{(Ak)(t), 0\}| \\ &\leq |(Ah)(t) - (Ak)(t)| < \varepsilon, \end{aligned}$$

we have

$$\|(\psi A)h - (\psi A)k\| < \varepsilon \;\text{ for }\; k \in X, \;\; ||h - k|| < \sigma,$$

and so ψA is continuous.

For any arbitrary bounded set $D \subset X$ and for all $\varepsilon > 0$, there are y_i, $i = 1, \ldots, m$ such that

$$AD \subset \bigcup_{i=1}^{m} \beta(y_i, \varepsilon),$$

where $\beta(y_i, \varepsilon) = \{x \in X : \|x - y_i\| < \varepsilon\}$. Then, if we denote ψy by $\bar{y}$, for all $\bar{y} \in (\psi o A)(D)$, there is $y \in AD$ such that $\bar{y}(t) = \max\{y(t), 0\}$. We choose $y_i \in \{y_1, .., y_m\}$ such that

$$\max_{t\in[0,1]} |y(t) - y_i(t)| < \varepsilon.$$

Thus

$$\max_{t\in[0,1]} |\bar{y}(t) - \bar{y}_i(t)| \leq \max_{t\in[0,1]} |y(t) - y_i(t)| < \varepsilon,$$

which implies

$$\bar{y} \in B(\bar{y}_i, \varepsilon),$$

and therefore $(\psi o A)(D)$ is relatively compact. ■

Lemma 8. *If u is a fixed point of operator T, then u is also a fixed point of operator A.*

Proof. Let u be a fixed point of operator T. We claim that u is also a fixed point of A in K_a. Suppose on the contrary, that there exists a $t^* \in [0,1]$ such that

$$Au(t^*) \neq u(t^*) = Tu(t^*) = \max\{Au(t^*), 0\},$$

and so, this forces

$$Au(t^*) < 0 = u(t^*).$$

Let (t_1, t_2) be the maximal interval which contain t^* and shuch that $Au(t) < 0$ for all $t \in (t_1, t_2)$, and $Au(t_1) = Au(t_2) = 0$. Note that

$$u(t) = Tu(t) = \max\{Au(t), 0\} = 0, \quad \text{for all } \ t \in [t_1, t_2].$$

Obviously, $(t_1, t_2) \neq [0,1]$, by the assumption $(H1)(ii)$. So we should have either $t_2 < 1$ or $t_1 > 0$. By definition of A and lemma (3) we have

$${}^cD^{\alpha}_{0+}Au(t) = -f(t,u), \ t \in [0,1].$$

For each $t \in [t_1, t_2]$, by $(H1)(ii)$, we have, ${}^cD^{\alpha}_{0+}Au(t) = -f(t,0) \leq 0$ $(1 < \alpha \leq 2)$. So $0 \geq D^2_{0+}Au(t) = A''u(t)$, in particular, this implies that $A'u(t)$ is decreasing on $[t_1, t_2]$.

- If $t_2 < 1$, since $Au(t) < 0$ for $t \in (t_1, t_2)$, and $Au(t_2) = 0$, we have $A'u(t_2) \geq 0$. We obtain $t_1 = 0$, and $A'u(t) > 0$, for $t \in [0, t_2)$ which contradicts with the first condition of (1.1),(1.2) $(A'u(0) = 0)$.
- If $t_1 > 0$, we have $u(t) = 0$ for $t \in [t_1, t_2]$, $Au(t) < 0$ for $t \in (t_1, t_2)$ and $(Au)(t_1) = 0$. Thus $(Au)'(t_1) \leq 0$. From $(H1)(ii)$ we have $(Au)''(t) \leq 0$ for $t \in [t_1, t_2]$. So $t_2 = 1$. By the concavity of $Au(t)$ on $[t_1, 1]$, we have

$$\frac{|(Au)(s)|}{s - t_1} \leq \frac{|(Au)(1)|}{1 - t_1}.$$

This implies that

$$|(Au)(s)| \leq \frac{s - t_1}{1 - t_1}|(Au)(1)| < s|(Au)(1)|,$$

From the above inequalities, we obtain

$$\int_0^1 |Au(s)|ds \leq \int_0^1 s|(Au)(1)|ds < |(Au)(1)|$$

which contradicts

$$|(Au)(1)| = \lambda \left| \int_0^1 Au(s)ds \right| \leq \int_0^1 |Au(s)|ds.$$

■

3. Main result

In this section, we show the existence of two positive solutions for the BVP (1)-(2). We denote

$$m = \frac{(\alpha-1)\lambda}{\alpha(1-\lambda)\Gamma(\alpha+1)}, \quad M = \frac{1}{(1-\lambda)\Gamma(\alpha)} > 0.$$

From Lemma 3 and Lemma 4, we have

$$\int_0^1 \max_{t\in I} G(t,s)ds \leq M$$

and

$$\int_0^1 \min_{t\in I} G(t,s)ds \geq m.$$

Theorem 2. *Suppose that condition $(H1)$ hold. If there exist positive numbers a, b, d such that*

$$0 < \frac{1}{\gamma}d < a < \gamma b < b,$$

and that f satisfies the following assumptions:
$(H2)$ $f(t,u) \geq 0$ *for* $(t,u) \in [0,1] \times [d,b]$.
$(H3)$ $f(t,u) < \frac{a}{M}$ *for* $(t,u) \in [0,1] \times [0,a]$.
$(H4)$ $f(t,u) \geq \frac{b}{m}$ *for* $(t,u) \in [0,1] \times [\gamma b, b]$.
Then, (1)-(2) *has at least two positive solutions u_1 and u_2 such that*

$$0 \leq ||u_1|| < a < ||u_2||, \ \theta(u_2) < b.$$

Proof. From $(H3)$, for all $u \in \partial K_a$; i.e., $||u|| = a$, we have

$$\begin{aligned} ||Tu|| &= \max_{t\in I}\left[\int_0^1 G(t,s)f(s,u(s))ds\right]^+ \\ &\leq \max_{t\in I}\max\left\{\int_0^1 G(t,s)f(s,u(s))ds, 0\right\} \\ &\leq \frac{a}{M}\int_0^1 \max_{t\in I} G(t,s)ds \\ &< \frac{a}{M}M = a. \end{aligned}$$

So, $(C1)$ of Theorem (1) is satisfied. For $u \in \partial K'_a$, from $(H3)$ we have

$$\begin{aligned} ||T^*u|| &= \max_{t\in I}\int_0^1 G(t,s)f^+(s,u(s))ds \\ &\leq \frac{a}{M}\int_0^1 \max_{t\in I} G(t,s)ds \\ &< \frac{a}{M}M = a. \end{aligned}$$

Let $u \in \partial K'(\gamma b)$; i.e., $u \in K'$ and $\theta(u) = \gamma b$, we have

$$\gamma b = \theta(u) = \min_{t\in[0,1]} u(t) \geq \gamma ||u||,$$

hence

$$||u|| \leq b.$$

On the other hand

$$u(t) \geq \min_{t\in[0,1]} u(t) = \theta(u) = \gamma b, \quad \text{for} \;\; t \in [0,1],$$

so

$$\gamma b \leq u(t) \leq ||u|| \leq b.$$

From $(H4)$,

$$f(s,u(s)) \geq \frac{b}{m}, \quad \text{for} \;\; s \in [0,1]. \tag{5}$$

So

$$\begin{aligned}
||T^*u|| &= \max_{t\in I} \int_0^1 G(t,s) f^+(s,u(s))ds \\
&\geq \frac{b}{m} \int_0^1 \max_{t\in I} G(t,s)ds \\
&\geq \frac{b}{m} \int_0^1 \min_{t\in I} G(t,s)ds \\
&\geq \frac{b}{m} m = b.
\end{aligned}$$

Thus

$$b < \|T^*u\|.$$

On the other hand

$$(\theta T^*u) \geq \gamma \|T^*u\| > \gamma b.$$

So, $(C2)$ of Theorem (1) is satisfied. Finally, we show that $(C3)$ of Theorem (1) is also satisfied.

Let $u \in K'_a(\gamma b) \cap \{u : T^*u = u\}$, then

$$||u|| > a > \frac{1}{\gamma} d,$$

and for all $t \in [0,1]$,

$$u(t) \geq \min_{t\in[0,1]} u(t) \geq \gamma ||u|| > \gamma \frac{1}{\gamma} d = d.$$

Therefore, for $u \in K'_a(\gamma b) \cap \{u : T^* u = u\}$, we have

$$d \leq u(t) \leq ||u|| \leq b.$$

From $(H2)$,

$$f^+(s, u(s)) = f(s, u(s)).$$

Which implies that

$$Tu = T^* u.$$

Then, the conditions of Theorem (1) are satisfied and, T has two fixed points u_1 and u_2 in K satisfying

$$0 \leq ||u_1|| < a < ||u_2||, \quad \theta(u_2) < \gamma b. \tag{6}$$

■

4. Example

In this section, we present an example to illustrate our result. Consider the fractional differential equation boundary value problems

$$ {}^c D^{\alpha}_{0^+} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \tag{7}$$

$$u'(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)ds, \tag{8}$$

where $\lambda = \frac{1}{2}$, $\alpha = \frac{3}{2}$ and

$$f(t, u) = \frac{1}{12} \begin{cases} (t^2 + 1)\sqrt{u^2 + 9}, & 0 \leq u \leq 4, \\ (t^2 + 1)(u + 1)e^{u-4} & 4 \leq u \leq 48, \\ 49(t^2 + 1)e^{\frac{11}{12}u}(49 - u) & u \geq 48. \end{cases}$$

Clearly f is allowed to change sign. We can let $\gamma = \frac{1}{6}$, $m = 0.25$, $M = 2.25$ $d = \frac{1}{2}$, $a = 4$, and $b = 48$, then

$$\begin{aligned} f(t, u) &> 0 \text{ for all } u \in [d, b], \\ f(t, u) &> \frac{a}{M} \text{ for all } u \in [0, a], \\ f(t, u) &\geq \frac{b}{m} \text{ for all } u \in [\gamma b, b]. \end{aligned}$$

So the conditions of Theorem (2) hold. Then (7)-(8) has at least two positive solutions.

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Y. Tabet Zatla
Department of Mathematics
University of Tlemcen
13000 Tlemcen, Algeria
e-mail: tab_dayas@yahoo.fr

N. Daoudi-Merzagui
Department of Mathematics
University of Tlemcen
13000 Tlemcen, Algeria
e-mail: nmerzagui@yahoo.fr

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