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ON $p\delta s$-IRRESOLUTE FUNCTIONS

Abstract. The object of the present paper is to define and investigate the concepts of $p\delta s$-irresolute functions in topological spaces. And we obtain some characterizations and fundamental properties of such functions.

Key words: α-open set, semi open set, preopen set, δ-semi open set and $p\delta s$-irresolute function.

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1. Preliminaries

Throughout this paper, spaces always mean topological spaces and $f : X \to Y$ denotes a single valued function from a space (X, τ) into a space (Y, υ). Let S be a subset of a space (X, τ). The closure and the interior of S are denoted by $Cl(S)$ and $Int(S)$, respectively. We recall some known definitions and properties.

Definition 1. *A subset S of a space (X, τ) is said to be α-open* [16] *(resp. semi open* [10]*, preopen* [12]*) if $S \subset Int(Cl(Int(S)))$ (resp. $S \subset Cl(Int(S))$, $S \subset Int(Cl(S))$).*

A point $x \in X$ is called the δ-cluster point of A if $A \cap Int(Cl(U)) \neq \varnothing$ for every open set U of X containing x. The set of all δ-cluster points of A is called the δ-closure [17] of A and is denoted by $Cl_\delta(A)$. A subset A of X is called δ-closed [22] if $A = Cl_\delta(A)$. The complement of a δ-closed set is called δ-open [22]. A subset A of X is said to be δ-semi open [18] if there exists a δ-open set U of X such that $U \subset A \subset Cl(U)$. The complement of a δ-semi open set is called δ-semi closed [18]. A point $x \in X$ is called the δ-semicluster point of A if $A \cap U \neq \varnothing$ for every δ-semi open set U of X containing x. The set of all δ-semicluster points of A is called the δ-semiclosure [18] of A and is denoted by $sCl_\delta(A)$.

The family of all α-open (resp. semi open, preopen, δ-semi open) sets in a space (X, τ) is denoted by $\alpha(X)$ (resp. $SO(X)$, $PO(X)$, $\delta SO(X)$). It is shown in [16] that $\alpha(X)$ is a topology for X. Moreover, $\tau \subset \alpha(X) = PO(X) \cap SO(X) \subset PO(X) \cup SO(X)$.

The complement of a preopen set is said to be preclosed [12]. The intersection of all preclosed sets in (X, τ) containing a subset A is called the preclosure [7] of A and is denoted by $pCl(A)$. The union of all δ-semi open sets of X contained in A is called the δ-semiinterior [18] of A and is denoted by $sInt_\delta(A)$.

2. Some properties of $p\delta s$-irresolute functions

Definition 2. *A function $f : (X, \tau) \to (Y, \upsilon)$ is said to be $p\delta s$-irresolute* [1] (*resp. $\alpha\delta s$-irresolute* [1], *semi δs-irresolute* [3]) *if $f^{-1}(V)$ is preopen* (*resp. α-open, semi open*) *in X for every δ-semi open subset V of Y.*

Remark 1 ([1])**.** It is obvious that

$p\delta s$-irresoluteness $\Leftarrow$ $\alpha\delta s$-irresoluteness $\Rightarrow$ semi δs-irresoluteness.

The above implications are not reversible in general. It is shown in Examples 2.1 and 2.2 of [1] that $p\delta s$-irresoluteness and semi δs-irresoluteness are independent of each other.

Here, we obtain the following characterizations of $p\delta s$-irresolute functions.

Theorem 1. *For a function $f : (X, \tau) \to (Y, \upsilon)$, the following are equivalent:*

(1) *f is $p\delta s$-irresolute,*

(2) *For each $x \in X$ and each δ-semi open subset V of Y containing $f(x)$, there exists a preopen set U of X containing x such that $f(U) \subset V$,*

(3) *$f^{-1}(V) \subset Int(Cl(f^{-1}(V)))$ for every δ-semi open subset V of Y,*

(4) *$f^{-1}(F)$ is preclosed in X for every δ-semi closed subset F of Y,*

(5) *$Cl(Int(f^{-1}(B))) \subset f^{-1}(sCl_\delta(B))$ for every subset B of Y,*

(6) *$f(Cl(Int(A))) \subset sCl_\delta(f(A))$ for every subset A of X.*

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and V be any δ-semi open subset of Y containing $f(x)$. By Definition 2, $f^{-1}(V)$ is preopen in X and contains x. Set $U = f^{-1}(V)$. Then by (1), the set U is a preopen subset of X containing x and $f(U) \subset V$.

(2) $\Rightarrow$ (3). Let V be any δ-semi open subset of Y and $x \in f^{-1}(V)$. By (2), there exists a preopen set U of X containing x such that $f(U) \subset V$. Therefore, we obtain $x \in U \subset Int(Cl(U)) \subset Int(Cl(f^{-1}(V)))$ and hence $f^{-1}(V) \subset Int(Cl(f^{-1}(V)))$.

(3) $\Rightarrow$ (4). Let F be any δ-semi closed subset of Y. Set $V = Y - F$. Then the set V is δ-semi open in Y. By (3), we have $f^{-1}(V) \subset Int(Cl(f^{-1}(V)))$ and hence $f^{-1}(F) = X - f^{-1}(Y - F) = X - f^{-1}(V)$ is preclosed in X.

(4) $\Rightarrow$ (5). Let B be any subset of Y. Since the set $sCl_\delta(B)$ is δ-semi closed in Y, by (4) $f^{-1}(sCl_\delta(B))$ is preclosed in X and hence $Cl(Int(f^{-1}(sCl_\delta (B)))) \subset f^{-1}(sCl_\delta(B))$. Thus, we have $Cl(Int(f^{-1}(B))) \subset f^{-1}(sCl_\delta(B))$.

(5) $\Rightarrow$ (6). Let A be any subset of X. By (5), we obtain $Cl(Int(A)) \subset Cl(Int(f^{-1}(f(A)))) \subset f^{-1}(sCl_\delta(f(A)))$. Therefore, we have $f(Cl(Int(Cl(A)))) \subset sCl_\delta(f(A))$.

(6) $\Rightarrow$ (1). Let V be any δ-semi open subset of Y. Since $f^{-1}(Y-V) = X - f^{-1}(V)$ is a subset of X, by (6) we have $f(Cl(Int(f^{-1}(Y-V)))) \subset sCl_\delta(f(f^{-1}(Y-V))) \subset sCl_\delta(Y-V) = Y - slnt_\delta(V) = Y - V$ and hence $X - Int(Cl(f^{-1}(V)) = Cl(Int(X - f^{-1}(V))) = Cl(Int(f^{-1}(Y-V))) \subset f^{-1}(Y-V) = X - f^{-1}(V)$. Thus, we obtain $f^{-1}(V) \subset Int(Cl(f^{-1}(V)))$ and hence $f^{-1}(V)$ is preopen in X. ■

Lemma 1 ([7], [9]). *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of spaces and U_{λ_i} be a nonempty subset of X_{λ_i} for each $i = 1, 2, ..., n$. Then $U = \Pi_{\lambda \neq \lambda_i} X_\lambda \times \Pi_{i=1}^n U_{\lambda_i}$ is a nonempty preopen* [7] *(resp. δ-semi open* [9]*) subset of ΠX_λ if and only if U_{λ_i} is preopen (resp. δ-semi open) in X_{λ_i} for each $i = 1, 2, ..., n$.*

Theorem 2. *A function $f : X \to Y$ is $p\delta s$-irresolute if the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is $p\delta s$-irresolute.*

Proof. Let $x \in X$ and V be any δ-semi open subset of Y containing $f(x)$. Then, by Lemma 1, $X \times V$ is a δ-semi open set of $X \times Y$ containing $g(x)$. Since g is $p\delta s$-irresolute, there exists a preopen subset U of X containing x such that $g(U) \subset X \times V$ and hence $f(U) \subset V$. Thus f is $p\delta s$-irresolute. ■

Theorem 3. *If the product function $f : \Pi X_\lambda \to \Pi Y_\lambda$ is $p\delta s$-irresolute, then $f_\lambda : X_\lambda \to Y_\lambda$ is $p\delta s$-irresolute for each $\lambda \in \Lambda$.*

Proof. Let $\lambda_0 \in \Lambda$ be an arbitrary fixed index and V_{λ_0} be any δ-semi open subset of Y_{λ_0}. Then, by Lemma 1, $\Pi Y_\gamma \times V_{\lambda_0}$ is δ-semi open in ΠY_λ, where $\lambda_0 \neq \gamma \in \Lambda$. Since f is $p\delta s$-irresolute, $f^{-1}(\Pi Y_\gamma \times V_{\lambda_0}) = \Pi X_\gamma \times f_{\lambda_0}^{-1}(V_{\lambda_0})$ is preopen in ΠX_λ and hence $f_{\lambda_0}^{-1}(V_{\lambda_0})$ is preopen in X_{λ_0} by Lemma 1. This shows that f_{λ_0} is $p\delta s$-irresolute. ■

Lemma 2 ([13]). *If $A \in SO(X)$ and $B \in PO(X)$, then $A \cap B \in PO(A)$.*

Theorem 4. *If $f : (X, \tau) \to (Y, \upsilon)$ is $p\delta s$-irresolute and A is a semi open subset of X, then the restriction $f_{/A} : A \to Y$ is $p\delta s$-irresolute.*

Proof. Let V be any δ-semi open subset of Y. Since f is $p\delta s$-irresolute, $f^{-1}(V)$ is a preopen set in X. Since A is semi open in X, by Lemma 2, $(f_{/A})^{-1}(V) = A \cap f^{-1}(V)$ is preopen in A. Therefore $f_{/A}$ is $p\delta s$-irresolute. ■

Lemma 3 ([13]). *If $A \subset B \subset X$, $A \in PO(B)$ and $B \in PO(X)$, then $A \in PO(X)$.*

Theorem 5. *Let $f : (X, \tau) \to (Y, \upsilon)$ be a function and $\{A_\lambda : \lambda \in \Lambda\}$ be a cover of X by preopen sets of (X, τ). Then f is $p\delta s$-irresolute if $f_{/A_\lambda} : A_\lambda \to Y$ is $p\delta s$-irresolute for each $\lambda \in \Lambda$.*

Proof. Let V be any δ-semi open subset of Y. Since $f_{/A_\lambda}$ is $p\delta s$-irresolute, $(f_{/A_\lambda})^{-1}(V)$ is preopen in A_λ. Since A_λ is preopen in X, $(f_{/A_\lambda})^{-1}(V)$ is preopen in X for each $\lambda \in \Lambda$ by Lemma 3. Thus, we have $f^{-1}(V) = X \cap f^{-1}(V) = \cup\{A_\lambda \cap f^{-1}(V) : \lambda \in \Lambda\} = \cup\{(f_{/A_\lambda})^{-1}(V) : \lambda \in \Lambda\}$ is preopen in X because the union of preopen sets is a preopen set. Therefore, f is $p\delta s$-irresolute. ■

We recall that a function $f : (X, \tau) \to (Y, \upsilon)$ is said to be δ-irresolute ([17], [2]) (resp. preirresolute [21]) if $f^{-1}(V)$ is δ-semi open (resp. preopen) in X for every δ-semi open (resp. preopen) set V of Y.

Theorem 6. *Let $f : X \to Y$ be a function and $g : Y \to Z$ be a δ-irresolute function. If f is $p\delta s$-irresolute, then the composition $gof : X \to Z$ is $p\delta s$-irresolute.*

Proof. Let W be any δ-semi open subset of Z. Since g is δ-irresolute, $g^{-1}(W)$ is δ-semi open in Y. Since f is $p\delta s$-irresolute, $(gof)^{-1}(W) = f^{-1}(g^{-1}(W))$ is preopen in X and hence gof is $p\delta s$-irresolute. ■

Theorem 7. *Let $f : X \to Y$ be a preirresolute function and $g : Y \to Z$ be a function. If g is $p\delta s$-irresolute, then the composition $gof : X \to Z$ is $p\delta s$-irresolute.*

Proof. Let W be any δ-semi open subset of Z. Since g is $p\delta s$-irresolute, $g^{-1}(W)$ is preopen in Y. Since f is preirresolute, $(gof)^{-1}(W) = f^{-1}(g^{-1}(W))$ is preopen in X and hence gof is $p\delta s$-irresolute. ■

We recall that a topological space (X, τ) is said to be pre-T_2 [8] (resp. δ-semi T_2 [2]) if for any two distinct points x and y in X, there exist disjoint preopen (resp. δ-semi open) sets U and V in X such that $x \in U$ and $y \in V$.

Theorem 8. *If a function $f : (X, \tau) \to (Y, \upsilon)$ is a $p\delta s$-irresolute injection and a space Y is δ-semi T_2, then X is a pre-T_2 space.*

Proof.. Let x_1 and x_2 be any two distinct points of X. Then $f(x_1) \neq f(x_2)$. Since Y is δ-semi T_2, there exist disjoint δ-semi open sets V and W of Y contaning $f(x_1)$ and $f(x_2)$, respectively. Since f is $p\delta s$-irresolute, there exist preopen sets U and G of X containing x_1 and x_2, respectively such that $f(U) \subset V$ and $f(G) \subset W$. It follows that $U \cap G = \varnothing$. This means that X is a pre-T_2 space. ■

Theorem 9. *If a function $f:(X,\tau)\to(Y,\upsilon)$ is $p\delta s$-irresolute and a space Y is δ-semi T_2, then a set $A=\{(x,y):f(x)=f(y)\}$ is preclosed in $X\times X$.*

Proof. Suppose that $(x,y)\notin A$. Then $f(x)\neq f(y)$. Since Y is δ-semi T_2, there exist disjoint δ-semi open sets V and W of Y contaning $f(x)$ and $f(y)$, respectively. Since f is $p\delta s$-irresolute, there exist preopen sets U and G of X containing x and y, respectively such that $f(U)\subset V$ and $f(G)\subset W$. Set $B=U\times G$. Then, the set B is preopen in $X\times X$ such that $(x,y)\in B$ and $A\cap B=\varnothing$. This shows that $pCl(A)\subset A$ and hence the set A is preclosed in $X\times X$. ■

Definition 3. *For a function $f:(X,\tau)\to(Y,\upsilon)$, the graph $G(f)=\{(x,f(x)):x\in X\}$ is called $p\delta s$-closed if for each $(x,y)\in(X\times Y)-G(f)$, there exist $U\epsilon PO(X)$ containing x and $V\epsilon\delta SO(Y)$ containing y such that $(U\times V)\cap G(f)=\varnothing$.*

Theorem 10. *If a function $f:(X,\tau)\to(Y,\upsilon)$ is $p\delta s$-irresolute and Y is δ-semi T_2, then $G(f)$ is $p\delta s$-closed in $X\times Y$.*

Proof. Let $(x,y)\in(X\times Y)-G(f)$. This implies that $f(x)\neq y$. Since Y is δ-semi T_2, there exist disjoint δ-semi open sets V and W in Y containing $f(x)$ and y, respectively. Since f is $p\delta s$-irresolute, there exists a preopen set U of X containing x such that $f(U)\subset V$. Therefore, $f(U)\cap W=\varnothing$ and hence $(U\times W)\cap G(f)=\varnothing$. Thus, $G(f)$ is $p\delta s$-closed in $X\times Y$. ■

Theorem 11. *If a function $f:(X,\tau)\to(Y,\upsilon)$ is a $p\delta s$-irresolute injection with a $p\delta s$-closed graph, then X is a pre-T_2 space.*

Proof. Let x and y be any distinct points of X. Then $f(x)\neq f(y)$ and hence $(x,f(y))\in(X\times Y)-G(f)$. Since $G(f)$ is $p\delta s$-closed, there exist $U\epsilon PO(X)$ containing x and $V\epsilon\delta SO(Y)$ containing $f(y)$ such that $f(U)\cap V=\varnothing$. Since f is $p\delta s$-irresolute, there exists $G\epsilon PO(X)$ containing y such that $f(G)\subset V$. Thus we have $f(U)\cap f(G)=\varnothing$ and hence $U\cap G=\varnothing$. This shows that X is a pre-T_2 space. ■

Definition 4. *A subset A of a topological space (X,τ) is said to be*

(1) *strongly compact relative to* (X,τ) [15] *if for every cover $\{V_\alpha:\alpha\in\Delta\}$ of A by preopen sets of X, there exists a finite subset Δ_0 of Δ such that $A\subset\cup_{\alpha\in\Delta_0}V_\alpha$,*

(2) *s-closed relative to* (X,τ) [6] *if for every cover $\{V_\alpha:\alpha\in\Delta\}$ of A by semi open sets of X, there exists a finite subset Δ_0 of Δ such that $A\subset\cup_{\alpha\in\Delta_0}sCl(V_\alpha)$,*

(3) (X,τ) *is said to be strongly compact* [15] (*resp. s-closed* [6]) *if X is strongly compact* (*resp. s-closed*) *relative to* (X,τ).

Recall that the complement of a semi open set is said to be semi closed [5].

Lemma 4. *For a subset A of a topological space (X, τ), the following properties hold:*
(1) *If A is semi open, then $sCl(A)$ is semi closed and semi open,*
(2) *If A is semi open and semi closed, then A is δ-semi open,*
(3) *If A is semi open, then $sCl(A)$ is δ-semi open.*

Proof.(1) This follows from Proposition 2.2 of [6].
(2) This follows from Lemma 3.1 of [17].
(3) This is an immediate consequence of (1) and (2). ■

Theorem 12. *Let $f : (X, \tau) \to (Y, \upsilon)$ be a pδs-irresolute function. If K is strongly compact relative to X, then $f(K)$ is s-closed relative to Y.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be any cover of $f(K)$ by semi open sets of Y. Since V_α is semi open, by Lemma 4 $sCl(V_\alpha)$ is δ-semi open in Y. Since f is pδs-irresolute, $f^{-1}(sCl(V_\alpha)))$ is preopen in X for each $\alpha \in \Delta$. Since $f(K) \subset \cup_{\alpha\in\Delta} V_\alpha \subset \cup_{\alpha\in\Delta} sCl(V_\alpha)$, $K \subset f^{-1}(f(K)) \subset f^{-1}(\cup_{\alpha\in\Delta} sCl(V_\alpha)) = \cup_{\alpha\in\Delta} f^{-1}(sCl(V_\alpha))$. Since K is strongly compact relative to X, there exists a finite subset Δ_0 of Δ such that $K \subset \cup_{\alpha\in\Delta_0} f^{-1}(sCl(V_\alpha))$. Therefore, we have $f(K) \subset \cup_{\alpha\in\Delta_0} sCl(V_\alpha)$. This shows that $f(K)$ is s-closed relative to Y. ■

Corollary 1. *Let $f : (X, \tau) \to (Y, \upsilon)$ be a pδs-irresolute surjection. If (X, τ) is strongly compact, then (Y, υ) is s-closed.*

Definition 5. *A topological space (X, τ) is said to be semi-connected* [19] (*resp. preconnected* [20]) *if X cannot be expressed as the disjoint union of two nonempty semi open* (*resp. preopen*) *sets.*

Theorem 13. *Let $f : (X, \tau) \to (Y, \upsilon)$ be a pδs-irresolute surjection. If (X, τ) is preconnected, then (Y, υ) is semi-connected.*

Proof. Suppose that (Y, υ) is not semi-connected. Then there exist two nonempty semi open sets U and V such that $U \cap V = \varnothing$ and $U \cup V = Y$. Since U and V are semi open, by Lemma 4 $sCl(U)$ is semi open and $sCl(U) \cap V = \varnothing$. Hence $sCl(U) \cap sCl(V) = \varnothing$. Moreover, by Lemma 4, $sCl(U)$ and $sCl(V)$ are δ-semi open and $sCl(U) \cup sCl(V) = Y$. Therefore, $X = f^{-1}(sCl(U)) \cup f^{-1}(sCl(V))$ and $f^{-1}(sCl(U)) \cap f^{-1}(sCl(V)) = \varnothing$. Moreover, $f^{-1}(sCl(U))$ and $f^{-1}(sCl(V))$ are nonempty preopen sets. This shows that (X, τ) is not preconnected. This completes the proof. ■

References

[1] Beceren Y., Noiri T., On $\alpha\delta s$-irresolute functions, *An. Univ. Oradea Fasc. Mat.,* (to appear).

[2] Caldas M., Georgiou D.N., Jafari S., Noiri T., More on δ-semiopen sets, *Note Mat.,* 22(2)(2003/2004), 113-126.

[3] Caldas M., Jafari S., On semi δs-irresolute functions, *Fasc. Math.,* 58(2017), 47-55.

[4] Chae G.I., Noiri T., Lee D.W., On *na*-continuous functions, *Kyungpook Math. J.,* 26(1986), 73-79.

[5] Crossley S.G., Hildebrand S.K., Semi-closure, *Texas J. Sci.,* 22(1971), 99-112.

[6] Di Maio G., Noiri T., On s-closed spaces, *Indian J. Pure Appl. Math.,* 18(1987), 226-233.

[7] El-Deeb N., Hasanein I.A., Mashhour A.S., Noiri T., On p-regular spaces, *Bull. Math. Soc. Sci. Math. (N.S.),* 27(75)(1983), 311-315.

[8] Kar A., Bhattacharyya P., Some weak separation axioms, *Bull. Calcutta Math. Soc.,* 82(1990), 415-422.

[9] Lee B.Y., Son M.J., Park J.H., δ-semiopen sets and its applications, *Far East J. Math. Sci. (FJMS),* 3(2001), 745-759.

[10] Levine N., Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly,* 70(1963), 36-41.

[11] Maheshwari S.N., Thakur S.S., On α-irresolute mappings, *Tamkang J. Math.,* 11(1980), 209-214.

[12] Mashhour A.S., Abd El-Monsef M.E., El-Deeb S.N., On precontinuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt,* 53(1982), 47-53.

[13] Mashhour A.S., Hasanein I.A., El-Deeb S.N., A note on semi-continuity and precontinuity, *Indian J. Pure Appl. Math.,* 13(1982), 1119-1123.

[14] Mashhour A.S., Hasanein I.A., El-Deeb S.N., α-continuous and α-open mappings, *Acta Math. Hungar,* 41(1983), 213-218.

[15] Mashhour A.S., Abd El-Monsef M.E., Hasanein I.A., Noiri T., Strongly compact spaces, *Delta J. Sci.,* 8(1984), 30-46.

[16] Njåstad O., On some classes of nearly open sets, *Pacific J. Math.,* 15(1965), 961-970.

[17] Noiri T., Remarks on δ-semi open sets and δ-preopen sets, *Demonstratio Math.,* 36(2003), 1007-1020.

[18] Park J.H., Lee B.Y., Son M.J., On δ-semiopen sets in topological spaces, *J. Indian Acad. Math.,* 19(1997), 59-67.

[19] Pipitone V., Russo G., Spazi semiconnessi e spazi semiaperti, *Rend. Circ. Mat. Palermo (2),* 24(1975), 273-285.

[20] Popa V., Properties of H-almost continuous functions, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.),* 31(79)(1987), 163-168.

[21] Reilly I.L., Vamanamurthy M.K., On α-continuity in topological spaces, *Acta Math. Hungar,* 45 (1985), 27-32.

[22] Veličko N.V., H-closed topological spaces, *Amer. Math. Soc. Transl. (2),* 78 (1968), 103-118.

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