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OSCILLATIONS IN SYSTEMS OF IMPULSIVE NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

Abstract. In this paper, we consider systems of impulsive nonlinear neutral delay partial differential equations with distributed deviating arguments and sufficient conditions for the oscillation of the system under the Dirichlet boundary condition. The main results are illustrated by one example.

Key words: neutral partial differential equations, oscillation, system, impulse, damping term.

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1. Introduction

In recent years considerable attention has been given to impulsive differential equations which appear as a natural description of observed evolution phenomena of several real world problems. The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects, and have consistent real world applications, since many models present forces acting abruptly, almost instantly, and at different times. The theory of impulsive differential equations has its beginning in [19] by V. Mil'man et.al.

The first investigation on the oscillation theory of impulsive differential equations was published in 1989 [7]. The first paper on impulsive partial differential equations [5] was published in 1991. The authors of [5] have shown that impulsive partial differential equations provide a natural framework for mathematical modeling of population growth. In the last two decades there has been and still is to this day a strong interest in studying the oscillatory behavior of partial differential equations with or without impulse [18], [21]-[26], [28], [29]. However very little attention has been given to systems of partial differential equations [1], [4], [11], [13] - [17] and systems of impulsive partial differential equations [3, 6, 12].

For the basic background on the oscillation theory of differential equations, we refer to the monographs see [2, 9, 10, 28, 29] and the references there in. It seems that there has been no work published on the oscillation of systems of impulsive partial differential equations with continuous distributed deviating arguments.

In this paper, we consider the system of impulsive nonlinear neutral delay partial differential equations with distributed deviating arguments of the form

(E)

$$\begin{cases} \dfrac{\partial}{\partial t}\left[r(t)\dfrac{\partial}{\partial t}\left(u_i(x,t)+\int_a^b g(t,\xi)u_i(x,\tau(t,\xi))d\eta(\xi)\right)\right] \\ \quad + p(t)\dfrac{\partial}{\partial t}\left(u_i(x,t)+\int_a^b g(t,\xi)u_i(x,\tau(t,\xi))d\eta(\xi)\right) \\ \quad + \sum\limits_{n=1}^{m}\sum\limits_{j=1}^{d}\int_a^b q_{inj}(x,t,\xi)f_{ij}(u_n(x,\sigma_j(t,\xi)))d\eta(\xi) = a_i(t)\Delta u_i(x,t) \\ \quad + \sum\limits_{n=1}^{m}\sum\limits_{h=1}^{l} a_{inh}(t)\Delta u_n(x,\rho_h(t)), \text{ for } t\neq t_k \text{ and } (x,t)\in\Omega\times\mathbb{R}_+ \\ u_i(x,t_k^+) = \alpha_{k_i}\left(x,u_i(x,t_k),t_k\right), \\ \dfrac{\partial u_i(x,t_k^+)}{\partial t} = \beta_{k_i}\left(x,\dfrac{\partial u_i(x,t_k)}{\partial t},t_k\right), \text{ for } k\geqslant 1, \end{cases}$$

for $i=1,2,\ldots,m$, where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in the Euclidean space $\mathbb{R}^N$, supplemented by the following Dirichlet boundary condition

$$\textbf{(B)} \qquad u_i(x,t)=0 \text{ for } (x,t)\in\partial\Omega\times\mathbb{R}_+ \quad\text{and}\quad i=1,2,\ldots,m.$$

We denote $\mathbb{R}_+=[0,+\infty)$, the sequence $\{t_k\}$ is a fixed strictly increasing sequence of positive real numbers with $t_k\to\infty$ as $k\to\infty$, $a<b$ and $\eta\colon[a,b]\to\mathbb{R}$ is a nondecreasing function.[1] Before presenting the hypotheses and detailed explanation for the terms of equation (E), we need the following definition.

Definition 1. *If $U\subset\mathbb{R}^p$ for some $p\geqslant 1$ and $J\subset\mathbb{R}$ we define the set $PC(U\times\mathbb{R}_+,J)$ as the set of functions $U\times\mathbb{R}_+\ni(x,t)\mapsto f(x,t)\in J$ which are continuous in the variable x and piecewise continuous in t, with discontinuities of the first kind in $t=t_k$ and left continuous at $t=t_k$.*

Also, we consider $PC(\mathbb{R}_+,J)$ as the set of functions $\mathbb{R}_+\ni t\mapsto f(t)\in J$ which are piecewise continuous, with discontinuities of the first kind in $t=t_k$ and left continuous at $t=t_k$.

[1] The integral in (E) is a Riemann-Stieltjes integral.

Now we present a set of conditions that will use throughout the paper.

($\mathbf{H_1}$) $p(t) \in C(\mathbb{R}_+, \mathbb{R})$, $r(t) \in C^1(\mathbb{R}_+, (0, +\infty))$ with $r'(t) \geqslant 0$, $\displaystyle\int_{t_0}^{+\infty} \frac{1}{R(s)} ds = +\infty$, where $R(t) = \exp\left(\displaystyle\int_{t_0}^{t} \frac{r'(s) + p(s)}{r(s)} ds\right)$, and $g \in C^2(\mathbb{R}_+ \times [a, b], \mathbb{R}_+)$.

($\mathbf{H_2}$) $a_i, a_{inh} \in PC(\mathbb{R}_+, \mathbb{R}_+)$ and for each $h = 1, 2, \ldots, l$ we have

$$A_h(t) = \min_{1 \leqslant i \leqslant m} \left\{ a_{iih}(t) - \sum_{n=1,\ n \neq i}^{m} |a_{nih}(t)| \right\} > 0.$$

($\mathbf{H_3}$) $\tau,\ \sigma_j \in C(\mathbb{R}_+ \times [a, b], \mathbb{R})$, $\tau(t, \xi) \leqslant t$ and $\sigma_j(t, \xi) \leqslant t$ for $\xi \in [a, b]$, $\tau(t, \xi)$ and $\sigma_j(t, \xi)$ are nondecreasing with respect to t and with respect to ξ. Moreover

$$\lim_{t \to +\infty} \tau(t, a) = \sigma_j(t, a) = +\infty.$$

Also $\rho_h \in C(\mathbb{R}_+, \mathbb{R})$, $\rho_h(t) \leqslant t$ and $\lim\limits_{t \to +\infty} \rho_h(t) = +\infty$ for $h = 1, 2, \ldots, l$.

($\mathbf{H_4}$) There exist functions $\theta_j \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfying $\theta_j(t) \leqslant \sigma_j(t, a)$, with $\theta_j'(t) > 0$ and

$$\lim_{t \to +\infty} \theta_j(t) = +\infty \text{ for } j = 1, 2, \cdots, d.$$

($\mathbf{H_5}$) $q_{inj} \in C\left(\bar{\Omega} \times \mathbb{R}_+ \times [a, b], \mathbb{R}_+\right)$, $q_{iij}(t, \xi) = \min\limits_{x \in \bar{\Omega}} q_{iij}(x, t, \xi)$, $\bar{q}_{inj}(t, \xi) = \max\limits_{x \in \bar{\Omega}} |q_{inj}(x, t, \xi)|$,
$Q_j(t, \xi) = \min\limits_{1 \leqslant i \leqslant m} \left\{ q_{iij}(t, \xi) - \sum\limits_{n=1,\ n \neq i}^{m} |\bar{q}_{nij}(t, \xi)| \right\} \geqslant 0$
for $i, n = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, d$. $f_{ij} \in C(\mathbb{R}, \mathbb{R})$ is convex in $\mathbb{R}_+$ and $\dfrac{f_{ij}(s)}{s} \geqslant M > 0$, for $s \neq 0$; $i, n = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, d$.

($\mathbf{H_6}$) $u_i,\ \dfrac{\partial u_i}{\partial t} \in PC(\overline{\Omega} \times \mathbb{R}_+, \mathbb{R})$.

($\mathbf{H_7}$) $\alpha_{k_i},\ \beta_{k_i} \in PC(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}_+, \mathbb{R})$ for $k = 1, 2, \cdots,\ i = 1, 2, \cdots, m$, and there exist positive constants $a_{k_i}, a_{k_i}^*, b_{k_i}, b_{k_i}^*$ such that for $i = 1, 2, \cdots, m$ and $k \geqslant 1$ we have $b_{k_i} \leqslant a_{k_i}^*$ and

$$a_{k_i}^* \leqslant \frac{\alpha_{k_i}\left(x, u_i(x, t_k), t_k\right)}{u_i(x, t_k)} \leqslant a_{k_i}, \qquad b_{k_i}^* \leqslant \frac{\beta_{k_i}\left(x, \frac{\partial u_i(x, t_k)}{\partial t}, t_k\right)}{\frac{\partial u_i(x, t_k)}{\partial t}} \leqslant b_{k_i}.$$

In the next section, we discuss the oscillation of problem (E)-(B) in detail, and in Section 3, we present one example to illustrate our main result.

3. Oscillations of problem (E)-(B)

In this section, we establish sufficient conditions for the oscillation of all solutions of problem (E)-(B). We begin with the definition of a solution of problem (E)-(B).

Definition 2 ([31]). *By a* **solution** *of* (E)-(B) *we mean a vector function* $\mathbf{u} = (u_1, \cdots, u_m)$ *such that* $u_i \in C^2(\overline{\Omega} \times [\omega_1, +\infty), \mathbb{R}) \cap C^1(\overline{\Omega} \times [\omega_2, +\infty), \mathbb{R}) \cap C(\overline{\Omega} \times [\omega_3, +\infty), \mathbb{R})$ *for each* $i = 1, \dots, m$ *and* u_i *satisfies* (E)-(B) *in* $\Omega \times \mathbb{R}_+$ *for each* $i = 1, \dots, m$, *where*

$$\omega_1 = \min\left\{0, \min_{1 \leqslant h \leqslant l}\left[\inf_{t \geqslant 0} \rho_h(t)\right]\right\}, \quad \omega_2 = \min\left\{0, \inf_{t \geqslant 0} \tau(t,a)\right\}$$
$$\text{and} \quad \omega_3 = \min\left\{0, \min_{1 \leqslant j \leqslant d}\left[\inf_{t \geqslant 0} \sigma_j(t,a)\right]\right\}.$$

Now with this definition of solution, we can precisely define what we mean by *oscillation*.

Definition 3 ([31]). *A nontrivial component* $u_i(x,t)$ *of a solution* $\mathbf{u}$ *is said to be* **oscillatory** *in* $\Omega \times [\delta_0, +\infty)$ *if there is a point* $(x_0, t_0) \in \Omega \times [\delta, +\infty)$ *such that* $u_i(x_0, t_0) = 0$, *for each* $\delta > \delta_0$.

Definition 4 ([31]). *A solution* $\mathbf{u}$ *is said to be* **oscillatory** *in* $\Omega \times [\delta_0, +\infty)$ *if at least one of its nontrivial component is oscillatory in* $\Omega \times [\delta_0, +\infty)$. *Otherwise, the vector solution* $u_i(x,t)$ *is said to be* **nonoscillatory** *in* $\Omega \times [\delta_0, +\infty)$.

Definition 5 ([31]). *A solution* $\mathbf{u}$ *is said to be* **strongly oscillatory** *in* $\Omega \times [\delta_0, +\infty)$ *if each of its nontrivial component is oscillatory in* $\Omega \times [\delta_0, +\infty)$.

Next, we state two lemmas that will help us establish our results.

Lemma 1 ([8]). *If* x *and* y *are nonnegative, then*

$$x^\lambda - \lambda x y^{\lambda-1} + (\lambda - 1) y^\lambda \geqslant 0, \quad \text{if} \quad \lambda > 1$$
$$x^\lambda - \lambda x y^{\lambda-1} - (1 - \lambda) y^\lambda \leqslant 0, \quad \text{if} \quad 0 < \lambda < 1,$$

while, in both cases, equality holds if and only if $x = y$.

Proposition 1 ([27]). *Consider the eigenvalue problem*

$$\begin{cases} \Delta w(x) + \lambda w(x) = 0 & \text{in } \Omega \\ w(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

Then its smallest eigenvalue λ_0 *is positive.*

Now we can begin our study of oscillations for the solutions of (E)-(B) in detail. We start with the following result, recalling that a function $Z\colon \mathbb{R}_+ \to \mathbb{R}$ is said to be **eventually positive** if there exists $\delta > 0$ such that $Z(t) > 0$ for all $t \geqslant \delta$.

Theorem 1. *If the functional impulsive differential inequality*

$$(2)\qquad \begin{cases} (r(t)Z'(t))' + p(t)Z'(t) + \sum\limits_{j=1}^{d} \int_a^b MQ_j(t,\xi)[1 - \int_a^b g(\sigma_j(t,\xi),\xi)] \\ \qquad\qquad\qquad \times Z(\theta_j(t))d\eta(\xi) \leqslant 0, \quad t \neq t_k, \\ a^*_{k_i} \leqslant \dfrac{Z(t_k^+)}{Z(t_k)} \leqslant a_{k_i}, \qquad b^*_{k_i} \leqslant \dfrac{Z'(t_k^+)}{Z'(t_k)} \leqslant b_{k_i} \\ \qquad\qquad\qquad k = 1,2,\cdots, \ i = 1,2,\cdots,m, \end{cases}$$

has no eventually positive solution, then every solution of the boundary value problem (E)-(B) *is oscillatory in* $\Omega \times \mathbb{R}_+$.

Proof. Suppose to the contrary that there is a non-oscillatory solution $\mathbf{u}$ of (E)-(B). We may assume that $|u_i(x,t)| > 0$ for $t \geqslant t_0$ for some $t_0 \in \mathbb{R}$ and $i = 1,2,\ldots m$. For $t \geqslant t_0$, let $\delta_i = \operatorname{sgn}(u_i(x,t))$ and $z_i(x,t) = \delta_i u_i(x,t)$. Then $z_i(x,t) > 0$, for $(x,t) \in \Omega \times [t_0,+\infty)$ and $i = 1,2,\cdots m$. From (H_3) there exists $t^* > t_0$ such that $\tau(t,\xi) \geqslant t_0$, $\sigma_j(t,\xi) \geqslant t_0$ for $(t,\xi) \in [t^*,+\infty) \times [a,b]$, and $\rho_h(t) \geqslant t_0$ for $t \geqslant t_0$, then

$$z_i(x,\tau(t,\xi)) > 0, \quad z_i(x,\sigma_j(t,\xi)) > 0 \quad \text{and} \quad z_i(x,\rho_h(t)) > 0,$$

for $x \in \Omega$, $t \in [t^*,+\infty)$, $\xi \in [a,b]$, $j = 1,\ldots,d$ and $h = 1,\ldots,l$.

For $t \geqslant t_0$ and $t \neq t_k$ for $k = 1,2,\cdots$, we multiply both sides of the equation in (E) by $\delta_i\Phi(x)$ and integrate with respect to x over the domain Ω to attain

$$\begin{aligned}(3)\qquad \frac{d}{dt}\Bigg[r(t)\frac{d}{dt}\Bigg(&\int_\Omega \delta_i u_i(x,t)\Phi(x)dx \\ &+ \int_\Omega\int_a^b \delta_i g(t,\xi)u_i(x,\tau(t,\xi))\Phi(x)d\eta(\xi)dx\Bigg)\Bigg] \\ &+ p(t)\frac{d}{dt}\Bigg(\int_\Omega \delta_i u_i(x,t)\Phi(x)dx \\ &+ \int_\Omega\int_a^b \delta_i g(t,\xi)u_i(x,\tau(t,\xi))\Phi(x)d\eta(\xi)dx\Bigg) \\ &+ \sum_{n=1}^{m}\sum_{j=1}^{d}\int_\Omega\int_a^b \delta_i q_{inj}(x,t,\xi)f_{ij}\left(u_n(x,\sigma_j(t,\xi))\right)\Phi(x)d\eta(\xi)dx\end{aligned}$$

$$
\begin{aligned}
&= a_i(t)\int_\Omega \delta_i \Delta u_i(x,t)\Phi(x)dx\\
&\quad + \sum_{n=1}^{m}\sum_{h=1}^{l}\int_\Omega a_{inh}(t)\delta_i\Delta u_n(x,\rho_h(t))\Phi(x)dx,
\end{aligned}
$$

for $i = 1, 2, \dots, m$. It is easy to see that

$$
\begin{aligned}
&\int_\Omega\int_a^b \delta_i g(t,\xi)u_i(x,\tau(t,\xi))\Phi(x)d\eta(\xi)dx\\
&\qquad = \int_a^b\int_\Omega \delta_i g(t,\xi)u_i(x,\tau(t,\xi))\Phi(x)dxd\eta(\xi),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega\int_a^b \delta_i q_{inj}(x,t,\xi)f_{ij}\left(u_n(x,\sigma_j(t,\xi))\right)\Phi(x)d\eta(\xi)dx\\
&\qquad = \int_a^b\int_\Omega \delta_i q_{inj}(x,t,\xi)f_{ij}\left(u_n(x,\sigma_j(t,\xi))\right)\Phi(x)dxd\eta(\xi).
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
(4)\quad &\frac{d}{dt}\left[r(t)\frac{d}{dt}\left(\int_\Omega z_i(x,t)\Phi(x)dx\right.\right.\\
&\qquad \left.\left. + \int_a^b\int_\Omega g(t,\xi)z_i(x,\tau(t,\xi))\Phi(x)dxd\eta(\xi)\right)\right]\\
&\qquad + p(t)\frac{d}{dt}\left(\int_\Omega z_i(x,t)\Phi(x)dx\right.\\
&\qquad \left. + \int_a^b\int_\Omega g(t,\xi)z_i(x,\tau(t,\xi))\Phi(x)dxd\eta(\xi)\right)\\
&\qquad + \sum_{j=1}^{d}\Big[\int_a^b\int_\Omega q_{iij}(x,t,\xi)f_{ii}\left(z_i(x,\sigma_j(t,\xi))\right)\Phi(x)dxd\eta(\xi)\\
&\qquad + \sum_{n=1,\ n\neq i}^{m}\delta_i\delta_n\int_a^b\int_\Omega q_{inj}(x,t,\xi)f_{in}\left(z_n(x,\sigma_j(t,\xi))\right)\Phi(x)dxd\eta(\xi)\Big]\\
&= a_i(t)\int_\Omega \Delta z_i(x,t)\Phi(x)dx + \sum_{h=1}^{l}\Big[\int_\Omega a_{iih}(t)\Delta z_i(x,\rho_h(t))\Phi(x)dx\\
&\qquad + \sum_{n=1,\ n\neq i}^{m}\int_\Omega a_{inh}(t)\delta_i\delta_n\Delta z_n(x,\rho_h(t))\Phi(x)dx\Big].
\end{aligned}
$$

From Green's formula and boundary condition (B), it follows that

$$\begin{aligned}(5)\qquad \int_\Omega \Delta z_i(x,t)\Phi(x)dx &= \int_\Omega z_i(x,t)\Delta\Phi(x)dx \\ &\quad + \int_{\partial\Omega}\left[\Phi(x)\frac{\partial z_i(x,t)}{\partial\gamma} - z_i(x,t)\frac{\partial\Phi(x)}{\partial\gamma}\right]dS \\ &= -\lambda_0\int_\Omega z_i(x,t)\Phi(x)dx \leqslant 0,\end{aligned}$$

and

$$\begin{aligned}(6)\qquad \int_\Omega \Delta z_n(x,\rho_h(t))\Phi(x)dx &= \int_\Omega z_n(x,\rho_h(t))\Delta\Phi(x)dx \\ &\quad + \int_{\partial\Omega}\left[\Phi(x)\frac{\partial z_n(x,\rho_h(t))}{\partial\gamma} - z_i(x,\rho_h(t))\frac{\partial\Phi(x)}{\partial\gamma}\right]dS \\ &= -\lambda_0\int_\Omega z_n(x,\rho_h(t))\Phi(x)dx,\end{aligned}$$

where $h = 1, 2, \ldots, l$ and dS is the surface element on $\partial\Omega$. Furthermore with Jensen's inequality and (H_5) we obtain

$$\begin{aligned}(7)\qquad &\int_a^b\int_\Omega q_{iij}(x,t,\xi)f_{ii}\left(z_i(x,\sigma_j(t,\xi))\right)\Phi(x)dxd\eta(\xi) \\ &\qquad \geqslant \int_a^b\int_\Omega q_{iij}(x,t,\xi)Mz_i(x,\sigma_j(t,\xi))\Phi(x)dxd\eta(\xi)\end{aligned}$$

and

$$\begin{aligned}(8)\qquad &\int_a^b\int_\Omega q_{inj}(x,t,\xi)f_{in}\left(z_n(x,\sigma_j(t,\xi))\right)\Phi(x)dxd\eta(\xi) \\ &\qquad \geqslant \int_a^b\int_\Omega q_{inj}(x,t,\xi)Mz_n(x,\sigma_j(t,\xi))\Phi(x)dxd\eta(\xi).\end{aligned}$$

Combining (4)-(8) we get

$$\begin{aligned}&\frac{d}{dt}\left[r(t)\frac{d}{dt}\left(\int_\Omega z_i(x,t)\Phi(x)dx + \int_a^b\int_\Omega g(t,\xi)z_i(x,\tau(t,\xi))\Phi(x)dxd\eta(\xi)\right)\right] \\ &\quad + p(t)\frac{d}{dt}\left(\int_\Omega z_i(x,t)\Phi(x)dx + \int_a^b\int_\Omega g(t,\xi)z_i(x,\tau(t,\xi))\Phi(x)dxd\eta(\xi)\right) \\ &\quad + \sum_{j=1}^{d}\Big[\int_a^b\int_\Omega q_{iij}(t,\xi)Mz_i(x,\sigma_j(t,\xi))\Phi(x)dxd\eta(\xi) \\ &\quad - \sum_{n=1,\ n\neq i}^{m}\int_a^b\int_\Omega \bar{q}_{inj}(t,\xi)Mz_n(x,\sigma_j(t,\xi))\Phi(x)dxd\eta(\xi)\Big]\end{aligned}$$

$$\leqslant \sum_{h=1}^{l} \Big[-\lambda_0 \int_\Omega a_{iih}(t) z_i(x,\rho_h(t))\Phi(x)dx + \lambda_0 \sum_{n=1,\ n\neq i}^{m} \int_\Omega |a_{inh}(t)|\, z_n(x,\rho_h(t))\Phi(x)dx \Big].$$

Setting $v_i(t) = \int_\Omega z_i(x,t)\Phi(x)dx$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big[r(t)\frac{d}{dt}\Big(v_i(t) + \int_a^b g(t,\xi) v_i(\tau(t,\xi)) d\eta(\xi)\Big)\Big] \\
&\quad + p(t)\frac{d}{dt}\Big(v_i(t) + \int_a^b g(t,\xi) v_i(\tau(t,\xi)) d\eta(\xi)\Big) \\
&\quad + \sum_{j=1}^{d} \Big[\int_a^b M q_{iij}(t,\xi) v_i(\sigma_j(t,\xi)) d\eta(\xi) \\
&\quad - \sum_{n=1,\ n\neq i}^{m} \int_a^b M \bar{q}_{inj}(t,\xi) v_n(\sigma_j(t,\xi)) d\eta(\xi)\Big] \\
&\leqslant \sum_{h=1}^{l} \Big[-\lambda_0 a_{iih}(t) v_i(\rho_h(t)) + \lambda_0 \sum_{n=1,\ n\neq i}^{m} |a_{inh}(t)|\, v_n(\rho_h(t))\Big].
\end{aligned} \tag{9}$$

Let $V(t) = \sum_{i=1}^{m} v_i(t)$. It follows from (9) that

$$\begin{aligned}
&\frac{d}{dt}\Big[r(t)\frac{d}{dt}\Big(V(t) + \int_a^b g(t,\xi) V(\tau(t,\xi)) d\eta(\xi)\Big)\Big] \\
&\quad + p(t)\frac{d}{dt}\Big(V(t) + \int_a^b V(\tau(t,\xi)) d\eta(\xi)\Big) \\
&\quad + \sum_{j=1}^{d} M\Big[\sum_{i=1}^{m} \Big(\int_a^b q_{iij}(t,\xi) v_i(\sigma_j(t,\xi)) d\eta(\xi) \\
&\quad - \sum_{n=1,\ n\neq i}^{m} \int_a^b \bar{q}_{inj}(t,\xi) v_n(\sigma_j(t,\xi)) d\eta(\xi)\Big)\Big] \\
&\quad + \sum_{h=1}^{l} \lambda_0 \Big[\sum_{i=1}^{m} \Big(\int_a^b a_{iih}(t) v_i(\rho_h(t)) \\
&\quad - \sum_{n=1,\ n\neq i}^{m} \int_a^b |a_{inh}(t)|\, v_n(\rho_h(t))\Big)\Big] \leqslant 0.
\end{aligned} \tag{10}$$

Noting that

$$\begin{aligned}
&\sum_{i=1}^{m}\int_a^b \Big(q_{iij}(t,\xi)v_i(\sigma_j(t,\xi)) - \sum_{n=1,\ n\neq i}^{m} \bar{q}_{inj}(t,\xi)v_n(\sigma_j(t,\xi))\Big)d\eta(\xi)\\
&\quad= \int_a^b \Big(q_{11j}(t,\xi)v_1(\sigma_j(t,\xi)) - \sum_{n=1,\ n\neq 1}^{m} \bar{q}_{1nj}(t,\xi)v_n(\sigma_j(t,\xi))\Big)d\eta(\xi)\\
&\qquad+ \int_a^b \Big(q_{22j}(t,\xi)v_2(\sigma_j(t,\xi)) - \sum_{n=1,\ n\neq 2}^{m} \bar{q}_{2nj}(t,\xi)v_n(\sigma_j(t,\xi))\Big)d\eta(\xi)\\
&\qquad+\cdots+ \int_a^b \Big(q_{mmj}(t,\xi)v_m(\sigma_j(t,\xi))\\
&\qquad- \sum_{n=1,\ n\neq m}^{m} \bar{q}_{mnj}(t,\xi)v_n(\sigma_j(t,\xi))\Big)d\eta(\xi)\\
&\quad= \int_a^b \Big(q_{11j}(t,\xi) - \sum_{n=1,\ n\neq 1}^{m} \bar{q}_{n1j}(t,\xi)\Big)v_1(\sigma_j(t,\xi))d\eta(\xi)\\
&\qquad+ \int_a^b \Big(q_{22j}(t,\xi) - \sum_{n=1,\ n\neq 2}^{m} \bar{q}_{n2j}(t,\xi)\Big)v_2(\sigma_j(t,\xi))d\eta(\xi)\\
&\qquad+\cdots+ \int_a^b \Big(q_{mmj}(t,\xi) - \sum_{n=1,\ n\neq m}^{m} \bar{q}_{nmj}(t,\xi)\Big)v_m(\sigma_j(t,\xi))d\eta(\xi)\\
&\quad\geqslant \int_a^b \min_{1\leqslant i\leqslant m}\Big(q_{iij}(t,\xi) - \sum_{n=1,\ n\neq i}^{m} \bar{q}_{nij}(t,\xi)\Big)\sum_{i=1}^{m} v_i(\sigma_j(t,\xi))d\eta(\xi)\\
&\quad= \int_a^b Q_j(t,\xi)V(\sigma_j(t,\xi))d\eta(\xi),
\end{aligned}$$

and similarly

$$\begin{aligned}
&\sum_{i=1}^{m}\Big(a_{iih}(t)v_i(\rho_h(t)) - \sum_{n=1,\ n\neq i}^{m} |a_{inh}(t)|\, v_n(\rho_h(t))\Big)\\
&\quad\geqslant \min_{1\leqslant i\leqslant m}\Big(a_{iih}(t) - \sum_{n=1,\ n\neq i}^{m} |a_{nih}(t)|\Big)\sum_{i=1}^{m} v_i(\rho_h(t)) = A_h(t)V(\rho_h(t)).
\end{aligned}$$

From (10), we have

$$\begin{aligned}
&\frac{d}{dt}\left[r(t)\frac{d}{dt}\left(V(t)+\int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)\right)\right]\\
&\quad+ p(t)\frac{d}{dt}\left(V(t)+\int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)\right)\\
&\quad+ \sum_{j=1}^{d} M\int_a^b Q_j(t,\xi)V(\sigma_j(t,\xi))d\eta(\xi) + \sum_{h=1}^{l}\lambda_0 A_h(t)V(\rho_h(t)) \leqslant 0.
\end{aligned}$$

It is easy to see that $V(\rho_h(t)) = \sum_{i=1}^{m} v_i(\rho_h(t)) \geqslant 0$, and therefore

$$\begin{aligned}\frac{d}{dt}\left[r(t)\frac{d}{dt}\left(V(t)+\int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)\right)\right]\\ + p(t)\frac{d}{dt}\left(V(t)+\int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)\right)\\ + \sum_{j=1}^{d} M\int_a^b Q_j(t,\xi)V(\sigma_j(t,\xi))d\eta(\xi) \leqslant 0.\end{aligned}$$

Setting $Z(t) = V(t) + \int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)$, we have

$$\big(r(t)Z'(t)\big)' + p(t)Z'(t) + \sum_{j=1}^{d} M\int_a^b Q_j(t,\xi)V(\sigma_j(t,\xi))d\eta(\xi) \leqslant 0. \tag{11}$$

Clearly $Z(t) > 0$ for $t \geqslant t^*$. Next, we show that $Z'(t) > 0$ for $t \geqslant t^*$. In fact assume that there exists $T \geqslant t^*$ such that $Z'(T) \leqslant 0$. Then we have

$$r(t)Z''(t) + \big(r'(t) + p(t)\big)\, Z'(t) \leq 0. \tag{12}$$

From (H_1), it follows that $R'(t) = R(t)\left(\frac{r'(t)+p(t)}{r(t)}\right)$, $R(t) > 0$ and $R'(t) \geqslant 0$ for $t \geqslant t^*$. Multiplying both sides of (12) by $\frac{R(t)}{r(t)}$, we obtain

$$R(t)Z''(t) + R'(t)Z'(t) = \big(R(t)Z'(t)\big)' \leqslant 0. \tag{13}$$

From (13) we have $R(t)Z'(t) \leq R(T)Z'(T) \leqslant 0$, $t \geqslant T$. Thus

$$Z(t) \leqslant Z(T) + R(T)Z'(T)\int_T^t \frac{ds}{R(s)} \text{ for } t \geqslant T.$$

Again from (H_1) we have $\lim_{t\to\infty} Z(t) = -\infty$ which contradicts $Z(t) > 0$ for $t \geqslant t^*$. Hence $Z'(t) > 0$ and since $\tau(t,\xi) \leqslant t$, we have

$$\begin{aligned} V(t) &= Z(t) - \int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)\\ &\geqslant Z(t) - \int_a^b g(t,\xi)Z(\tau(t,\xi))d\eta(\xi)\\ &\geqslant Z(t)\left(1 - \int_a^b g(t,\xi)d\eta(\xi)\right)\end{aligned}$$

and

$$V(\sigma_j(t,\xi)) \geqslant Z(\sigma_j(t,\xi))\left(1 - \int_a^b g(\sigma_j(t,\xi),\xi)d\eta(\xi)\right).$$

Therefore from (11), we have

$$\big(r(t)Z'(t)\big)' + p(t)Z'(t) + \sum_{j=1}^{d}\int_a^b MQ_j(t,\xi)\left(1-\int_a^b g(\sigma_j(t,\xi),\xi)d\eta(\xi)\right)Z(\sigma_j(t,\xi))d\eta(\xi) \leqslant 0.$$

From (H_3) and (H_4), we have

$$Z(\sigma_j(t,\xi)) \geqslant Z(\sigma_j(t,a)) > 0, \quad \xi\in[a,b] \quad \text{and} \quad \theta_j(t)\leqslant \sigma_j(t,a)\leqslant t,$$

thus $Z(\theta_j(t)) \leqslant Z(\sigma_j(t,a))$ and therefore

$$\big(r(t)Z'(t)\big)' + p(t)Z'(t) + \sum_{j=1}^{d} M\int_a^b Q_j(t,\xi)\left(1-\int_a^b g(\sigma_j(t,\xi),\xi)d\eta(\xi)\right)Z(\theta_j(t))d\eta(\xi) \leqslant 0.$$

For $t\geqslant t_0$, $t=t_k$, $k=1,2,\cdots$, multiplying both sides of equation (E) by $\delta_i\Phi(x)$, integrating with respect to x over the domain Ω, and using (H_7), we obtain

$$a^*_{k_i} \leqslant \frac{u_i(x,t_k^+)}{u_i(x,t_k)} \leqslant a_{k_i}, \qquad b^*_{k_i} \leqslant \frac{\frac{\partial u_i(x,t_k^+)}{\partial t}}{\frac{\partial u_i(x,t_k)}{\partial t}} \leqslant b_{k_i}.$$

Since $z_i(t) = \delta_i\int_\Omega u_i(x,t_k)\Phi(x)dx$, we have

$$a^*_{k_i} \leqslant \frac{V(t_k^+)}{V(t_k)} \leqslant a_{k_i}, \qquad b^*_{k_i} \leqslant \frac{V'(t_k^+)}{V'(t_k)} \leqslant b_{k_i},$$

and since $Z(t) = V(t) + \int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)$, we obtain

$$a^*_{k_i} \leqslant \frac{Z(t_k^+)}{Z(t_k)} \leqslant a_{k_i}, \qquad b^*_{k_i} \leqslant \frac{Z'(t_k^+)}{Z'(t_k)} \leqslant b_{k_i}.$$

Therefore $Z(t)$ is an eventually positive solution of (2), which contradicts the hypothesis and completes the pf. ■

Theorem 2. *Assume that if there exists $j_0\in\{1,2,\cdots,d\}$ and a function $\varphi(t)\in C^1(\mathbb{R}_+,(0,+\infty))$ which is nondecreasing with respect to t and such that*

$$\int_{t_0}^{+\infty}\prod_{t_0\leqslant t_k<s}\left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1}\left[\varphi(s)B(s)-\frac{A^2(s)}{4C(s)}\right]ds = +\infty, \tag{14}$$

where

$$A(t) = \frac{\varphi'(t)}{\varphi(t)} - \frac{p(t)}{r(t)},$$

$$B(t) = M\int_a^b Q_{j_0}(t,\xi)\Big(1-\int_a^b g(\sigma_{j_0}(t,\xi),\xi)d\eta(\xi)\Big)d\eta(\xi)$$

and $E(t) = \dfrac{\theta'_{j_0}(t)}{\varphi(\theta_{j_0}(t))r(\theta_{j_0}(t))}$.

Then every solution of the boundary value problem (E)-(B) *is oscillatory in* $\Omega \times \mathbb{R}_+$.

Proof. We prove that the inequality (2) has no eventually positive solution if the conditions of Theorem 2 hold. Suppose that $Z(t)$ is an eventually positive solution of inequality (2). Then there exists a number $t^* \geqslant t_0$ such that $Z(\theta_{j_0}(t)) > 0,\ j = 1, 2, \cdots, d$ for $t \geqslant t^*$. Thus we have

$$(15)\quad \big(r(t)Z'(t)\big)' + p(t)Z'(t) + M\int_a^b Q_{j_0}(t,\xi)\left(1-\int_a^b g(\sigma_{j_0}(t,\xi),\xi)d\eta(\xi)\right)Z(\theta_{j_0}(t))d\eta(\xi) \leqslant 0.$$

Define

$$W(t) := \varphi(t)\frac{r(t)Z'(t)}{Z(\theta_{j_0}(t))},$$

then $W(t) \geqslant 0$ and

$$\begin{aligned} W'(t) \leqslant{}& \left(\frac{\varphi'(t)}{\varphi(t)} - \frac{p(t)}{r(t)}\right)W(t) \\ &- M\varphi(t)\int_a^b Q_{j_0}(t,\xi)\left(1-\int_a^b g(\sigma_{j_0}(t,\xi),\xi)d\eta(\xi)\right)d\eta(\xi) \\ &- \frac{W^2(t)}{\varphi(\theta_{j_0}(t))}\frac{\theta'_{j_0}(t)}{r(\theta_{j_0}(t))}. \end{aligned}$$

Thus $W'(t) \leqslant\ A(t)W(t) - B(t)\varphi(t) - W^2(t)C(t)$ and $W(t_k^+) \leqslant \frac{b_{k_i}}{a^*_{k_i}}W(t_k)$.
Define

$$U(t) = \prod_{t_0\leqslant t_k<t}\left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1}W(t).$$

It is clear that $W(t)$ is continuous on every interval $(t_k, t_{k+1}]$ and since $W(t_k^+) \leq \dfrac{b_{k_i}}{a^*_{k_i}}W(t_k)$, it follows that

$$U(t_k^+) = \prod_{t_0\leqslant t_j\leqslant t_k}\left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1}W(t_k^+) \leqslant \prod_{t_0\leqslant t_j<t_k}\left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1}W(t_k) = U(t_k)$$

and for all $t \geqslant t_0$,

$$U(t_k^-) = \prod_{t_0 \leqslant t_j \leqslant t_{k-1}} \left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1} W(t_k^-) \leq \prod_{t_0 \leqslant t_j < t_k} \left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1} W(t_k) = U(t_k)$$

which implies that $U(t)$ is continuous on $[t_0, +\infty)$.

$$\begin{aligned}
U'(t) &+ \prod_{t_0 \leqslant t_k < t} \left(\frac{b_{k_i}}{a^*_{k_i}}\right) U^2(t)C(t) + \prod_{t_0 \leqslant t_k < t} \left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1} B(t)\varphi(t) - A(t)U(t) \\
&= \prod_{t_0 \leqslant t_k < t} \left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1} W'(t) + \prod_{t_0 \leqslant t_k < t} \left(\frac{b_{k_i}}{a^*_{k_i}}\right) \prod_{t_0 \leqslant t_k < t} \left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-2} C(t)W^2(t) \\
&+ \prod_{t_0 \leqslant t_k < t} \left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1} B(t)\varphi(t) - \prod_{t_0 \leqslant t_k < t} \left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1} A(t)W(t) \\
&= \prod_{t_0 \leqslant t_k < t} \left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1} \left[W'(t) + W^2(t)C(t) - W(t)A(t) + B(t)\varphi(t)\right] \leqslant 0,
\end{aligned}$$

that is

$$\begin{aligned}
U'(t) \leqslant &- \prod_{t_0 \leqslant t_k < t} \left(\frac{b_{k_i}}{a^*_{k_i}}\right) C(t)U^2(t) \\
&+ A(t)U(t) - \prod_{t_0 \leqslant t_k < t} \left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1} B(t)\varphi(t).
\end{aligned} \tag{16}$$

Taking

$$x(t) = \left(\prod_{t_0 \leqslant t_k < t} \left(\frac{b_{k_i}}{a^*_{k_i}}\right) C(t)\right)^{\frac{1}{2}} U(t)$$

$$\text{and} \quad y(t) = \frac{A(t)}{2} \left(\prod_{t_0 \leq t_k < t} \left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1} \frac{1}{C(t)}\right)^{\frac{1}{2}}$$

and using Lemma 1, we have

$$A(t)U(t) - \prod_{t_0 \leqslant t_k < t} \left(\frac{b_{k_i}}{a^*_{k_i}}\right) C(t)U^2(t) \leqslant \frac{A^2(t)}{4C(t)} \prod_{t_0 \leqslant t_k < t} \left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1}.$$

Thus

$$U'(t) \leqslant -\prod_{t_0\leqslant t_k<t}\left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1}\left[B(t)\varphi(t)-\frac{A^2(t)}{4C(t)}\right].$$

Integrating both sides from t_0 to t, we have

$$U(t) \leqslant U(t_0)-\int_{t_0}^{t}\prod_{t_0\leqslant t_k<s}\left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1}\left[B(s)\varphi(s)-\frac{A^2(s)}{4C(s)}\right]ds.$$

Letting $t\to\infty$ and using (14) we have $\lim\limits_{t\to\infty} U(t)=-\infty$, which contradicts $U(t)\geqslant 0$. The pf of the theorem is complete. ■

We assume that there exist two functions $H(t,s)$, $h(t,s)\in C^1(D,\mathbb{R})$, in which $D=\{(t,s)|t\geq s\geq t_0>0\}$, such that

($\mathbf{H_8}$) $H(t,t)=0,\quad t\geq t_0;\qquad H(t,s)>0,\quad t>s\geq t_0,$
($\mathbf{H_9}$) $H_t^{'}(t,s)\geq 0,\ H_s^{'}(t,s)\leq 0,$
($\mathbf{H_{10}}$) $-\dfrac{\partial}{\partial s}H(t,s)\phi(s)-A(s)H(t,s)\phi(s)=h(t,s).$

Theorem 3 ([31]). *Assume that there exist functions $\varphi(t)$ and $\phi(s)\in C^1(\mathbb{R}_+,(0,+\infty))$ such that $\varphi(t)$ is nondecreasing. If there exist two functions $H(t,s)$, $h(t,s)\in C^1(D,\mathbb{R})$ satisfying $(H_8)-(H_{10})$ and*

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\prod_{t_0\leqslant t_k<s}\left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1}\Pi(s)ds=+\infty, \tag{17}$$

where

$$\Pi(s)=B(s)\varphi(s)H(t,s)\phi(s)-\frac{1}{4}\frac{|h(t,s)|^2}{C(s)H(t,s)\phi(s)},$$

then every solution of the boundary value problem $(E)-(B)$ is oscillatory in $\Omega\times\mathbb{R}_+$.

Choosing $\phi(s)=\varphi(s)\equiv 1$, in Theorem 3, we establish the following corollary.

Corollary 1. *Assume that the conditions of Theorem 3 hold, and*

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1}\Gamma(s)ds=\infty,$$

where

$$\Gamma(s)=B(s)H(t,s)-\frac{1}{4}\frac{|h(t,s)|^2}{C(s)H(t,s)},$$

then every solution of the boundary value problem $(E)-(B)$ is oscillatory in $\Omega\times\mathbb{R}_+$.

Remark 1. Using Theorem 3 and Corollary 1, we can obtain various oscillatory criteria by different choices of the weighted function $H(t,s)$. For example, choosing $H(t,s)=(t-s)^{\lambda-1}$, $t\geq s\geq t_0$, in which $\lambda>2$ is an integer, then $h(t,s)=(t-s)^{\lambda-1}\left(\frac{\lambda-1}{t-s}-A(s)\right)$, $t\geq s\geq t_0$. From Corollary 1, we have the following Kamenev type result.

Corollary 2. *If there exists an integer $\lambda>2$ such that*

$$\limsup_{t\to+\infty}\frac{1}{(t-t_0)^{\lambda-1}}\int_{t_0}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1}(t-s)^{\lambda-1}\times\left\{B(s)-\frac{1}{4C(s)}\left(\frac{(\lambda-1)^2}{(t-s)^2}+A^2(s)-2A(s)\frac{\lambda-1}{t-s}\right)\right\}ds=+\infty, \tag{18}$$

then every solution of the boundary value problem $(E)-(B)$ is oscillatory in $\Omega\times\mathbb{R}_+$.

Theorem 4. *Let the functions $H(t,s)$, $h(t,s)$, $\varphi(s)$ and $\phi(s)$ be as defined in Theorem 3. Additionally, suppose that $0<\inf_{s\geq t_0}\left\{\liminf_{t\to+\infty}\frac{H(t,s)}{H(t,t_0)}\right\}\leq+\infty$, and*

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1}\frac{|h(t,s)|^2}{C(s)H(t,s)\phi(s)}ds<+\infty.$$

If there exists a function $A(t)\in C([t_0,+\infty),\mathbb{R})$ such that

$$\limsup_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_{k_i}}{a_{k_i}}\right)\frac{C(s)(A_+(s))^2}{\phi(s)}ds=+\infty,$$

and for every $T\geq t_0$

$$\limsup_{t\to+\infty}\frac{1}{H(t,T)}\int_{T}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1}\times\left[B(s)H(t,s)\varphi(s)\phi(s)-\frac{1}{4}\frac{|h(t,s)|^2}{C(s)H(t,s)\phi(s)}\right]ds\geq A(T),$$

where $A_+(s)=\max\{A(s),0\}$, then every solution of the boundary value problem $(E)-(B)$ is oscillatory in $\Omega\times\mathbb{R}_+$.

Choosing $\phi(s)=\varphi(s)\equiv 1$, in Theorem 4, we establish the following corollary.

Corollary 3. *Assume that the conditions of Theorem 4 hold and assume that* $\phi(s) = \varphi(s) \equiv 1$. *If*

$$\limsup_{t\to+\infty} \frac{1}{H(t,T)} \int_T^t \prod_{t_0\le t_k<s} \left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1} \times \left[B(s)H(t,s) - \frac{1}{4}\frac{|h(t,s)|^2}{C(s)H(t,s)}\right] ds \ge A(T),$$

for every $T \ge t_0$, *where* $A_+(s) = \max\{A(s), 0\}$, *then every solution of the boundary value problem* $(E) - (B)$ *is oscillatory in* $\Omega \times \mathbb{R}_+$.

As with Corollaries 1 and 2, we can obtain the following corollary from Corollary 3.

Corollary 4. *Assume that the conditions of Theorem 4 hold, and*

$$\limsup_{t\to+\infty} \frac{1}{(t-t_0)^{\lambda-1}} \int_{t_0}^t \prod_{t_0\le t_k<s} \left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1} \frac{(t-s)^{\lambda-1}}{C(s)} \times \left[\frac{(\lambda-1)^2}{(t-s)^2} + A^2(s) - 2A(s)\frac{\lambda-1}{t-s}\right] ds < +\infty.$$

If there exists an integer $\lambda > 2$ *and a function* $A(t) \in C([t_0, +\infty), \mathbb{R})$ *such that*

$$\limsup_{t\to+\infty} \int_{t_0}^t \prod_{t_0\le t_k<s} \left(\frac{b_{k_i}}{a^*_{k_i}}\right) C(s)(A_+(s))^2 ds = +\infty,$$

and for every $T \ge t_0$

$$\limsup_{t\to+\infty} \frac{1}{(t-t_0)^{\lambda-1}} \int_T^t \prod_{t_0\le t_k<s} \left(\frac{b_{k_i}}{a^*_{k_i}}\right)^{-1} (t-s)^{\lambda-1} \times \left\{B(s) - \frac{1}{4C(s)}\left(\frac{(\lambda-1)^2}{(t-s)^2} + A^2(s) - 2A(s)\frac{\lambda-1}{t-s}\right)\right\} ds \ge A(T),$$

where $A_+(s) = \max\{A(s), 0\}$, *then every solution of the boundary value problem* $(E) - (B)$ *is oscillatory in* $\Omega \times \mathbb{R}_+$.

4. Example

In this section we present an example to illustrate the results established in Section 2. To this end, consider the following equation

$$
(19)\quad \begin{cases}
\dfrac{\partial}{\partial t}\left[4\dfrac{\partial}{\partial t}\left(u_1(x,t)+\dfrac{1}{2}\int_{\pi/4}^{\pi/2} u_1(x,t-2\xi)d\xi\right)\right] \\
\quad +\left(-\dfrac{4}{5}\right)\dfrac{\partial}{\partial t}\left(u_1(x,t)+\dfrac{1}{2}\int_{\pi/4}^{\pi/2} u_1(x,t-2\xi)d\xi\right) \\
\quad +6\int_{\pi/4}^{\pi/2} u_1(x,t-2\xi)d\xi+4\int_{\pi/4}^{\pi/2} u_2(x,t-2\xi)d\xi=5\Delta u_1(x,t) \\
\quad +\dfrac{23}{5}\Delta u_1(x,t-\tfrac{3\pi}{2})+\dfrac{4}{5}\Delta u_2(x,t-\tfrac{3\pi}{2}),\quad t\neq t_k,\ k=1,2,\cdots, \\
\dfrac{\partial}{\partial t}\left[4\dfrac{\partial}{\partial t}\left(u_2(x,t)+\dfrac{1}{2}\int_{\pi/4}^{\pi/2} u_2(x,t-2\xi)d\xi\right)\right] \\
\quad +\left(-\dfrac{4}{5}\right)\dfrac{\partial}{\partial t}\left(u_2(x,t)+\dfrac{1}{2}\int_{\pi/4}^{\pi/2} u_2(x,t-2\xi)d\xi\right) \\
\quad +4\int_{\pi/4}^{\pi/2} u_1(x,t-2\xi)d\xi+6\int_{\pi/4}^{\pi/2} u_2(x,t-2\xi)d\xi=7\Delta u_2(x,t) \\
\quad +\dfrac{6}{5}\Delta u_1(x,t-\tfrac{3\pi}{2})+\dfrac{3}{5}\Delta u_2(x,t-\tfrac{3\pi}{2}),\quad t\neq t_k,\ k=1,2,\cdots, \\
u_i(x,t_k^+)=\dfrac{k+1}{k}u_i(x,t_k), \\
\dfrac{\partial}{\partial t}u_i(x,t_k^+)=\dfrac{\partial}{\partial t}u_i(x,t_k),\quad k=1,2,\cdots,\quad i=1,2,\cdots,m
\end{cases}
$$

for $(x,t)\in(0,\pi)\times\mathbb{R}_+$, with the boundary condition

$$
(20)\qquad u_i(0,t)=u_i(\pi,t)=0,\quad t\neq t_k,\quad i=1,2,\cdots,m.
$$

Here $\Omega=(0,\pi)$, $N=1$, $m=2$, $d=1$, $l=1$, $a_{k_i}=a^*_{k_i}=\dfrac{k+1}{k}$, $b_{k_i}=b^*_{k_i}=1$, $i=1,2$, $r(t)=4$, $g(t,\xi)=\dfrac{1}{2}$, $\tau(t,\xi)=t-2\xi$, $p(t)=-\dfrac{4}{5}$, $\sigma_1(t,\xi)=t-2\xi$, $\eta(\xi)=\xi$, $f_{ij}(u_n)=u_n$, $M=1$, $q_{111}(x,t,\xi)=6$, $q_{121}(x,t,\xi)=4$, $a_1(t)=5$, $a_{111}(t)=\dfrac{23}{5}$, $a_{121}(t)=\dfrac{4}{5}$, $q_{211}(x,t,\xi)=4$, $q_{221}(x,t,\xi)=6$, $a_2(t)=7$, $a_{211}(t)=\dfrac{6}{5}$, $a_{221}(t)=\dfrac{3}{5}$, $\rho_1(t)=t-\frac{3\pi}{2}$, $Q_1(t,\xi)=2$, $[a,b]=[\pi/4,\pi/2]$, $\lambda=3$, $\theta_1(t)=t$, $\theta_1'(t)=1$. Since $t_0=1$, $t_k=2^k$, $A(s)=\dfrac{1}{5}$, $B(s)=\dfrac{8\pi-\pi^2}{16}$, $C(s)=\frac{1}{4}$.Then hypotheses $(H_1)-(H_7)$

hold, and moreover

$$\begin{aligned}
\lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0\leqslant t_k<s}\frac{b^*_{k_i}}{a_{k_i}}ds &= \int_1^{+\infty}\prod_{1<t_k<s}\frac{k}{k+1}ds \\
&= \int_1^{t_1}\prod_{1<t_k<s}\frac{k}{k+1}ds+\int_{t_1^+}^{t_2}\prod_{1<t_k<s}\frac{k}{k+1}ds \\
&\quad +\int_{t_2^+}^{t_3}\prod_{1<t_k<s}\frac{k}{k+1}ds+\cdots \\
&= 1+\frac{1}{2}\times 2+\frac{1}{2}\times\frac{2}{3}\times 2^2+\cdots \\
&= \sum_{n=0}^{\infty}\frac{2^n}{n+1}=+\infty.
\end{aligned}$$

Thus

$$\begin{aligned}
&\limsup_{t\to+\infty}\frac{1}{(t-1)^2}\left\{\int_1^t\prod_{1<t_k<s}\frac{k+1}{k}(t-s)^2\right. \\
&\qquad\left.\times\left[\frac{8\pi-\pi^2}{16}-\frac{1}{25}+\frac{4}{5(t-s)}-\frac{4}{(t-s)^2}\right]ds\right\}=+\infty.
\end{aligned}$$

Therefore all the conditions of the Corollary 2 are satisfied and hence every solution of equation (19)-(20) is oscillatory in $\Omega\times\mathbb{R}_+$. In fact $u_1(x,t)=\sin x\sin t,\ u_2(x,t)=\sin x\cos t$ is such a solution.

Conclusion: In this paper, we have established some new oscillation criteria for systems of impulsive nonlinear partial differential equations with distributed deviating arguments. Through four theorems and corresponding corollaries, in the main results section, we have established sufficient conditions for the oscillation of a system of such equations, constrained by the Dirichlet boundary condition. Through an example, we have shown the effectiveness of Corollary 2. The present results complement and extend those derived for problems without impulses.

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