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SOME REVERSES OF HÖLDER VECTOR OPERATOR INEQUALITY

Abstract.In this paper we obtain some new reverses of Hölder vector inequality for positive operators on Hilbert spaces.

Key words: Young's inequality, Hölder operator inequality, arithmetic mean-geometric mean inequality.

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1. Introduction

Throughout this paper A, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notation

$$A\sharp_\nu B := A^{1/2} \left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},$$

the *weighted geometric mean.* When $\nu = \frac{1}{2}$ we write $A\sharp B$ for brevity.

In [6] the authors obtained the following result:

$$\begin{aligned} \left\langle B^q \sharp_{1/p} A^p x, x\right\rangle &\le \langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} \\ &\le \lambda^{1/p}\left(p; \frac{m_1}{M_2^{q-1}}, \frac{M_1}{m_2^{q-1}}\right)\left\langle B^q \sharp_{1/p} A^p x, x\right\rangle \end{aligned} \tag{1}$$

for any $x \in H$, where $0 < m_1 I \le A \le M_1 I$, $0 < m_2 I \le B \le M_2 I$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, I is the identity operator and

$$\lambda(p; m, M) := \left[\frac{1}{p^{1/p} q^{1/q}} \frac{M^p - m^p}{(M-m)^{1/p} (mM^p - Mm^p)^{1/q}}\right]^p$$

for $0 < m < M$.

In particular, one can obtain from (1) the following noncommutative version of *Greub-Rheinboldt inequality*

$$\left\langle A^2 \sharp B^2 x, x\right\rangle \le \left\langle A^2 x, x\right\rangle^{1/2} \left\langle B^2 x, x\right\rangle^{1/2} \le \frac{m_1 m_2 + M_1 M_2}{2\sqrt{m_1 m_2 M_1 M_2}} \left\langle A^2 \sharp B^2 x, x\right\rangle \tag{2}$$

for any $x \in H$.

Moreover, if A and B are replaced by $C^{1/2}$ and $C^{-1/2}$ in (2), then we get the *Kantorovich inequality* [20]

$$\langle Cx, x\rangle^{1/2} \left\langle C^{-1}x, x\right\rangle^{1/2} \leq \frac{m+M}{2\sqrt{mM}}, \quad x \in H \text{ with } \|x\| = 1,$$

provided $mI \leq C \leq MI$ for some $0 < m < M$.

For various related inequalities, see [5]-[12] and [16]- [17].

In this paper, by making use of some recent Young's type inequalities outlined below, we establish some reverses and a refinement of Hölder's inequality for the positive operators A, B

$$\left\langle B^q \sharp_{1/p} A^p x, x\right\rangle \leq \langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q}, \quad x \in H$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0, 1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{3}$$

with equality if and only if $a = b$. The inequality (3) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [21]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} & \text{if } h \in (0,1) \cup (1, \infty) \\ 1 \text{ if } h = 1. \end{cases} \tag{4}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0, 1)$ and increasing on $(1, \infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^r\right) a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \leq S\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{5}$$

where $a, b > 0$, $\nu \in [0, 1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (5) is due to Tominaga [22] while the first one is due to Furuichi [7].

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \quad h > 0. \tag{6}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h>0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^{r}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \leq (1-\nu)a+\nu b \leq K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{7}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (7) was obtained by Zou et al. in [23] while the second by Liao et al. [19].

Kittaneh and Manasrah [14], [15] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^{2} \leq (1-\nu)a+\nu b - a^{1-\nu}b^{\nu} \leq R\left(\sqrt{a}-\sqrt{b}\right)^{2} \tag{8}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$. The case $\nu=\frac{1}{2}$ reduces (8) to an identity.

In the recent paper [1] we obtained the following reverses of Young's inequality as well:

$$0 \leq (1-\nu)a+\nu b - a^{1-\nu}b^{\nu} \leq \nu(1-\nu)(a-b)(\ln a - \ln b) \tag{9}$$

and

$$1 \leq \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} \leq \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{10}$$

where $a,b>0$, $\nu\in[0,1]$.

In [2] we obtained the following inequalities that improve the corresponding results of Furuichi and Minculete from [9]

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^{2}\min\{a,b\} &\leq (1-\nu)a+\nu b - a^{1-\nu}b^{\nu}\\ &\leq \frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^{2}\max\{a,b\}\end{aligned} \tag{11}$$

and

$$\begin{aligned}\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^{2}\right] &\leq \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\\ &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^{2}\right]\end{aligned} \tag{12}$$

for any $a,b>0$ and $\nu\in[0,1]$.

2. Some reverse inequalities

We have the following reverse of Hölder's inequality:

Theorem 1. *Let A and B be two positive invertible operators, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m,M>0$ such that*

$$m^p B^q \le A^p \le M^p B^q. \tag{13}$$

Then for any $x\in H$ we have the inequality

$$\langle A^p x,x\rangle^{1/p}\langle B^q x,x\rangle^{1/q} \le S\left(\left(\frac{M}{m}\right)^p\right)\left\langle B^q \sharp_{1/p} A^p x,x\right\rangle. \tag{14}$$

Proof. Assume that $\nu\in(0,1)$. Let $a,\ b\in[t,T]\subset(0,\infty)$, then $\frac{t}{T}\le\frac{a}{b}\le\frac{T}{t}$ with $\frac{t}{T}<1<\frac{T}{t}$. If $\frac{a}{b}\in\left[\frac{t}{T},1\right)$ then $S\left(\frac{a}{b}\right)\le S\left(\frac{t}{T}\right)=S\left(\frac{T}{t}\right)$. If $\frac{a}{b}\in\left(1,\frac{T}{t}\right]$ then also $S\left(\frac{a}{b}\right)\le S\left(\frac{T}{t}\right)$. Therefore for any $a,\ b\in[t,T]$ we have by Tominaga's inequality (5) that

$$(1-\nu)\,a+\nu b\le S\left(\frac{T}{t}\right)a^{1-\nu}b^{\nu}. \tag{15}$$

Now, if C is an operator with $tI\le C\le TI$ then for $p>1$ we have $t^pI\le C^p\le T^pI$. Using the functional calculus we get from (15) for $\nu=\frac{1}{p}$ that

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}C^p\le S\left(\left(\frac{T}{t}\right)^p\right)d^{1-\frac{1}{p}}C,$$

namely, the vector inequality,

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}\langle C^p y,y\rangle\le S\left(\left(\frac{T}{t}\right)^p\right)d^{1-\frac{1}{p}}\langle Cy,y\rangle, \tag{16}$$

for any $y\in H$, $\|y\|=1$ and $d\in[t^p,T^p]$.

Since $d:=\langle C^p y,y\rangle\in[t^p,T^p]$ for any $y\in H$, $\|y\|=1$, hence by (16) we have

$$\left(1-\frac{1}{p}\right)\langle C^p y,y\rangle+\frac{1}{p}\langle C^p y,y\rangle\le S\left(\left(\frac{T}{t}\right)^p\right)\langle C^p y,y\rangle^{1-\frac{1}{p}}\langle Cy,y\rangle,$$

that is equivalent to

$$\langle C^p y,y\rangle\le S\left(\left(\frac{T}{t}\right)^p\right)\langle C^p y,y\rangle^{1-\frac{1}{p}}\langle Cy,y\rangle$$

and to

$$\langle C^p y,y\rangle\le S^p\left(\left(\frac{T}{t}\right)^p\right)\langle Cy,y\rangle^p \tag{17}$$

for any $y \in H$, $\|y\| = 1$.

If $z \in H$ with $z \neq 0$, then by taking $y = \frac{z}{\|z\|}$ in (17) we get

$$\langle C^p z, z \rangle \|z\|^{2p-2} \leq S^p \left(\left(\frac{T}{t} \right)^p \right) \langle Cz, z \rangle^p$$

for any $z \in H$, and by taking the power $\frac{1}{p}$ we get

$$\langle C^p z, z \rangle^{1/p} \langle z, z \rangle^{1/q} \leq S \left(\left(\frac{T}{t} \right)^p \right) \langle Cz, z \rangle \tag{18}$$

for any $z \in H$.

Now, from (13) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^p I \leq B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \leq M^p I$ and by taking the power $\frac{1}{p}$ we get $mI \leq \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \right)^{\frac{1}{p}} \leq MI$.

By writing the inequality (18) for $C = \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \right)^{\frac{1}{p}}$, $t = m$, $T = M$ and $z = B^{\frac{q}{2}} x$, with $x \in H$, we have

$$\begin{aligned} &\left\langle B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} B^{\frac{q}{2}} x, B^{\frac{q}{2}} x \right\rangle^{1/p} \left\langle B^{\frac{q}{2}} x, B^{\frac{q}{2}} x \right\rangle^{1/q} \\ &\qquad \leq S \left(\left(\frac{M}{m} \right)^p \right) \left\langle \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \right)^{\frac{1}{p}} B^{\frac{q}{2}} x, B^{\frac{q}{2}} x \right\rangle, \end{aligned}$$

namely

$$\langle A^p x, x \rangle^{1/p} \langle B^q x, x \rangle^{1/q} \leq S \left(\left(\frac{M}{m} \right)^p \right) \left\langle B^{\frac{q}{2}} \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \right)^{\frac{1}{p}} B^{\frac{q}{2}} x, x \right\rangle$$

for any $x \in H$, and the inequality (14) is proved. ■

Remark 1. We observe, for A and B two positive invertible operators, that the condition (13) is equivalent to following condition

$$mI \leq \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \right)^{\frac{1}{p}} \leq MI. \tag{19}$$

If we assume that

$$rB^q \leq A^p \leq RB^q, \tag{20}$$

for r, $R > 0$, then by (14) we have the inequality

$$\langle A^p x, x \rangle^{1/p} \langle B^q x, x \rangle^{1/q} \leq S \left(\frac{R}{r} \right) \langle B^q \sharp_{1/p} A^p x, x \rangle \tag{21}$$

for any $x \in H$.

Corollary 1. *Let A and B be two positive invertible operators and m, $M>0$ such that*

$$mI \le \left(B^{-1}A^2B^{-1}\right)^{\frac{1}{2}} \le MI. \tag{22}$$

Then for any $x\in H$ we have the inequality

$$\left\langle A^2x,x\right\rangle^{1/2}\left\langle B^2x,x\right\rangle^{1/2} \le S\left(\left(\frac{M}{m}\right)^2\right)\left\langle A^2\sharp B^2x,x\right\rangle, \tag{23}$$

where

$$A^2\sharp B^2 = A\left(A^{-1}B^2A^{-1}\right)^{1/2}A = B^2\sharp A^2.$$

Now, by taking $A=C^{1/2}$ and $B=C^{-1/2}$, then the condition (22) becomes

$$mI \le C \le MI \tag{24}$$

and by (23) we get

$$\langle Cx,x\rangle^{1/2}\left\langle C^{-1}x,x\right\rangle^{1/2} \le S\left(\left(\frac{M}{m}\right)^2\right),$$

for any $x\in H$ with $\|x\|=1$.

Corollary 2. *Assume that A and B satisfy the conditions*

$$m_1I \le A \le M_1I, \qquad m_2I \le B \le M_2I \tag{25}$$

for some $0<m_1<M_1$ and $0<m_2<M_2$. Then we have

$$\langle A^px,x\rangle^{1/p}\langle B^qx,B^qx\rangle^{1/q} \le S\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)\left\langle B^q\sharp_{1/p}A^px,x\right\rangle, \tag{26}$$

for any $x\in H$.

In particular, we have

$$\left\langle A^2x,x\right\rangle^{1/2}\left\langle B^2x,x\right\rangle^{1/2} \le S\left(\left(\frac{M_1M_2}{m_1m_2}\right)^2\right)\left\langle A^2\sharp B^2x,x\right\rangle, \tag{27}$$

for any $x\in H$.

Proof. We have from (25) that

$$m_1^pI \le A^p \le M_1^pI.$$

Then

$$m_1^p M_2^{-q} I \leq m_1^p B^{-q} \leq B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \leq M_1^p B^{-q} \leq M_1^p m_2^{-q} I,$$

which implies that

$$m_1 M_2^{-\frac{q}{p}} I \leq \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} \leq M_1 m_2^{-\frac{q}{p}} I.$$

Now, on using the inequality (14) for $m = m_1 M_2^{-\frac{q}{p}}$ and $M = M_1 m_2^{-\frac{q}{p}}$, we get the desired result (26). ■

Using Kantorovich's constant (6) we also have:

Theorem 2. *With the assumptions of Theorem 1 we have the inequality*

$$\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} \leq K^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(\left(\frac{M}{m}\right)^p\right) \left\langle B^q \sharp_{1/p} A^p x, x\right\rangle \tag{28}$$

for any $x \in H$.

Proof. Assume that $\nu \in (0,1)$ and $R = \max\{1-\nu, \nu\}$. Let $a, b \in [t,T] \subset (0,\infty)$, then $\frac{t}{T} \leq \frac{a}{b} \leq \frac{T}{t}$ with $\frac{t}{T} < 1 < \frac{T}{t}$. If $\frac{a}{b} \in \left[\frac{t}{T}, 1\right)$ then $K^R\left(\frac{a}{b}\right) \leq K^R\left(\frac{t}{T}\right) = K^R\left(\frac{T}{t}\right)$. If $\frac{a}{b} \in \left(1, \frac{T}{t}\right]$ then also $K^R\left(\frac{a}{b}\right) \leq K^R\left(\frac{T}{t}\right)$. Therefore for any a, $b \in [t,T]$ we have by inequality (7) that

$$(1-\nu)\, a + \nu b \leq K^R\left(\frac{T}{t}\right) a^{1-\nu} b^{\nu}. \tag{29}$$

Now, if C is an operator with $tI \leq C \leq TI$ then for $p > 1$ we have $t^p I \leq C^p \leq T^p I$. Using the functional calculus we get from (29) for $\nu = \frac{1}{p}$ that

$$\left(1 - \frac{1}{p}\right) d + \frac{1}{p} C^p \leq K^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(\left(\frac{T}{t}\right)^p\right) d^{1-\frac{1}{p}} C,$$

namely, the vector inequality,

$$\left(1 - \frac{1}{p}\right) d + \frac{1}{p}\langle C^p y, y\rangle \leq K^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(\left(\frac{T}{t}\right)^p\right) d^{1-\frac{1}{p}} \langle Cy, y\rangle,$$

for any $y \in H$, $\|y\| = 1$ and $d \in [t^p, T^p]$.

Now, by employing a similar argument to the one in the proof of Theorem 1 we deduce the desired result (28). The details are omitted. ■

Corollary 3. *With the assumptions of Corollary 1 we have*

$$\left\langle A^2 x, x\right\rangle^{1/2} \left\langle B^2 x, x\right\rangle^{1/2} \leq \left[K\left(\left(\frac{M}{m}\right)^2\right)\right]^{1/2} \left\langle A^2 \sharp B^2 x, x\right\rangle, \tag{30}$$

for any $x \in H$.

We also have:

Corollary 4. *With the assumptions of Corollary 2 we have*

$$\langle A^p x, x\rangle^{1/p} \langle B^q x, B^q x\rangle^{1/q} \le K^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}} \left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right) \left\langle B^q \sharp_{1/p} A^p x, x\right\rangle, \tag{31}$$

for any $x \in H$.

In particular, we have

$$\left\langle A^2 x, x\right\rangle^{1/2} \left\langle B^2 x, x\right\rangle^{1/2} \le \left[K\left(\left(\frac{M_1 M_2}{m_1 m_2}\right)^2\right)\right]^{1/2} \left\langle A^2 \sharp B^2 x, x\right\rangle, \tag{32}$$

for any $x \in H$.

3. Exponential reverses

We have:

Theorem 3. *With the assumptions of Theorem 1 we have the inequality*

$$\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} \le \exp\left[\frac{4}{pq}\left(K\left[\left(\frac{M}{m}\right)^p\right] - 1\right)\right] \left\langle B^q \sharp_{1/p} A^p x, x\right\rangle \tag{33}$$

for any $x \in H$.

Proof. Assume that $\nu \in (0,1)$. Let a, $b \in [t, T] \subset (0, \infty)$, then by the inequality (10) we have

$$(1-\nu)\, a + \nu b \le a^{1-\nu} b^{\nu} \exp\left[4\nu\,(1-\nu)\left(K\left(\frac{T}{t}\right) - 1\right)\right]. \tag{34}$$

Now, if C is an operator with $tI \le C \le TI$ then for $p > 1$ we have $t^p I \le C^p \le T^p I$. Using the functional calculus we get from (34) for $\nu = \frac{1}{p}$ that

$$\left(1 - \frac{1}{p}\right) d + \frac{1}{p} C^p \le \exp\left[\frac{4}{pq}\left(K\left[\left(\frac{T}{t}\right)^p\right] - 1\right)\right] d^{1-\frac{1}{p}} C,$$

namely, the vector inequality,

$$\left(1 - \frac{1}{p}\right) d + \frac{1}{p}\left\langle C^p y, y\right\rangle \le \exp\left[\frac{4}{pq}\left(K\left[\left(\frac{T}{t}\right)^p\right] - 1\right)\right] d^{1-\frac{1}{p}} \left\langle C y, y\right\rangle,$$

for any $y \in H$, $\|y\| = 1$ and $d \in [t^p, T^p]$.

Now, by employing a similar argument to the one in the proof of Theorem 1 we deduce the desired result (33). The details are omitted. ■

We have:

Corollary 5. *With the assumptions of Corollary 1 we have*

$$\left\langle A^2x,x\right\rangle^{1/2}\left\langle B^2x,x\right\rangle^{1/2}\leq \exp\left(K\left[\left(\frac{M}{m}\right)^2\right]-1\right)\left\langle A^2\sharp B^2x,x\right\rangle, \tag{35}$$

for any $x\in H$.

Corollary 6. *With the assumptions of Corollary 2 we have*

$$\begin{aligned}&\left\langle A^px,x\right\rangle^{1/p}\left\langle B^qx,B^qx\right\rangle^{1/q}\\&\qquad\leq \exp\left[\frac{4}{pq}\left(K\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]-1\right)\right]\left\langle B^q\sharp_{1/p}A^px,x\right\rangle,\end{aligned} \tag{36}$$

for any $x\in H$.

In particular, we have

$$\left\langle A^2x,x\right\rangle^{1/2}\left\langle B^2x,x\right\rangle^{1/2}\leq \exp\left(K\left[\left(\frac{M_1M_2}{m_1m_2}\right)^2\right]-1\right)\left\langle A^2\sharp B^2x,x\right\rangle, \tag{37}$$

for any $x\in H$.

Finally, we have the following reverse of Hölder's inequality as well:

Theorem 4. *With the assumptions of Theorem 1 we have the inequality*

$$\left\langle A^px,x\right\rangle^{1/p}\left\langle B^qx,x\right\rangle^{1/q}\leq \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p-1\right)^2\right]\left\langle B^q\sharp_{1/p}A^px,x\right\rangle \tag{38}$$

for any $x\in H$.

Proof. If $a,b\in[t,T]\subset(0,\infty)$ and since

$$0<\frac{\max\{a,b\}}{\min\{a,b\}}-1\leq\frac{T}{t}-1,$$

hence

$$\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\leq\left(\frac{T}{t}-1\right)^2.$$

Therefore, by (12) we get

$$(1-\nu)\,a+\nu b\leq a^{1-\nu}b^{\nu}\exp\left[\frac{1}{2}\nu\,(1-\nu)\left(\frac{T}{t}-1\right)^2\right], \tag{39}$$

for any $a,b\in[t,T]$ and $\nu\in(0,1)$.

Now, if C is an operator with $tI \leq C \leq TI$ then for $p > 1$ we have $t^p I \leq C^p \leq T^p I$. Using the functional calculus we get from (34) for $\nu = \frac{1}{p}$ that

$$\left(1 - \frac{1}{p}\right) d + \frac{1}{p} C^p \leq \exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p - 1\right)^2\right] d^{1-\frac{1}{p}} C,$$

namely, the vector inequality,

$$\left(1 - \frac{1}{p}\right) d + \frac{1}{p}\left\langle C^p y, y\right\rangle \leq \exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p - 1\right)^2\right] d^{1-\frac{1}{p}}\left\langle Cy, y\right\rangle,$$

for any $y \in H$, $\|y\| = 1$ and $d \in [t^p, T^p]$.

Now, by employing a similar argument to the one in the proof of Theorem 1 we deduce the desired result (39). The details are omitted. ■

We have:

Corollary 7. *With the assumptions of Corollary 1 we have*

$$\left\langle A^2 x, x\right\rangle^{1/2}\left\langle B^2 x, x\right\rangle^{1/2} \leq \exp\left[\frac{1}{8}\left(\left(\frac{M}{m}\right)^2 - 1\right)^2\right]\left\langle A^2 \sharp B^2 x, x\right\rangle \tag{40}$$

for any $x \in H$.

If $mI \leq C \leq MI$ for some m, M with $0 < m < M$, then by (40) we get

$$\left\langle Cx, x\right\rangle^{1/2}\left\langle C^{-1} x, x\right\rangle^{1/2} \leq \exp\left[\frac{1}{8}\left(\left(\frac{M}{m}\right)^2 - 1\right)^2\right]\|x\|^2, \tag{41}$$

for any $x \in H$.

Corollary 8. *With the assumptions of Corollary 2 we have*

$$\begin{aligned}&\left\langle A^p x, x\right\rangle^{1/p}\left\langle B^q x, x\right\rangle^{1/q}\\&\qquad \leq \exp\left[\frac{1}{2pq}\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q - 1\right)^2\right]\left\langle B^q \sharp_{1/p} A^p x, x\right\rangle,\end{aligned}$$

for any $x \in H$.

In particular, we have

$$\begin{aligned}&\left\langle A^2 x, x\right\rangle^{1/2}\left\langle B^2 x, x\right\rangle^{1/2}\\&\qquad \leq \exp\left[\frac{1}{8}\left(\left(\frac{M_1 M_2}{m_1 m_2}\right)^2 - 1\right)^2\right]\left\langle A^2 \sharp B^2 x, x\right\rangle,\end{aligned} \tag{42}$$

for any $x \in H$.

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