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DIFFERENCES OF OPERATORS OF BASKAKOV TYPE

Abstract. In the present article, we study the approximation of difference of operators and find the quantitative estimates for the differences of Baskakov with Baskakov-Szász and genuine Baskakov-Durrmeyer operators. We also estimate the result for the difference of Baskakov-Szász and genuine Baskakov-Durrmeyer operators.

Key words: difference of operators, Baskakov operators, Baskakov-Szász operators, genuine Baskakov-Durrmeyer operators, modulus of continuity.

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1. Introduction

The study on the difference of linear positive operators is an active area of research in recent years. Such problem was initiated by A. Lupaş [11]. In the starting Acu-Rasa [1] and Aral et al. [3] established certain estimates for the difference of operators. Some of the recent results on this topic can be found in [2], [6], [9, Ch. 7] and [10] etc.

Let us consider $F_{n,k}, G_{n,k} : D \to \mathbb{R}$, where D is a subspace of $C[0,\infty)$, which contains polynomials of degree upto 4, we define the operators

$$U_n(f,x) = \sum_{k=0}^{\infty} v_{n,k}(x) F_{n,k}(f), \qquad V_n(f,x) = \sum_{k=0}^{\infty} v_{n,k}(x) G_{n,k}(f).$$

with $F_{n,k}(e_0) = G_{n,k}(e_0) = 1$. Throughout the paper, we use the notations

$$b^F := F(e_1), \quad \mu_r^F = F(e_1 - b^F e_0)^r, \ \ r \in \mathbb{N}$$

Very recently Gupta in [5] established the following result for difference of operators.

Theorem 1 ([5]). *Let $f^{(s)} \in C_B[0,\infty), s \in \{0,1,2\}$ and $x \in [0,\infty)$, then for $n \in \mathbb{N}$, we have*

$$|(U_n - V_n)(f,x)| \leq \frac{\alpha(x)}{2}||f''|| + \frac{(1+\alpha(x))}{2}\omega(f'',\delta_1) + 2\omega(f,\delta_2(x)),$$

where $C_B[0,\infty)$ *be the class of bounded continuous functions defined for* $x \geq 0$, $||\cdot|| = \sup_{x\in[0,\infty)} |f(x)| < \infty$,

$$\alpha(x) = \sum_{k=0}^{\infty} v_{n,k}(x)(\mu_2^{F_{n,k}} + \mu_2^{G_{n,k}})$$

and

$$\delta_1^2 = \sum_{k=0}^{\infty} v_{n,k}(x)(\mu_4^{F_{n,k}} + \mu_4^{G_{n,k}}), \quad \delta_2^2 = \sum_{k=0}^{\infty} v_{n,k}(x)(b^{F_{n,k}} - b^{G_{n,k}})^2.$$

Corollary 1. *If the operators* U_n *and* V_n *satisfy* $F_{n,k}(e_1) = G_{n,k}(e_1) = \frac{k}{n}$, *then under the assumptions of Theorem 1, we have*

$$|(U_n - V_n)(f,x)| \leq \frac{\alpha(x)}{2}||f''|| + \frac{(1+\alpha(x))}{2}\omega(f'',\delta_1).$$

The Baskakov operators are defined as

$$\begin{aligned} V_n(f,x) &= \sum_{k=0}^{\infty} v_{n,k}(x)F_{n,k}(f) \\ &= \sum_{k=0}^{\infty} v_{n,k}(x) f\left(\frac{k}{n}\right), \end{aligned} \tag{1}$$

where the Baskakov basis function is given by

$$v_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}.$$

Remark 1. With $e_r(t) = t^r, r \in \mathbb{N}^0$ we consider

$$b^{F_{n,k}} = F_{n,k}(e_1) = \frac{k}{n}.$$

Also, for $r \in \mathbb{N}$, we have

$$\mu_r^{F_{n,k}} := F_{n,k}(e_1 - b^{F_{n,k}}e_0)^r = 0.$$

Lemma 1. *The following recurrence relation holds for moments*

$$V_n(e_{m+1},x) = \frac{x(1+x)}{n}V_n'(e_m,x) + xV_n(e_m,x).$$

Some of the moments of Baskakov operators defined by (1) are given as:

$$
\begin{aligned}
&V_n(e_0, x) = 1\\
&V_n(e_1, x) = x\\
&V_n(e_2, x) = \frac{x^2(n+1)+x}{n}\\
&V_n(e_3, x) = \frac{x^3(n+1)(n+2)+3x^2(n+1)+x}{n^2}\\
&V_n(e_4, x) = \frac{x^4(n+1)(n+2)(n+3)+6x^3(n+1)(n+2)+7x^2(n+1)+x}{n^3}.
\end{aligned}
$$

In the present paper, which is in continuation of our previous papers [5], [7], we establish here quantitative estimates for the difference of Baskakov type operators and their variants.

2. Difference of operators for Baskakov type

In this section, we estimate quantitative result for the difference of Baskakov with Baskakov-Szász and genuine Baskakov-Durrmeyer operators. We also estimate the result for the difference of Baskakov-Szász and genuine Baskakov- Durrmeyer operators.

2.1. Baskakov and Baskakov-Szász operators

The Baskakov-Szász operators considered in [8] are defined as

$$
M_n(f;x) = \sum_{k=0}^{\infty} v_{n,k}(x) G_{n,k}(f), \tag{2}
$$

where $v_{n,k}(x)$ is defined in (1) and

$$
G_{n,k}(f) = n\int_0^\infty s_{n,k}(t) f(t)dt, \qquad s_{n,k}(t) = e^{-nt}\frac{(nt)^k}{k!}.
$$

Remark 2. By simple computation with $e_r(t) = t^r, r \in \mathbb{N}^0$, we have

$$
G_{n,k}(e_r) = n\int_0^\infty s_{n,k}(t) t^r dt = \frac{(k+r)!}{k! n^r}.
$$

Thus

$$
b^{G_{n,k}} = G_{n,k}(e_1) = \frac{k+1}{n}
$$

and

$$\begin{aligned}
\mu_2^{G_{n,k}} &:= G_{n,k}(e_1 - b^{G_{n,k}} e_0)^2 \\
&= G_{n,k}(e_2,x) + \left(\frac{k+1}{n}\right)^2 - 2G_{n,k}(e_1,x)\left(\frac{k+1}{n}\right) \\
&= \frac{(k+2)(k+1)}{n^2} - \left(\frac{k+1}{n}\right)^2 \\
&= \frac{k+1}{n^2}
\end{aligned}$$

and

$$\begin{aligned}
\mu_4^{G_{n,k}} &:= G_{n,k}(e_1 - b^{G_{n,k}} e_0)^4 \\
&= G_{n,k}(e_4,x) - 4G_{n,k}(e_3,x)\left(\frac{k+1}{n}\right) + 6G_{n,k}(e_2,x)\left(\frac{k+1}{n}\right)^2 \\
&\quad - 4G_{n,k}(e_1,x)\left(\frac{k+1}{n}\right)^3 + G_{n,k}(e_0,x)\left(\frac{k+1}{n}\right)^4 \\
&= \frac{(k+1)(k+2)(k+3)(k+4)}{n^4} \\
&\quad - 4\frac{(k+1)(k+2)(k+3)}{n^3}\left(\frac{k+1}{n}\right) + 6\frac{(k+1)(k+2)}{n^2}\left(\frac{k+1}{n}\right)^2 \\
&\quad - 4\frac{(k+1)}{n}\left(\frac{k+1}{n}\right)^3 + \left(\frac{k+1}{n}\right)^4 \\
&= \frac{3(k^2+4k+3)}{n^4}.
\end{aligned}$$

Below, we present the application of Theorem 1, i.e. exact estimate for difference of Baskakov-Szász- and Baskakov operators.

Theorem 2. *Let $f^{(s)} \in C_B[0,\infty), s \in \{0,1,2\}$ and $x \in [0,\infty)$, then for $n \in \mathbb{N}$, we have*

$$|(M_n - V_n)(f,x)| \le \frac{\alpha(x)}{2}||f''|| + \frac{(1+\alpha(x))}{2}\omega(f'',\delta_1) + 2\omega(f,\delta_2(x)),$$

where

$$\alpha(x) = \frac{nx+1}{n^2}, \quad \delta_1^2(x) = \frac{3x^2 n(n+1) + 15nx + 9}{n^4}, \quad \delta_2^2(x) = \frac{1}{n^2}.$$

Proof. Following Theorem 1, and using Remark 1, Remark 2 and Lemma 1, we have the following estimates

$$\begin{aligned}
\alpha(x) &:= \sum_{k=0}^{\infty} v_{n,k}(x)(\mu_2^{F_{n,k}} + \mu_2^{G_{n,k}}) = \sum_{k=0}^{\infty} v_{n,k}(x)\frac{k+1}{n^2} \\
&= \frac{1}{n}V_n(e_1,x) + \frac{1}{n^2} = \frac{nx+1}{n^2}.
\end{aligned}$$

$$\begin{aligned}
\delta_1^2(x) &= \sum_{k=0}^{\infty} v_{n,k}(x)(\mu_4^{F_{n,k}} + \mu_4^{G_{n,k}}) \\
&= \sum_{k=0}^{\infty} v_{n,k}(x)\mu_4^{G_{n,k}} \\
&= \frac{3x^2 n(n+1) + 15nx + 9}{n^4}. \\
\delta_2^2(x) &= \sum_{k=0}^{\infty} v_{n,k}(x)(b^{F_{n,k}} - b^{G_{n,k}})^2 \\
&= \sum_{k=0}^{\infty} v_{n,k}(x)\left[\frac{k}{n} - \frac{k+1}{n}\right]^2 = \frac{1}{n^2}.
\end{aligned}$$

The theorem follows by collecting the above values. ■

2.2. Baskakov and Genuine Baskakov-Durrmeyer operators

The genuine Baskakov operators operators (see [4]) are defined as

$$P_n(f;x) = \sum_{k=0}^{\infty} v_{n,k}(x) H_{n,k}(f), \tag{3}$$

where and

$$H_{n,k}(f) = \frac{1}{B(k,n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} f(t)dt,$$

$1 \le k < \infty$, $H_{n,0}(f) = f(0)$.

Remark 3. By simple computation with $e_r(t) = t^r, r \in \mathbb{N}^0$, we have

$$H_{n,k}(e_r) = \frac{(k+r-1)!(n-r)!}{(k-1)!n!}.$$

Thus

$$b^{H_{n,k}} = H_{n,k}(e_1) = \frac{k}{n}.$$

$$\begin{aligned}
\mu_2^{H_{n,k}} &:= H_{n,k}(e_1 - b^{H_{n,k}} e_0)^2 \\
&= H_{n,k}(e_2) - 2H_{n,k}(e_1)\left(\frac{k}{n}\right) + H_{n,k}(e_0)\left(\frac{k}{n}\right)^2 \\
&= \frac{k^2+k}{n(n-1)} - \frac{k^2}{n^2} = \frac{k^2+nk}{n^2(n-1)}
\end{aligned}$$

and

$$\begin{aligned}
\mu_4^{H_{n,k}} &:= H_{n,k}(e_1 - b^{H_{n,k}}e_0)^4\\
&= H_{n,k}(e_4) - 4H_{n,k}(e_3)\left(\frac{k}{n}\right) + 6H_{n,k}(e_2)\left(\frac{k}{n}\right)^2\\
&\quad - 4H_{n,k}(e_1)\left(\frac{k}{n}\right)^3 + H_{n,k}(e_0)\left(\frac{k}{n}\right)^4\\
&= \frac{(k+3)(k+2)(k+1)k}{n(n-1)(n-2)(n-3)} - 4\frac{(k+2)(k+1)k}{n(n-1)(n-2)}\left(\frac{k}{n}\right)\\
&\quad + 6\frac{(k+1)k}{n(n-1)}\left(\frac{k}{n}\right)^2 - 4\frac{k}{n}\left(\frac{k}{n}\right)^3 + \left(\frac{k}{n}\right)^4\\
&= \frac{3}{n^4(n-1)(n-2)(n-3)}\\
&\quad \times \Big[k^4(n+6) + 2nk^3(n+6) + n^2k^2(n+8) + 2n^3k\Big].
\end{aligned}$$

We present below the application of Theorem 1, i.e. exact estimate for difference of genuine Baskakov-Durrmeyer and Baskakov operators.

Theorem 3. *Let $f^{(s)} \in C_B[0,\infty), s \in \{0,1,2\}$ and $x \in [0,\infty)$, then for $n \in \mathbb{N}$, we have*

$$|(P_n - V_n)(f,x)| = \frac{\alpha(x)}{2}||f''|| + \frac{(1+\alpha(x))}{2}\omega(f'',\delta_1),$$

where

$$\alpha(x) = \frac{(n+1)x(1+x)}{n(n-1)}$$

and

$$\begin{aligned}
\delta_1^2 := &\frac{3(n+1)x(x+1)}{n^3(n-1)(n-2)(n-3)}\Big[n^3x(x+1)\\
&+ n^2(11x^2+11x+3) + n(36x^2+36x+7) + 6(6x^2+6x+1)\Big].
\end{aligned}$$

Proof. Using Remark 1, Remark 3 and Lemma 1, we have the following estimates

$$\alpha(x) := \sum_{k=0}^{\infty} v_{n,k}(x)(\mu_2^{F_{n,k}} + \mu_2^{H_{n,k}}) = \frac{(n+1)x(1+x)}{n(n-1)}.$$

$$\begin{aligned}
\delta_1^2(x) &= \sum_{k=0}^{\infty} v_{n,k}(x)(\mu_4^{F_{n,k}} + \mu_4^{H_{n,k}}) \\
&= \sum_{k=0}^{\infty} v_{n,k}(x)\mu_4^{H_{n,k}} \\
&= \frac{3(n+1)x(x+1)}{n^3(n-1)(n-2)(n-3)}\Big[n^3x(x+1) + n^2(11x^2+11x+3) \\
&\quad + n(36x^2+36x+7) + 6(6x^2+6x+1)\Big].
\end{aligned}$$

Combining these values, the result follows from Corollary 1. ■

2.3. Baskaov-Szász and Genuine Baskakov-Durrmeyer operators

We present below the application of Theorem 1, i.e. exact estimate for difference of Baskakov-Szász and genuine Baskakov-Durrmeyer operators.

Theorem 4. *Let $f^{(s)} \in C_B[0,\infty), s \in \{0,1,2\}$ and $x \in [0,\infty)$, then for $n \in \mathbb{N}$, we have*

$$|(M_n - P_n)(f,x)| = \frac{\alpha(x)}{2}||f''|| + \frac{(1+\alpha(x))}{2}\omega(f'',\delta_1) + 2\omega(f,\delta_2(x)),$$

where

$$\alpha(x) = \frac{nx+1}{n^2} + \frac{(n+1)x(1+x)}{n(n-1)}, \quad \delta_2^2(x) = \frac{1}{n^2}$$

and

$$\begin{aligned}
\delta_1^2(x) &= \frac{3x^2n(n+1)+15nx+9n}{n^4} \\
&+ \frac{3(n+1)x(x+1)}{n^3(n-1)(n-2)(n-3)}\Big[n^3x(x+1) + n^2(11x^2+11x+3) \\
&+ n(36x^2+36x+7) + 6(6x^2+6x+1)\Big].
\end{aligned}$$

Proof. Following Theorem 1, using Remark 2, Remark 3 and Lemma 1, we have

$$\begin{aligned}
\alpha(x) &:= \sum_{k=0}^{\infty} v_{n,k}(x)(\mu_2^{G_{n,k}} + \mu_2^{H_{n,k}}) \\
&= \frac{nx+1}{n^2} + \frac{(n+1)x(1+x)}{n(n-1)}.
\end{aligned}$$

$$\begin{aligned}\delta_1^2(x) &= \sum_{k=0}^{\infty} v_{n,k}(x)(\mu_4^{G_{n,k}}+\mu_4^{H_{n,k}})\\ &= \frac{3x^2n(n+1)+15nx+9n}{n^4}\\ &+ \frac{3(n+1)x(x+1)}{n^3(n-1)(n-2)(n-3)}\Big[n^3x(x+1)+n^2(11x^2+11x+3)\\ &+ n(36x^2+36x+7)+6(6x^2+6x+1)\Big].\end{aligned}$$

and by using above identities, we have

$$\delta_2^2(x) = \sum_{k=0}^{\infty} v_{n,k}(x)(b^{G_{n,k}}-b^{H_{n,k}})^2 = \frac{1}{n^2}.$$

Combining the above estimates, the result follows from Theorem 1. ■

Remark 4. In the present paper, we considered $v_{n,k}(x)$ as Baskakov basis function, one may consider any other basis function analogously.

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