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REMARKS ON SUBMULTIPLICATIVE FUNCTIONS

Abstract. The real functions satisfying the inequality $\Phi(uv) \leq K\Phi(u)\Phi(v)$ for some positive K which occur among others in [5], [3], [4], and referred there as submultiplicative, are discussed. A simplifying remark that Φ satisfies this inequality iff $K\Phi$ is submultiplicative in the standard sense, is done. It is shown that, under general conditions, the standard submultiplicativity of Φ and the inequality $\Phi(u)\Phi\left(\frac{1}{u}\right) \leq 1$ imply that Φ must be multiplicative. Applying a result of Bhatt [1], we observe that if p is a nontrivial seminorm on a Banach algebra X such that the set $\{\frac{p(x^2)}{[p(x)]^2} : x \in X, p(x) \neq 0\}$ is a singleton $\{\lambda\}$, then $s = \lambda p$ is a submultiplicative seminorm on X.

Key words: submultiplicativitive function, seminorm, Orlicz function, square property.

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1. Introduction

Submultiplicative functions, similarly as subadditive ones, frequently appear in applications, and have well-developed theories (see, for instance, Hille and Phillips, [2], Kuczma [6]). In some parts of functional analysis, especially concerned the Orlicz spaces, a nonstandard form of submultiplicativity occurs. In Krasnoselskij and Rutickij [5], (see also Hudzik and Maligranda, Mastyło, Persson [3], [4]) a function $\Phi : [0, \infty) \to \mathbb{R}$ is referred to as submultiplicative on $[0, \infty)$, if there exists a positive constant K such that

$$\Phi(uv) \leq K\Phi(u)\Phi(v) \quad \text{for all} \quad u, v \geq 0.$$

As for $K = 1$ we get the classical submultiplicativity, one could treat it as a generalization and, for convenience, we call it the *submultiplicativity in the sense of Krasnoselskij and Rutickij.*

We observe that Φ is submultiplicative in this sense iff the function $K\Phi$ is submultiplicative (see Theorem 1 in section 2). This fact allows to simplify notations and avoid introducing new notions of submultiplicativity. Moreover, in section 2 devoted to the standard submultiplicative functions, we prove that if $\Phi : (0,\infty) \to (0,\infty)$ is submultiplicative on $(0,\infty)$ and $\Phi(u) \le \frac{1}{\Phi(\frac{1}{u})}$ for all $u \in [1,\infty)$, then Φ is multiplicative on $(0,\infty)$.

In section 4, applying theorem of Bhatt [1] and Theorem 1, we conclude that if p is a nontrivial seminorm on a Banach algebra X such that the set $\{\frac{p(x^2)}{[p(x)]^2} : x \in X, p(x) \ne 0\}$ is a singleton $\{\lambda\}$, then $s = \lambda p$ is a submultiplicative seminorm on X.

2. Remark on classical submultiplicativity

A real valued function Φ defined on a set C that is closed under multiplication, is called *multiplicative on* C, if

$$\Phi(uv) = \Phi(u)\Phi(v) \quad \text{for all} \quad u, v \in C;$$

submultiplicative on C, if

$$\Phi(uv) \le \Phi(u)\Phi(v) \quad \text{for all} \quad u, v \in C,$$

and *supermultiplicative on* C, if the reversed inequality holds.

In the case of submultiplicative functions, simple considerations show that, without any loss of generality, one can assume that $O \notin C$ and the range of Φ is contained in the set of positive numbers.

Therefore, in this section, we assume the following

Definition 1. *Let $I \subset (0,\infty)$ be an interval that is closed under multiplication. A function $\Phi : (0,\infty) \to (0,\infty)$ is called:*

(i) multiplicative on I, if

$$\Phi(uv) = \Phi(u)\Phi(v), \quad u, v \in I; \tag{1}$$

(ii) submultiplicative on I, if

$$\Phi(uv) \le \Phi(u)\Phi(v) \quad u, v \in I, \tag{2}$$

(iii) supermultiplicative on I, if the reversed inequality holds.

Setting $u = v = 1$, respectively, in (1) and (2) leads to

Remark 1. Let $\Phi : (0,\infty) \to (0,\infty)$.

(i) If Φ is multiplicative on $(0,\infty)$ then $\Phi(1) = 1$.

(ii) If Φ is submultiplicative on $(0,\infty)$ $\Phi(1) \ge 1$.

Let us note the following

Proposition 1. *If a function* $\Phi : (0, \infty) \to (0, \infty)$ *is submultiplicative on* $(0, \infty)$ *and*

$$\Phi(u) \le \frac{1}{\Phi\left(\frac{1}{u}\right)}, \quad u \in [1, \infty), \tag{3}$$

then Φ *is multiplicative on* $(0, \infty)$.

Proof. From Remark 1 we have $1 \le \Phi(1)$. Hence, the submultiplicativity of Φ implies that for all $u > 0$,

$$1 \le \Phi\left(\frac{1}{u}u\right) \le \Phi\left(\frac{1}{u}\right)\Phi(u),$$

whence

$$\Phi(u) \ge \frac{1}{\Phi\left(\frac{1}{u}\right)}, u \ge 1.$$

This inequality and (3) imply that, for all $u \ge 1$

$$\Phi(u) = \frac{1}{\Phi\left(\frac{1}{u}\right)}, u \in [1, \infty),$$

whence, obviously,

$$\Phi(u) = \frac{1}{\Phi\left(\frac{1}{u}\right)}, u \in (0, \infty). \tag{4}$$

Applying in turn: the submultiplicativity of Φ; twice (4); the submultiplicativity of Φ; and again (4), we get, for all $u, v \in (0, \infty)$,

$$\Phi(uv) \le \Phi(u)\Phi(v) = \frac{1}{\Phi\left(\frac{1}{u}\right)\Phi\left(\frac{1}{v}\right)} \le \frac{1}{\Phi\left(\frac{1}{uv}\right)} = \Phi(uv).$$

■

Remark 2. If $\Phi : (0, \infty) \to (0, \infty)$ multiplicative the graph of Φ is not dense in $(0, \infty)^2$ or Φ is Lebesgue measurable, then there is $p \in \mathbb{R}$ such that

$$\Phi(u) = u^p, \quad u \in (0, \infty).$$

The theory of subadditive function (cf. Hille-Phillips [2], Kuczma [6]) leads to the following

Remark 3. If $\Phi : (0, \infty) \to (0, \infty)$ is submultiplicative continuous at 1 and $\Phi(1) \le 1$ then Φ is continuous.

Similarly, making use of the main result of [7] one gets the following

Remark 4. If $\Phi : (1, \infty) \to (1, \infty)$ is one-to-one, submultiplicative on $(1, \infty)$ and $\lim_{u \to 1+} \Phi(u) = 1$, then Φ is continuous.

Example 1. Let $p, q \in (0, \infty)$, $0 < q \leq 1 \leq p$ be arbitrarily fixed. Then the function $\Phi : (0, \infty) \to (0, \infty)$ defined by

$$\Phi(u) := \begin{cases} u^q & \text{if} \quad u \in (0, 1) \\ u^p & \text{if} \quad u \in [1, \infty) \end{cases}$$

is submultiplicative on $(0, \infty)$.

3. Submultiplicativity in the sense of Krasnoselskij and Rutickij

In Krasnoselskij and Rutickij [5], Hudzik and Maligranda [3]), a function $\Phi : [0, \infty) \to \mathbb{R}$ is referred to as submultiplicative on $[0, \infty)$, if it satisfies the following condition:

there exists a positive constant K such that

$$\Phi(uv) \leq K\Phi(u)\Phi(v) \quad \text{for all} \quad u, v \geq 0. \tag{5}$$

Let us note the following obvious

Remark 5. Every nonpositive function $\Phi : [0, \infty) \to \mathbb{R}$ satisfies inequality (5) with arbitrary $K \geq 0$.

Remark 6. Let $\Phi : [0, \infty) \to \mathbb{R}$ be an arbitrary function satisfying (5) with some $K \geq 0$.

If $\Phi(u_0) = 0$ for some $u_0 > 0$ then $\Phi(u) \leq 0$ for all $u \geq 0$.

Proof. For every $u \geq 0$, making use of (5), we have

$$\Phi(u) = \Phi\left(u_0 \frac{u}{u_0}\right) \leq K\Phi(u_0)\Phi(u_0) = 0.$$

∎

Replacing in (5): "$u, v \geq 0$" by "$u, v > 0$" we obtain a weaker condition than (5). Moreover, as the interval $(0, \infty)$ is a multiplicative group, the set of all positive real numbers seems to be more convenient in examination of submultiplicativity than $[0, \infty)$.

Taking into account the above remarks, one can propose the following

Definition 2. *A function $\Phi : (0, \infty) \to [0, \infty)$ is submultiplicative in the Krasnoselskij-Rutickij sense on $(0, \infty)$, if there exists a positive constant K such that*

$$\Phi(uv) \leq K\Phi(u)\Phi(v), \quad u, v \in (0, \infty). \tag{6}$$

Theorem 1. *Let* $\Phi : (0, \infty) \to [0, \infty)$. *Then*

(i) Φ *is submultiplicative in the Krasnoselskij-Rutickij sense on* $(0, \infty)$ *if, and only if, for some positive real* K, *the function* $K\Phi$ *is submultiplicative in the classical sense (Definition 1,* (ii)*);*

(ii) *if* Φ *is submultiplicative, then for every* $K \geq 1$, *the function* $K\Phi$ *is submultiplicative;*

(iii) *if* $K \in (0, 1]$ *and* $K\Phi$ *is submultiplicative, then* Φ *is submultiplicative;*

(iv) *if* $K > 1$ *and* $K\Phi$ *is submultiplicative, then* Φ *need not be submultiplicative.*

Proof. To show (i) note that inequality (6) is equivalent to the inequality

$$K\Phi(uv) \leq [K\Phi(u)][K\Phi(v)] \quad \text{for all} \quad u, v \geq 0,$$

that is equivalent to the sumultiplicativity of the function $K\Phi$.

(ii) and (iii) are easy to verify.

To prove (iv) take arbitrary $K > 1$ and $p \in \mathbb{R}$, an consider the function $\Phi : (0, \infty) \to (0, \infty)$, $\Phi(u) := \frac{1}{K} u^p$. Of course the function $K\Phi(u) = u^p$, being multiplicative, is submultiplicative. Since the inequality (6) holds iff $\frac{1}{K}(uv)^p \leq \frac{1}{K^2} u^p v^p$ for all $u, v > 0$, that is iff $K \leq 1$, the function Φ is not submultiplicative. ■

Remark 7. Hudzik and Maligranda [3] gave a negative answer to the question posed in [5], p. 301, whether or not for any Orlicz function Φ which is submultiplicative at infinity in the Krasnoselskij-Rutickij sense (i.e. such that $\Phi(uv) \leq K\Phi(u)\Phi(v)$ for all $u, v \geq u_0$ for some positive K and nonnegative u_0) there exists an Orlicz function Ψ which is equivalent to Φ at infinity, submultiplicative on $[0, \infty)$, and such that

$$\lim_{u \to 0} \frac{\Psi(u)}{u} = 0.$$

4. Remark on submultiplicative seminorms on Banach algebra

Let X be an algebra over the real or complex numbers $\mathbb{K}$. A seminorm on X is a function $s : X \to [0, \infty)$ such that it is homogeneous and subadditive, i.e.

$$s(tx) = |t|\, s(x), \quad s(x + y) \leq s(x) + s(y)$$

for all $t \in \mathbb{K}$ and $x, y \in X$. A seminorm s is called submultiplicative, if

$$s(xy) \leq s(x)\, s(y), \quad x, y \in X.$$

We prove the following

Theorem 2. *Let X be an algebra over $\mathbb{K}$. If $p : X \to \mathbb{R}$ is a function such that*

$$p(tx) \leq |t|\, p(x), \quad x \in X, \;\; t \in \mathbb{K}; \tag{7}$$

$$p(x+y) \leq p(x) + p(y), \quad x, y \in X, \tag{8}$$

and there is a positive constant λ such that

$$p(xy) \leq \lambda p(x)\, p(y), \quad x, y \in X, \tag{9}$$

then $s := \lambda p$ is a submultiplicative seminorm on X.

Proof. Take arbitrary $t \in \mathbb{K}$, $t \neq 0$ and $x \in X$. Replacing t by $\frac{1}{t}$ and x by tx in inequality (7) we get $|t|\, p(x) \leq p(tx)$ so, taking into account (7), we get

$$p(tx) = |t|\, p(x) \tag{10}$$

for all $x \in X$, $t \in \mathbb{K}$, $t \neq 0$.

From (7), for $t = 0$ and $x \in X$ we have $p(0x) = p(\mathbf{0}) \leq 0$. On the other hand, from (8) with $x = y = \mathbf{0}$, we get $0 \leq p(\mathbf{0})$. So, equality (10) holds for all $x \in X$, $t \in \mathbb{K}$, which proves that p is homogeneous.

Hence, applying in turn: subadditivity of p an homogeneity, we get, for all $x \in X$,

$$0 = p(\mathbf{0}) = p(x + (-x)) \leq p(x) + p(-x) = 2p(x),$$

which shows that $p : X \to [0, \infty)$, that is p is nonnegative.

Since λ is positive, clearly, the function $s := \lambda p$ is nonnegative, homogeneous and subadditive, so s is a seminorm on X. Moreover, multiplying both sides of inequality (9) by λ, we get

$$(\lambda p)(xy) \leq (\lambda p)(x)\,(\lambda p)(y), \quad x, y \in X,$$

that is

$$s(xy) \leq s(x)\, s(y), \quad x, y \in X,$$

which shows that s is submultiplicative. ■

Remark 8. Let $p : X \to \mathbb{K}$ be a nonzero seminorm on a Banach algebra X such that the set

$$\left\{ \frac{p(x^2)}{[p(x)]^2} : \; x \in X, \, p(x) \neq 0 \right\}$$

is a singleton $\{\lambda\}$. Then $s = \lambda p$ is a submultiplicative seminorm on X.

Proof. By Theorem 1 the function $s := \lambda p$ is seminorm. By the definition of λ we have, for all $x \in X$,

$$\left(x^2\right) = \lambda p\left(x^2\right) = \lambda\left(\lambda\left[p(x)\right]^2\right) = \left(\lambda\left[p(x)\right]\right)\left(\lambda\left[p(x)\right]\right) = \left[s\left(x\right)\right]^2,$$

which shows that s has the so called square property. In view of theorem of Bhatt [1], the seminorm s is submultiplicative. ■

References

[1] Bhatt S.J., A seminorm with square property on a Banach algebra is submultiplicative, *Proc. Amer. Math. Soc.* (2), 117(1993), 435-438.

[2] Hille E., Phillips R.S., Functional analysis and semi-groups, *American Mathematical Society Colloquium Publications,* vol. 31. American Mathematical Society, Providence, R. I., 1957.

[3] Hudzik H., Maligranda L., Some remarks on submultiplicative Orlicz functions, *Indag. Mathem., N.S.,* 3(1992), 313-321.

[4] Hudzik H., Mastyło M., Maligranda L., Persson L.E., Extension of submultiplicativity and supermultiplicativity of Orlicz functions, *Real Analysis Exchange,* 24(2)(1998/9), 567-578.

[5] Krasnoselskij M.A., Rutickij Ya.B., Convex functions and Orlicz spaces, *Groningen,* 1961.

[6] Kuczma M., *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality,* Prace Naukowe Uniwersytetu Śląskiego w Katowicach [Scientific Publications of the University of Silesia], 489. Uniwersytet Śląski, Katowice, Państwowe Wydawnictwo Naukowe (PWN), Warsaw, 1985.

[7] Matkowski J., Świątkowski T., On subadditive functions, *Proc. Amer. Math. Soc.,* (1)(119)(1993), 187-197.

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