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THE EQUIVALENCE OF CONVERGENCE RESULTS BETWEEN MODIFIED ISHIKAWA AND MODIFIED MANN ITERATIONS *

Abstract. In [11], the author discussed a new class of nearly weak uniformly L-Lipschitzian mappings and prove some strong convergence results of the modified Ishikawa iteration with errors in real Banach spaces. And the author has given the open problem as follows: Are there any difference on convergence between the Mann iteration and Ishikawa iteration? Can we prove the equivalence on convergence between these two iterations? In this paper, we given an affirmative answer to the open problem.

Key words: modified Mann iteration process with errors, Modified Ishikawa iteration process with error, Banach space, fixed point, nearly weak uniformly L-Lipschitzian.

AMS Mathematics Subject Classification: 47H09, 47H10, 46A03L.

1. Introduction

Let X be an arbitrary real normed space with the dual X^*. We denote by J the normalized duality mapping from X into 2^{X^*} by

$$J(x) = \{f \in \mathrm{X}^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where $\langle ., .\rangle$ denotes the generalized duality pairing between elements of X and X^*. We first recall and define some concepts as follows (see [15]).

Definition 1. *Let K be a subset of a real normed linear space X and $\{\sigma_n\}_{n\geq 1}$ be a sequence in $[0, \infty)$ such that $\lim_{n\to\infty} \sigma_n = 0$. A mapping $T : K \to K$ is said to be nearly Lipschitzian with respect to the sequence $\{\sigma_n\}$ if for each $n \in N$, there exists a constant $k_n \geq 1$ such that*

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + \sigma_n), \quad \forall\ x, y \in K. \tag{1}$$

Observe that for any sequence $\{k_n\}_n \geq 1$ satisfying (1), $\eta(T^n) \leq k_n$ $\forall n \in N$. $\eta(T^n)$ is the infimum of k_n and is called the nearly Lipschitz constant of the mapping T. A nearly Lipschitzian mapping T with sequence

* This paper is dedicated to my dearest mother Mrs. F. M. Mogbademu on her 75th Birthday.

$\{(\sigma_n, \eta(T^n)\}$ is said to be nearly uniformly L-Lipschitzian if $k_n = L$ for all $n \in N$, i.e., if

$$\|T^n x - T^n y\| \le L(\|x-y\| + \sigma_n), \quad \forall\ x, y \in K.$$

The class of nearly uniformly L-Lipschitzian have been studied extensively by many authors: for results in this area, see e.g. Kim et al. [6], Mogbademu [8] and Sahu[15]; for uniformly L-Lipschitzian mappings, see e.g., Chang [2], Chang et al. [3] and Ofoedu [13]; see also Mogbademu [9] and the references there in. In [10], one of the authors introduced the following new concept which generalize the notion of nearly uniformly L-Lipschitzian mappings:

Definition 2. *Let K be a subset of a real normed linear space X and $\{\sigma_n\}_{n\ge 1}$ be a sequence in $[0, \infty)$ such that $\lim_{n\to\infty} \sigma_n = 0$. A mapping $T : K \to K$ is called nearly weak uniformly Lipschitzian with respect to the sequence $\{\sigma_n\}$ if for each $n \in N$, there exists a constant $L \ge 1$ such that*

$$\|T^n x - T^n y\| \le L(\|x-y\| + \sigma_n), \quad \forall\ x \in K,\ y \in F(T). \tag{2}$$

It is clear that the class of nearly weak uniformly L-Lipschitzian mappings is a generalization of the class of nearly uniformly L-Lipschitzian mappings which inturn is a generalization of the class of uniformly L-Lipschitzian mappings (see also [9]).

It is well known that the modified Ishikawa (see [5]) and modified Mann (see [7]) iterations with errors for a mapping $T : K \to K$ are defined respectively as:

For arbitrary $x_1 \in K$,

$$\begin{aligned} x_{n+1} &= (1 - a_n - c_n)x_n + a_n T^n y_n + c_n u_n, \\ y_n &= (1 - b_n - d_n)x_n + b_n T^n x_n + d_n v_n, \quad n \ge 1, \end{aligned} \tag{3}$$

where $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ are real sequences in $[0, 1]$ satisfying $a_n + c_n \le 1$, $b_n + d_n \le 1$ and $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ are two bounded sequences in K, and for $z_1 \in K$,

$$z_{n+1} = (1 - a_n - c_n)z_n + a_n T^n z_n + c_n u_n, \quad n \ge 1. \tag{4}$$

The sequences $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ satisfy

$$\sum_{n\ge 1} a_n = \infty, \quad c_n = o(a_n). \tag{5}$$

The following problem was given in [11]: Can we prove the equivalence on convergence between the iterations (3) and (4) for the most general class of nearly weak uniformly L-Lipschitzian mappings? For results on equivalence, see e.g., Banarjee and Choudhury [1] and Rhoades and Soltuz [14]. The purpose of this paper is to give an affirmative answer for the question.

Lemma 1 ([5]). *Let X be real Banach Space and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then, for any $x, y \in X$*

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y, j(x+y) >, \quad \forall j(x+y) \in J(x+y).$$

Lemma 2 ([12]). *Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing function with $\Phi(x) = 0 \Leftrightarrow x = 0$ and let $\{b_n\}_{n=1}^{\infty}$ be a positive real sequence satisfying*

$$\sum_{n=1}^{\infty} b_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a nonnegative real sequence. If there exists an integer $N_0 > 0$ satisfying

$$a_{n+1}^2 < a_n^2 + o(b_n) - b_n \Phi(a_{n+1}), \quad \forall n \geq N_0$$

where $\lim_{n\to\infty} \frac{o(b_n)}{b_n} = 0$, then $\lim_{n\to\infty} a_n = 0$.

2. The Main results

Now, we are in a position to introduce and prove the main results of this paper.

Theorem 1. *Let X be a real Banach space, K be a nonempty closed convex subset of X, and $T : K \to K$ be a nearly weak uniformly L-Lipschitzian mapping with $\rho \in F(T) = \{\rho \in K : T\rho = \rho\} \neq \emptyset$ and sequences $\{\sigma_n\}_{n=1}^{\infty}$, $k_n \subset [1,\infty)$ and μ_n be such that $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} \mu_n = 0$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ satisfy (5), and $x_1 = z_1 \in K$. Suppose that there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that*

$$(*) \qquad \begin{aligned} &< T^n x_{n+1} - T^n z_{n+1}, j(x_{n+1} - z_{n+1}) > \\ &\qquad \leq k_n \|x_{n+1} - z_{n+1}\|^2 - \Phi(\|x_{n+1} - z_{n+1}\|) + \mu_n, \end{aligned}$$

for all $n \geq 0$, where $j(x_{n+1} - z_{n+1}) \in J(x_{n+1} - z_{n+1})$, then the following are equivalent:

(i) the modified Mann iteration with errors (4) converges (to $\rho \in F(T)$),

(ii) the modified Ishikawa iteration with errors (3) converges (to $\rho \in F(T)$).

Proof. It is obvious that (ii) implies (i) by setting, $b_n = d_n = 0$, for all $n \geq 1$ in equation (4). We will prove that (i) implies (ii). Let ρ be the fixed point of T. Suppose that $\lim_{n\to\infty} z_n = \rho$. Applying

$$(6) \qquad \lim_{n\to\infty} ||x_n - z_n|| = 0,$$

$$0 \le ||\rho - x_n|| \le ||z_n - \rho|| + ||x_n - z_n||, \tag{7}$$

we have

$$\lim_{n\to\infty} x_n = \rho.$$

This completes the proof if (6) is proved. Set $z_n = \rho$, for all $n \geq 1$, then (*) becomes

$$\langle T^n_{n+1} - T^n\rho, j(x_{n+1} - \rho)\rangle \le k_n\|x_{n+1} - \rho\|^2 - \Phi(\|x_{n+1} - \rho\|) + \mu_n, \tag{8}$$

for $\forall n \geq 0$, then this implies that

$$\mu_n + \langle k_n(x_{n+1} - \rho) - (T^n x_{n+1} - \rho), j(x_{n+1} - \rho)\rangle \ge \Phi(\|x_{n+1} - \rho\|), \forall n \ge 0. \tag{9}$$

Firstly, we will prove that there exists $x_1 \in K$ with $x_1 \neq Tx_1$ such that $r_0 = \mu_n + (k_n + L)\|x_1 - \rho\|^2 + L\|x_1 - \rho\|^2 \in R(\Phi)$, where $R(\Phi)$ is the range of Φ. Infact, if $x_1 = Tx_1$, then we are done. Otherwise, there exists the smallest positive integer $n_1 \in N$ such that $x_{n_1} \neq Tx_{n_1}$. We denote $x_{n_1} = x_1$, and then we obtain that $r_0 = \mu_n + (k_n + L)\|x_1 - \rho\|^2 + L\|x_1 - \rho\|^2 \in R(\Phi)$. Indeed, if $\Phi(r) \to +\infty$ as $r \to \infty$, then $r_0 \in R(\Phi)$; If $\sup\{\Phi(r) : r \in [0, \infty]\} = r_1 < +\infty$ with $r_1 < r_0$, then for $\rho \in K$, there exists a sequence $\{\eta_n\}$ in K such that $\eta_n \to \rho$ as $n \to \infty$ with $\eta_n \neq \rho$. Clearly, we have that $T\eta_n \to T\rho$ as $n \to \infty$ thus $\{\eta_n - T\eta_n\}$ is a bounded sequence. So therefore, there exists a natural number n_0 such that $\mu_n + (k_n + L)\|\eta_n - \rho\|^2 + L\|\eta_n - \rho\|^2 < \frac{r_1}{2}$ for $n \geq n_1$, and then we redefine $x_1 = \eta_{n_0}$ to have $\mu_n + (k_n + L)\|x_1 - \rho\|^2 + L\|x_1 - \rho\|^2 \in R(\Phi)$.

Secondly, we will show that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence using induction process. Set $R = \Phi^{-1}(r_0)$, then from (9), we obtain that $\|x_1 - \rho\| \le R$. Denote $B_1 = \{x \in K : \|x - \rho\| \le R\}$, $B_2 = \{x \in K : \|x - \rho\| \le 2R\}$, $M^* = \sup_n\{\|u_n - \rho\| + \|v_n - \rho\|\}$. Now, we want to prove that $x_n \in B_1$. If $n = 1$, then $x_1 \in B_1$. Now, assume that it holds for some n, that is, $x_n \in B_1$. Suppose that, it is not the case, then $\|x_{n+1} - \rho\| > R$.

Denote

$$\begin{aligned} \tau_0 = \min\Big\{ & 1, \frac{R}{8R(1+L)}, \frac{\Phi(R)}{8R(1+L)M}, \\ & \frac{\Phi(R)}{16(1+L)R(2(1+L)R + (ML + M^*))}, \\ & \frac{\Phi(R)}{(3R^2 + 4MR + 2)}, \frac{R}{(L(R+M) + M^*)}\Big\}. \end{aligned} \tag{10}$$

Since $\{\sigma_n\} \in [0, \infty)$ with $\sigma_n \to 0$ as $n \to \infty$, set $M = \sup_n\{\sigma_n : n \in N\}$. Since $\lim_{n\to\infty} a_n, b_n, c_n, d_n, \mu_n = 0$ and $\lim_{n\to\infty} k_n = 1$, without loss of generality, we assume that $0 \le a_n, b_n, c_n, d_n, \frac{c_n}{a_n}, k_n - 1, \mu_n \le \tau_0$, for any

$n \geq 1$. Given $c_n = o(a_n)$, we denote $c_n < a_n \tau_0$ for any $n \geq 1$. Observe that if $x_n \in B_1$, we get $y_n \in B_2$. That is

$$\begin{aligned}\|y_n - \rho\| &\leq (1 - b_n - d_n)\|x_n - \rho\| + b_n\|T^n x_n - T^n \rho\| + d_n\|v_n - \rho\| \\ &\leq (1 - b_n - d_n)\|x_n - \rho\| + b_n L(\|x_n - \rho\| + \sigma_n) + d_n\|v_n - \rho\| \\ &\leq \|x_n - \rho\| + \tau_0(L(\|x_n - \rho\| + M) + M^*) \\ &\leq R + \tau_0(L(R + M) + M^*) \leq 2R.\end{aligned}$$

From (3), we have the following estimates

$$\begin{aligned}\|x_{n+1} - \rho\| &\leq (1 - a_n - c_n)\|x_n - \rho\| \\ &\quad + a_n\|T^n y_n - T^n \rho\| + c_n\|u_n - \rho\| \\ &\leq (1 - a_n - c_n)\|x_n - \rho\| + a_n L(\|y_n - \rho\| + \sigma_n) + c_n\|u_n - \rho\| \\ &\leq R + \tau_0(L(2R + M) + M^*) < 2R,\end{aligned}$$

$$\begin{aligned}\|x_{n+1} - y_n\| &\leq a_n||x_n - T^n y_n|| + b_n||x_n - T^n x_n|| \\ &\quad + c_n||u_n - x_n|| + d_n||v_n - x_n|| \\ &\leq a_n(||x_n - \rho|| + L(||y_n - \rho|| + \sigma_n)) \\ &\quad + b_n(||x_n - \rho|| + L(||x_n - \rho|| + \sigma_n)) \\ &\quad + c_n(||u_n - \rho) + ||x_n - \rho||) \\ &\quad + d_n(||v_n - \rho) + ||x_n - \rho||) \\ &\leq a_n(R + L(2R + M)) + b_n(R + L(2R + M)) \\ &\quad + c_n(M^* + R) + d_n(M^* + R) \\ &= ((a_n + b_n)(1 + 2L) + (c_n + d_n))R \\ &\quad + (a_n + b_n)ML + (c_n + d_n)M^* \\ &\leq 2\tau_o(2(1 + L)R + (ML + M^*)) \leq \frac{\Phi(R)}{8R}.\end{aligned} \tag{11}$$

So,

$$\begin{aligned}\|T^n x_{n+1} - T^n y_n\| &\leq L(\|x_{n+1} - y_n\| + \sigma_n) \\ &\leq L(\frac{\Phi(R)}{8R} + M) \\ &\leq L(\frac{\Phi(R)}{8R(1 + L)} + M\tau_o) \\ &\leq L\frac{\Phi(R)}{4R(1 + L)}.\end{aligned} \tag{12}$$

Using Lemma 1 and the above estimates, we get

$$\begin{aligned}\|x_{n+1} - \rho\|^2 &= \|(1 - a_n - c_n)(x_n - \rho) \\ &\quad + a_n(T^n y_n - \rho) + c_n(u_n - \rho)\|^2\end{aligned} \tag{13}$$

$$
\begin{aligned}
&\leq (1-a_n)^2\|x_n-\rho\|^2 + 2a_n < T^n y_n - \rho, j(x_{n+1}-\rho) > \\
&\quad + 2c_n < u_n - x_n, j(x_{n+1}-\rho) > \\
&= (1-a_n)^2\|x_n-\rho\|^2 + 2a_n < T^n x_{n+1} - \rho, j(x_{n+1}-\rho) > \\
&\quad + 2a_n < T^n y_n - T^n x_{n+1}, j(x_{n+1}-\rho) > \\
&\quad + 2c_n < u_n - x_n, j(x_{n+1}-\rho) > \\
&\leq (1-a_n)^2\|x_n-\rho\|^2 \\
&\quad +2a_n(k_n\|x_{n+1}-\rho\|^2 - \Phi(\|x_{n+1}-\rho\|) + \mu_n) \\
&\quad +2a_n(\|T^n x_{n+1} - T^n y_n\|)\|x_{n+1}-\rho\| \\
&\quad +2c_n\|u_n - x_n\|\|x_{n+1}-\rho\| \\
&\leq (1-a_n)^2 R^2 + 2a_n(k_n\|x_{n+1}-\rho\|^2 - \Phi(R) + \mu_n) \\
&\quad +2a_n\frac{\Phi(R)}{4R}2R + 2c_n M2R \\
&\leq (1-a_n)^2 R^2 + 2a_n(k_n R^2 - \Phi(R) + \mu_n) \\
&\quad +2a_n\frac{\Phi(R)}{2} + 2c_n M2R \\
&\leq R^2 + 2a_n[(k_n-1) + \frac{a_n}{2}]R^2 - 2a_n\Phi(R) + 2a_n\mu_n \\
&\quad +2a_n\frac{\Phi(R)}{2} + 4c_n MR \\
&\leq R^2 + 2a_n[\tau_0 + \frac{\tau_0}{2}]R^2 - 2a_n\Phi(R) + 2a_n\tau_0 \\
&\quad +2a_n\frac{\Phi(R)}{2} + 4a_n\tau_0 MR \\
&\leq R^2 - 2a_n\Phi(R) + 2a_n\frac{\Phi(R)}{2} + a_n\tau_0(3R^2+4MR+2) \leq R^2,
\end{aligned}
$$

which is a contradiction with the assumption $\|x_{n+1}-\rho\| > R$. Hence $x_{n+1} \in B_1$, i.e., the sequence $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence. From above estimate, the sequence $\{y_n\}_{n=1}^{\infty}$ is also bounded. Since $||z_n - \rho|| \to 0$ as $n \to \infty$, without loss of generality, we let $||z_n - \rho|| \leq 1$. Therefore $\{||x_n - z_n||\}$ is also a bounded sequence.

Thirdly, we will prove that $\|x_n - z_n\| \to 0$ as $n \to \infty$.

Denote

$$
\begin{aligned}
M_0 &= \sup_n\{\|x_n-\rho\|\} + \sup_n\{\|y_n-\rho\|\} \\
&\quad + \sup_n\{\|u_n-\rho\|\} + \sup_n\{\|v_n-\rho\|\} + \sup_n\{\|x_n-z_n\|\} + \sup_n\{\sigma_n\}
\end{aligned}
$$

Employing Lemma 2, (12) and (13), we have

$$
\begin{aligned}
\|x_{n+1}-z_{n+1}\|^2 &\leq (1-a_n)^2\|x_n-z_n\|^2 \\
&\quad + 2a_n < T^n y_n - z_n, j(x_{n+1}-z_{n+1}) >
\end{aligned} \tag{14}
$$

$$\begin{aligned}
&= (1-a_n)^2\|x_n - z_n\|^2 \\
&\quad + 2a_n < T^n x_{n+1} - T^n z_{n+1}, j(x_{n+1} - z_{n+1}) > \\
&\quad + 2a_n < T^n y_n - T^n x_{n+1}, j(x_{n+1} - z_{n+1}) > \\
&\quad + 2a_n < T^n z_{n+1} - T^n z_n, j(x_{n+1} - z_{n+1}) > \\
&\leq (1-a_n)^2\|x_n - z_n\|^2 \\
&\quad + 2a_n(k_n\|x_{n+1} - z_{n+1}\|^2 - \Phi(\|x_{n+1} - z_{n+1}\|) + \mu_n) \\
&\quad + 2a_n(\|T^n x_{n+1} - T^n y_n\|)\|x_{n+1} - z_{n+1}\| \\
&\quad + 2a_n(\|T^n z_{n+1} - T^n z_n\|)\|x_{n+1} - z_{n+1}\| \\
&\leq (1-a_n)^2\|x_n - z_n\|^2 \\
&\quad + 2a_n(k_n\|x_{n+1} - z_{n+1}\|^2 - \Phi(\|x_{n+1} - z_{n+1}\|) + \mu_n) \\
&\quad + 2a_n L(\|x_{n+1} - y_n\| + \sigma_n)\|x_{n+1} - z_{n+1}\| \\
&\quad + 2a_n L(\|z_{n+1} - z_n\| + \sigma_n)\|x_{n+1} - z_{n+1}\|.
\end{aligned}$$

$$\begin{aligned}
\text{(15)} \qquad \|x_{n+1} - y_n\| &= ||(1-b_n-d_n)x_n + b_n T^n x_n + d_n v_n \\
&\quad - (1-a_n-c_n)x_n - a_n T^n y_n - c_n u_n)|| \\
&\leq a_n||x_n - T^n y_n|| + b_n||x_n - T^n x_n|| \\
&\quad + d_n||v_n - x_n|| + c_n||x_n - u_n|| \\
&\leq a_n(||x_n - \rho|| + L(||x_n - \rho|| + \sigma_n)) \\
&\quad + b_n(||x_n - \rho|| + L(||x_n - \rho|| + \sigma_n)) \\
&\quad + d_n(||v_n - \rho|| + ||x_n - \rho||) \\
&\quad + c_n(||x_n - \rho|| + ||u_n - \rho||)
\end{aligned}$$

Observe that

$$\begin{aligned}
\text{(16)} \qquad &\leq ((1+L)a_n + (1+L)b_n + c_n + d_n)||x_n - \rho|| \\
&\quad + (a_n + b_n)\sigma_n L + (c_n + d_n)M^* \\
&= A_n||x_n - \rho|| + B_n \\
&\leq A_n(||x_n - z_n|| + ||z_n - \rho||) + B_n \\
&\leq A_n(||x_n - z_n|| + 1) + B_n,
\end{aligned}$$

where

$$A_n = (1+L)a_n + (1+L)b_n + c_n + d_n$$

and

$$B_n = (a_n + b_n)\sigma_n L + (c_n + d_n)M^*.$$

In a similar way,

$$\begin{aligned}
\text{(17)} \qquad \|z_{n+1} - z_n\| &= ||(1-a_n-c_n)z_n + a_n T^n z_n + c_n u_n - z_n|| \\
&\leq a_n||T^n z_n - z_n|| + c_n||u_n - z_n||
\end{aligned}$$

$$
\begin{aligned}
&\leq a_n(||z_n-\rho||+L(||z_n-\rho||+\sigma_n)) \\
&\quad + c_n(||z_n-\rho||+||u_n-\rho||) \\
&= (a_n(1+L)+c_n)||z_n-\rho||+a_nL\sigma_n+c_nM^* \\
&= C_n||z_n-\rho||+D_n \\
&\leq C_n+D_n,
\end{aligned}
$$

where

$$C_n=(a_n(1+L)+c_n) \quad \text{and} \quad D_n=a_nL\sigma_n+c_nM^*.$$

Substituting equations (15) and (16) in (14), we obtain

$$
\begin{aligned}
(18)\qquad & \|x_{n+1}-z_{n+1}\|^2 \\
&\leq (1-a_n)^2\|x_n-z_n\|^2 \\
&\quad + 2a_n(k_n\|x_{n+1}-z_{n+1}\|^2-\Phi(\|x_{n+1}-z_{n+1}\|)+\mu_n) \\
&\quad + 2a_nL(A_n(||x_n-z_n||+1)+B_n+\sigma_n)\|x_{n+1}-z_{n+1}\| \\
&\quad + 2a_nL(C_n+B_n+\sigma_n)\|x_{n+1}-z_{n+1}\| \\
&\leq (1-a_n)^2\|x_n-z_n\|^2 \\
&\quad + 2a_n(k_n\|x_{n+1}-z_{n+1}\|^2-\Phi(\|x_{n+1}-z_{n+1}\|)+\mu_n) \\
&\quad + 2a_nLA_n||x_n-z_n||\|x_{n+1}-z_{n+1}\| \\
&\quad + 2a_nL(A_n+B_n+\sigma_n)\|x_{n+1}-z_{n+1}\| \\
&\quad + 2a_nL(C_n+B_n+\sigma_n)\|x_{n+1}-z_{n+1}\| \\
&\leq ||x_n-z_n||^2+2a_n(k_N-1)M_o^2+a_n^2M_o \\
&\quad - 2a_n\Phi(||x_{n+1}-z_{n+1}) \\
&\quad + 2a_n\mu_n+2a_nA_nLM_o+(A_n+B_n+\sigma_n)2a_nL \\
&\quad + (C_n+D_n+\sigma_n)2a_nLM_o \\
&\leq ||x_n-z_n||-a_n\Phi(||x_{n+1}-z_{n+1})+o(a_n),
\end{aligned}
$$

where

$$
\begin{aligned}
(19)\qquad o(a_n) &= 2a_n(k_n-1)M_o^2+a_n^2M_o+2a_n\mu_n+2a_nA_nLM_o \\
&\quad + (A_n+B_n+\sigma_n)2a_nL+(C_n+D_n+\sigma_n)2a_nLM_o.
\end{aligned}
$$

By Lemma 2, we obtain $||x_n-z_n||=0$ as $n\to\infty$. That is, $||x_n-z_n||\to 0$ as $n\to\infty$. From the inequalities $0\leq ||x_n-\rho||\leq ||x_n-z_n||+||z_n-\rho||$, we get $||x_n-\rho||\to 0$ as $n\to\infty$. This completes the proof. ■

Remark 1. Theorem 1 also holds if the nearly weak uniformly L-Lipschitzian mapping (see [6], [9]) and the uniformly L-Lipschitzian mapping ([4], [13]).

3. An equivalence result of the T- stability

The concepts employed for the equivalence of the T-stabilities between modified Ishikawa and modified Mann iterations with errors are similar to those from [10]. The following sequences are well defined for all $n \geq 1$:

$$\epsilon_n := ||x_{n+1} - (1 - a_n - c_n)x_n - a_n T^n y_n - c_n u_n||, \tag{20}$$

$$\delta_n := ||z_{n+1} - (1 - a_n - c_n)z_n - a_n T^n z_n - c_n u_n||. \tag{21}$$

Definition 3 (see [4]). *If* $\lim_{n\to\infty} \epsilon_n = 0$ (*resp.*, $\lim_{n\to\infty} \delta_n = 0$) *implies that* $\lim_{n\to\infty} x_n = \rho$ (*resp.*, $\lim_{n\to\infty} z_n = \rho$), *then (4) (resp., (3)) is said to be T-stable.*

Theorem 2. *Let X be a real Banach space, K be a nonempty closed convex subset of X, and $T : K \to K$ be a nearly weak uniformly L-Lipschitzian mapping with $\rho \in F(T) = \{\rho \in K : T\rho = \rho\} \neq \emptyset$ and sequences $\{\sigma_n\}_{n=1}^{\infty}$, $k_n \subset [1, \infty)$ and μ_n such that $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} \mu_n = 0$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ satisfy (5), and $x_1 = z_1 \in K$. Suppose that there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that*

$$\begin{aligned} &< T^n x_{n+1} - T^n z_{n+1}, j(x_{n+1} - z_{n+1}) > \\ &\qquad \leq k_n \|x_{n+1} - z_{n+1}\|^2 - \Phi(\|x_{n+1} - z_{n+1}\|) + \mu_n, \end{aligned}$$

for all $n \geq 0$, where $j(x_{n+1} - z_{n+1}) \in J(x_{n+1} - z_{n+1})$, then the following are equivalent:

(i) *the modified Mann iteration with errors (4) is T-stable,*

(ii) *the modified Ishikawa iteration with errors (3) is T-stable.*

Proof. The proof follows easily using Definition 3 and the assurance of Theorem 1. ■

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