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A GENERAL FIXED POINT THEOREM FOR WEAKLY SUBSEQUENTIALLY CONTINUOUS MAPPINGS

Abstract. In this paper a general fixed point theorem for two pairs of subsequentially mappings compatible of type E is proved, which generalize the results by [2]-[4], [6] and other results. As applications, new results for mappings satisfying contractive conditions of integral type, ϕ-contractive conditions and weak contractive conditions are obtained.

Key words: Fixed point, almost altering distance, implicit relation, weakly subsequentially continuous, compatible mappings of type E.

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1. Introduction

Let (X, d) be a metric space and S, T be two self mappings of X. In [15], Jungck defined S and T to be compatible if

$$\lim_{n \to \infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = u$$

for some $u \in X$.

This concept has been frequently used to prove existence theorems in fixed point theory.

After that, many authors introduced various types of compatibility, compatibility of type (A), (B), (C), (P) for two self mappings in metric spaces respectively in [16], [22], [24], [23].

In [34], [35], M. R. Singh and Y. M. Singh introduced the notions of compatible mappings of type (E) and proved some common fixed point theorems. In [20], Pant introduced the notion of reciprocally continuous functions.

Bouhadjera and Godet [8], [9] introduced the notion of subsequentially continuous functions, which generalize the notions of continuity and reciproc continuity.

Quite recently, Beloul [3]-[5] and Bouhadjera [7] introduced independently the notion of weakly subsequentially continuous functions, which generalize the notion of subsequentially continuous functions.

Some results for two pairs of weakly subsequentially continuous functions in metric spaces are obtained in [2]-[7], [12].

In [10], Branciari proved a general fixed point theorem for a mapping satisfying a contractive condition of integral type.

The notion of altering distance is introduced in [17]. In [29], Popa and Mocanu proved that the study of fixed points for mappings satisfying contractive conditions of integral type is reduced to the study of fixed points for mappings involving altering distances.

The study of fixed points for pairs of mappings satisfying implicit relations is initiated in [26], [27].

A general fixed point theorem for weakly subsequentially mappings and compatible mappings of type (E) using implicit relations is proved in [5].

In this paper a general fixed point theorem for two pairs of weakly subsequentially mappings compatible of type (E) is proved, generalizing the results from [2]-[4] and [6].

2. Preliminaries

First we recall the definitions of some known notions.

Definition 1 ([20]). *A pair $\{A, S\}$ of self mappings of a metric space (X, d) is called reciprocally continuous if*

$$\lim_{n \to \infty} ASx_n = At \quad and \quad \lim_{n \to \infty} SAx_n = St,$$

whenever $\{x_n\}$ is a subsequence in X such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t$ for some $t \in X$.

Remark 1. If two self mappings of (X, d) are continuous, then they are obviously reciprocally continuous, but the converse is not true (Example 2, [21]).

A general fixed point theorem for two pairs of compatible and reciprocally continuous mappings satisfying a implicit relation is proved in [28].

Definition 2 ([8], [9]). *A pair $\{A, S\}$ of self mappings of a metric space (X, d) is called subsequentially continuous if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z$$

for some $z \in X$ and $\lim\limits_{n\to\infty} ASx_n = Az$ and $\lim\limits_{n\to\infty} SAx_n = Sz$.

Remark 2. If two self mappings are continuous or reciprocally continuous, then they are subsequentially continuous. Moreover, there exist subsequentially continuous pairs of mappings which are neither continuous or reciprocally continuous (Example 2.6, [6]).

Two fixed point theorems for pairs of subcompatible and subsequentially continuous functions are proved in [2].

Definition 3 ([3], [4], [7]). *A pair $\{A, S\}$ of self mappings of a metric space (X, d) is said to be weakly subsequentially continuous if there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} ASx_n = Az$ or $\lim\limits_{n\to\infty} SAx_n = Sz$ for $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = z$ for some $z \in X$.*

Remark 3. The subsequentially continuous mappings or reciprocally continuous mappings are weakly subsequentially continuous, but the converse in not true (Example 2.8, [6]).

Definition 4 ([34], [35]). *Two self mappings $\{A, S\}$ of a metric space (X, d) are compatible of type (E) if $\lim\limits_{n\to\infty} S^2 x_n = \lim\limits_{n\to\infty} SAx_n = At$ and $\lim\limits_{n\to\infty} A^2 x_n = \lim\limits_{n\to\infty} ASx_n = St$ whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = t$ for some $t \in X$.*

Remark 4. If $At = St$, then compatible of type (E) implies compatible, compatible of type (A), compatible of type (B), compatible of type (C), compatible of type (P), but the converse is not true. Generally, compatibility of type (E) implies the compatibility of type (B).

Definition 5. *Two self mappings A and S of a metric space (X, d) are said to be S-compatible of type (E) if*

$$\lim_{n\to\infty} S^2 x_n = \lim_{n\to\infty} SAx_n = At,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$.

Definition 6. *Two self mappings A and S of a metric space (X, d) are said to be A-compatible of type (E) if*

$$\lim_{n\to\infty} A^2 x_n = \lim_{n\to\infty} ASx_n = St,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$.

Remark 5. If A and S are compatible of type (E), then they are A-compatible and S-compatible of type (E), but the converse is not true (Example 2.2, [35]).

Definition 7 ([17]). *An altering distance is a function $\psi : [0, \infty) \to [0, \infty)$ satisfying:*
(ψ_1) :   *ψ is increasing and continuous,*
(ψ_2) :   *$\psi(t) = 0$ if and only if $t = 0$.*

Fixed point theorems involving altering distances have been studied in [29], [32], [33] and in other papers.

Definition 8 ([30]). *A function $\psi : [0, \infty) \to [0, \infty)$ is an almost altering distance if:*
(ψ'_1) :   *ψ is continuous,*
(ψ'_2) :   *$\psi(t) = 0$ if and only if $t = 0$.*

Remark 6. Every altering distance is an almost altering distance, but the converse is not true.

Example 1. $\psi(t) = \begin{cases} t, & t \in [0, 1] \\ \frac{1}{t}, & t \in (1, \infty). \end{cases}$

3. Implicit relations

Several fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit function in [26], [27] and in other papers.

Definition 9. *Let $\mathcal{F}$ be the set of all continuous functions $F : \mathbb{R}^6_+ \to \mathbb{R}$ which are satisfying:*
(F_1) :   *F is nondecreasing in variable t_1 and nonincreasing in variables $t_2, t_3, ..., t_6$;*
(F_2) :   *For all $u > 0$, $F(u, u, 0, 0, u, u) > 0$.*

The following theorems in proved in [3].

Theorem 1. *Let (X, d) be a metric space and A, B, S and T be four mappings on X such that for all $x, y \in X$,*

$$F\left(\begin{array}{c} d(Ax, By), d(Sx, Ty), d(Sx, Ax), \\ d(Ty, By), d(Sx, By), d(Ty, Ax) \end{array}\right) \leq 0,$$

for some $F \in \mathcal{F}$. If the pairs $\{A, S\}$ and $\{B, T\}$ are weakly subsequentially continuous and compatible of type (E), then A, B, S and T have a unique common fixed point.

Let $\mathcal{F}^*$ be the family of continuous functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying only the condition (F_2).

Example 2. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}$, where $k \in [0, 1)$.

Example 3. $F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \dfrac{t_5 + t_6}{2}\right\}$, where $k \in [0, 1)$.

Example 4. $F(t_1, ..., t_6) = t_1 - a \max\{t_2, t_3, t_4\} - b(t_5 + t_6)$, where $a, b \geq 0$ and $a + 2b < 1$.

Example 5. $F(t_1, ..., t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)(at_5 + bt_6)$, where $\alpha \in (0, 1)$, $a, b \geq 0$ and $a + b < 1$.

Example 6. $F(t_1, ..., t_6) = (1 - at_1)t_2 - a(t_3 t_4 + t_5 t_6) - \alpha t_1 - (1 - \alpha) \times \max\left\{t_3, t_4, t_5^{1/2}, t_6^{1/2}\right\}$, where $a \geq 0$ and $\alpha \in (0, 1)$.

Example 7. $F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \dfrac{1}{2}(t_5 + t_6), \dfrac{t_3 t_4}{1 + t_2}, \dfrac{t_5 t_6}{1 + t_1}\right\}$, where $k \in [0, 1)$.

Example 8. $F(t_1, ..., t_6) = t_1^2 - at_2^2 - \dfrac{bt_5 t_6}{1 + t_3^2 + t_4^2}$, where $a, b \geq 0$ and $a + b < 1$.

Example 9. $F(t_1, ..., t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5 t_6$, where $a, b, c, d \geq 0$ and $a + d < 1$.

Example 10. $F(t_1, ..., t_6) = (1 + \alpha t_1^p)t_2^p - \alpha(t_3^p t_4^p + t_5^p t_6^p) - \alpha t_1^p - \beta \tau(t_3^p, t_4^p, t_5^p, t_6^p)$, where $\alpha, \beta \geq 0$, $\alpha + \beta < 1$, $p \geq 1$ and $\tau \in \Omega$, where $\Omega = \{\tau :_+^4 \to \mathbb{R}_+ \mid \tau$ continuous and $\tau(0, 0, x, x) = x\}$.

Example 11. $F(t_1, ..., t_6) = (1 + \alpha t_1^p)t_2^p - \lambda(t_3, t_4, t_5, t_6, t_1)$, $\lambda \in \Lambda$, where $\Lambda = \{\lambda : \mathbb{R}_+^5 \to \mathbb{R}_+, \lambda$ is continuous and $\lambda(0, 0, t, t, t) = kt$, $k \in (0, 1)\}$.

Remark 7. In Examples 7-9, 11, $F \in \mathcal{F}^*$ but $F \notin \mathcal{F}$.

The purpose of this paper is to prove a generalization of Theorem 1 for two pairs of weakly subsequentially continuous compatible mappings of type (E) with $F \in \mathcal{F}^*$. As application we obtain unique common fixed points for mappings satisfying contractive conditions of integral type, satisfying φ-contractive type, satisfying (ψ, φ)-contractive type.

Example 12. $F(t_1, ..., t_6) = (1 + at_1^p)t_2^p + \lambda\,(t_3^p, t_4^p, t_5^p, t_6^p, t_1^p)$, where a, λ, p are as in Example 11.

4. Main results

Theorem 2. *Let A, B, S and T be self mappings of a metric space (X, d) such that for all $x, y \in X$,*

$$
(1) \qquad F\left(\begin{matrix} \psi\left(d\left(Ax, By\right)\right), \psi\left(d\left(Sx, Ty\right)\right), \psi\left(d\left(Sx, Ax\right)\right), \\ \psi\left(d\left(Ty, By\right)\right), \psi\left(d\left(Sx, By\right)\right), \psi\left(d\left(Ty, Ax\right)\right) \end{matrix}\right) \le 0,
$$

for some $F \in \mathcal{F}^*$ and ψ is an almost altering distance. If the pair $\{A, S\}$ is weakly subsequentially continuous and compatible of type (E), as well $\{B, T\}$, then A, B, S and T have a unique common fixed point.

Proof. Suppose that $\{A, S\}$ is weakly subsequentially continuous, there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n \to \infty} Ax_n = \lim\limits_{n \to \infty} Sx_n = z$ and $\lim\limits_{n \to \infty} ASx_n = Az$. Also, the pair $\{A, S\}$ is compatible of type (E), so $\lim\limits_{n \to \infty} A^2 x_n = \lim\limits_{n \to \infty} ASx_n = Sz$ and $\lim\limits_{n \to \infty} S^2 x_n = \lim\limits_{n \to \infty} SAx_n = Az$. Hence, $Sz = Az$.

Similarly, if $\{B, T\}$ is weakly subsequentially continuous, then there exists a sequence $\{y_n\}$ in X such that $\lim\limits_{n \to \infty} By_n = \lim\limits_{n \to \infty} Ty_n = t$ and $\lim\limits_{n \to \infty} BTy_n = Bt$. Also, the pair $\{B, T\}$ is compatible of type (E), which implies $\lim\limits_{n \to \infty} B^2 y_n = \lim\limits_{n \to \infty} BTy_n = Tt$ and $\lim\limits_{n \to \infty} T^2 y_n = \lim\limits_{n \to \infty} TBy_n = Bt$. Hence, $Bt = Tt$.

First we prove that $z = t$.

By (1) for $x = x_n$ and $y = y_n$ we obtain

$$
F\left(\begin{matrix} \psi\left(d\left(Ax_n, By_n\right)\right), \psi\left(d\left(Sx_n, Ty_n\right)\right), \psi\left(d\left(Sx_n, Ax_n\right)\right), \\ \psi\left(d\left(Ty_n, By_n\right)\right), \psi\left(d\left(Sx_n, By_n\right)\right), \psi\left(d\left(Ty_n, Ax_n\right)\right) \end{matrix}\right) \le 0.
$$

Letting n tend to infinity we obtain

$$
F\left(\psi\left(d\left(z, t\right)\right), \psi\left(d\left(z, t\right)\right), 0, 0, \psi\left(d\left(z, t\right)\right), \psi\left(d\left(z, t\right)\right)\right) \le 0,
$$

a contradiction of $F \in \mathcal{F}^*$ if $\psi\left(d\left(z, t\right)\right) > 0$. Hence, $\psi\left(d\left(z, t\right)\right) = 0$ which implies $d\left(z, t\right) = 0$, i.e. $z = t$.

Now we prove that $z = Az = Sz$.

By (1) for $x = z$ and $y = y_n$ we have

$$
F\left(\begin{matrix} \psi\left(d\left(Az, By_n\right)\right), \psi\left(d\left(Sz, Ty_n\right)\right), \psi\left(d\left(Sz, Az\right)\right), \\ \psi\left(d\left(Ty_n, By_n\right)\right), \psi\left(d\left(Sz, By_n\right)\right), \psi\left(d\left(Ty_n, Az\right)\right) \end{matrix}\right) \le 0.
$$

Letting n tend to infinity we have

$$F\left(\psi\left(d\left(Az,z\right)\right),\psi\left(d\left(Az,z\right)\right),0,0,\psi\left(d\left(Az,z\right)\right),\psi\left(d\left(Az,z\right)\right)\right)\leq 0,$$

a contradiction of $F\in\mathcal{F}^{*}$ if $\psi\left(d\left(Az,z\right)\right)>0$. Hence $\psi\left(d\left(Az,z\right)\right)=0$ which implies $z=Az=Sz$ and z is a common fixed point of A and S.

Similarly, by (1) for $x=x_{n}$ and $y=z$ we obtain $z=Bz=Tz$. Therefore, $z=Az=Sz=Bz=Tz$ and z is a common fixed point of A,B,S and T.

If there exists a new common fixed point w for A,B,S and T, by (1) we have

$$F\left(\begin{array}{c}\psi\left(d\left(Az,Bw\right)\right),\psi\left(d\left(Sz,Tw\right)\right),\psi\left(d\left(Sz,Az\right)\right),\\\psi\left(d\left(Tw,Bw\right)\right),\psi\left(d\left(Sz,Bw\right)\right),\psi\left(d\left(Az,Tw\right)\right)\end{array}\right)\leq 0,$$

$$F\left(\psi\left(d\left(z,w\right)\right),\psi\left(d\left(z,w\right)\right),0,0,\psi\left(d\left(z,w\right)\right),\psi\left(d\left(z,w\right)\right)\right)\leq 0,$$

a contradiction of $F\in\mathcal{F}^{*}$ if $\psi\left(d\left(z,w\right)\right)>0$. Hence $\psi\left(d\left(z,w\right)\right)=0$ which implies $z=w$. ∎

If $\psi\left(t\right)=t$, by Theorem 2 we obtain

Theorem 3. *Let A, B, S and T be self mappings of a metric space (X,d) such that for all $x,y\in X$*

(2) $$F\left(\begin{array}{c}d\left(Ax,By\right),d\left(Sx,Ty\right),d\left(Sx,Ax\right),\\d\left(Ty,By\right),d\left(Sx,By\right),d\left(Ty,Ax\right)\end{array}\right)\leq 0$$

for some $F\in\mathcal{F}^{*}$.

If the pair $\{A,S\}$ is weakly subsequentially continuous and compatible of type (E), as well $\{B,T\}$, then A, B, S and T have a common fixed point.

Theorem 4. *Let A, B, S and T be self mappings of a metric space (X,d) such that (1) holds for all $x,y\in X$ and some $F\in\mathcal{F}^{*}$. If:*

1) the pair $\{A,S\}$ is S-subsequentially continuous and S-compatible of type (E), and

2) the pair $\{B,T\}$ is T-subsequentially continuous and T-compatible of type (E),

then A, B, S and T have a unique common fixed point.

Proof. The proof it follows by the proof of Theorem 2. ∎

By Theorem 4 for $\psi\left(t\right)=t$, we have

Theorem 5. *Let A, B, S and T be self mappings of a metric space (X,d) such that (2) holds for all $x,y\in X$ and some $F\in\mathcal{F}^{*}$. If:*

1) the pair $\{A,S\}$ is S-subsequentially continuous and S-compatible of type (E), and

2) *the pair $\{B, T\}$ is T-subsequentially continuous and T-compatible of type (E),*
    then A, B, S and T have a unique common fixed point.

Remark 8. 1) Theorem 3 is a generalization of Theorem 1 because $F \in \mathcal{F}^*$ (see Remark 7).
    2) By Theorem 3 and Example 2 we obtain Corollary 4 [4].
    3) By Theorem 3 and Example 11 we obtain Corollary 3.6 [6].
    4) By Theorem 5 and Example 4 we obtain Corollary 3.2 [3].
    5) By Theorem 3 and Example 7 we obtain Theorem 3.1 [2].

5. Applications

5.1. Fixed points for two pairs of mappings satisfying contractive conditions of integral type

In [10], Branciari established the following theorem, which opened the way to the study of fixed points for mappings satisfying a contractive condition of integral type.

Theorem 6. *Let (X, d) be a metric space, $c \in (0, 1)$ and $f : X \to X$ such that for all $x, y \in X$*

$$\int_0^{d(fx, fy)} h(t)dt \le c \int_0^{d(x, y)} h(t)dt,$$

where $h : [0, \infty) \to [0, \infty)$ is a Lebesgue measurable mapping which is summable (i.e. with finite integral) on each compact subset of $[0, \infty)$ such that $\int_0^\varepsilon h(t)dt > 0$, for $\varepsilon > 0$.
    Then, f has a unique fixed point $z \in X$ such that for all $x \in X$, $z = \lim_{n \to \infty} f^n x$.

There exists a vast literature concerning the existence of fixed points for mappings satisfying integral conditions. Quite recently, some fixed points theorems of Gregus type for weakly subsequentially continuous mappings satisfying strict contractive conditions of integral type are published in [6].

Lemma 1. *Let $h : [0, \infty) \to [0, \infty)$ be as in Theorem 6. Then $\psi(t) = \int_0^t h(x)dx$ is an almost altering distance.*

Proof. The proof it follows by Lemma 2.5 [29].   ∎

Theorem 7. *Let A, B, S and T be self mappings of a metric space (X, d) such that for all $x, y \in X$,*

$$(3) \quad F \left(\begin{array}{l} \int_0^{d(Ax, By)} h\,(t)\,dt, \int_0^{d(Sx, Ty)} h\,(t)\,dt, \int_0^{d(Sx, Ax)} h\,(t)\,dt, \\ \int_0^{d(Ty, By)} h\,(t)\,dt, \int_0^{d(Sx, By)} h\,(t)\,dt, \int_0^{d(Ty, Ax)} h\,(t)\,dt \end{array} \right) \leq 0,$$

for some $F \in \mathcal{F}^*$ and $h\,(t)$ as in Theorem 6. If the pair $\{A, S\}$ is weakly subsequentially continuous and compatible of type (E), as well $\{B, T\}$, then A, B, S and T have a unique common fixed point.

Proof. Let $\psi\,(t)$ as in Lemma 1. Then

$$\psi\,(d\,(Ax, By)) = \int_0^{d(Ax, By)} h\,(t)\,dt, \psi\,(d\,(Sx, Ty)) = \int_0^{d(Sx, Ty)} h\,(t)\,dt,$$

$$\psi\,(d\,(Sx, Ax)) = \int_0^{d(Sx, Ax)} h\,(t)\,dt, \psi\,(d\,(Ty, By)) = \int_0^{d(Ty, By)} h\,(t)\,dt,$$

$$\psi\,(d\,(Sx, By)) = \int_0^{d(Sx, By)} h\,(t)\,dt, \psi\,(d\,(Ty, Ax)) = \int_0^{d(Ty, Ax)} h\,(t)\,dt.$$

By (3) we obtain

$$F \left(\begin{array}{l} \psi\,(d\,(Ax, By))\,, \psi\,(d\,(Sx, Ty))\,, \psi\,(d\,(Sx, Ax))\,, \\ \psi\,(d\,(Ty, By))\,, \psi\,(d\,(Sx, By))\,, \psi\,(d\,(Ty, Ax)) \end{array} \right) \leq 0,$$

which is inequality (1). Hence, all the conditions of Theorem 2 are satisfied and the proof of Theorem 7 follows by Theorem 2. ∎

Corollary 1 (Theorem 3.1 [6])**.** *Let A, B, S and T be self mappings of a metric space (X, d) such that for all $x, y \in X$ we have*

$$\left(1 + a \left(\int_0^{d(Ax, By)} h\,(t)\,dt \right)^p \right) \left(\int_0^{d(Sx, Ty)} h\,(t)\,dt \right)^p$$

$$\leq a \left[\left(\int_0^{d(Sx, Ax)} h\,(t)\,dt \right)^p \left(\int_0^{d(Ty, By)} h\,(t)\,dt \right)^p \right.$$

$$\left. + \left(\int_0^{d(Ax, Ty)} h\,(t)\,dt \right)^p \left(\int_0^{d(Sx, By)} h\,(t)\,dt \right)^p \right]$$

$$+ \alpha \left(\int_0^{d(Ax, By)} h\,(t)\,dt \right)^p$$

$$+ \beta \tau \left(\left(\int_0^{d(Sx, Ax)} h\,(t)\,dt \right)^p , \left(\int_0^{d(Ty, By)} h\,(t)\,dt \right)^p , \right.$$

$$\left. \left(\int_0^{d(Ax, Ty)} h\,(t)\,dt \right)^p , \left(\int_0^{d(Ty, Ax)} h\,(t)\,dt \right)^p \right)$$

where $a, \alpha, \beta \geq 0$ such that $\alpha + \beta < 1$, $p \geq 1$, $h(t)$ as in Theorem 6 and $\tau \in \Omega$, where $\Omega = \{\tau : \mathbb{R}_+^4 \to \mathbb{R}_+ \mid \tau \text{ continuous and } \tau(0, 0, t, t) = t\}$. If the pair $\{A, S\}$ is weakly subsequentially continuous and compatible of type (E), as well $\{B, T\}$, then A, B, S and T have a unique common fixed point.

Proof. The proof it follows by Theorem 7 and Example 10.   ∎

Remark 9. Corollary 1 improves Theorem 2 [12], some results by [14] and Theorem 2.5 [25].

Corollary 2 (Corollary 3.3 [6]). *Let A, B, S and T be self mappings of a metric space (X, d) such that*

$$\left(1 + a \int_0^{d(Ax,By)} h(t)\, dt\right) \int_0^{d(Sx,Ty)} h(t)\, dt$$

$$\leq a \int_0^{d(Sx,Ax)} h(t)\, dt \int_0^{d(By,Ty)} h(t)\, dt + a \int_0^{d(Ax,By)} h(t)\, dt$$

$$+ (1 - \alpha) \max\left\{ \int_0^{d(Sx,Ax)} h(t)\, dt, \int_0^{d(By,Ty)} h(t)\, dt, \right.$$

$$\left. \left(\int_0^{d(Ax,Ty)} h(t)\, dt\right)^{1/2} \left(\int_0^{d(Sx,By)} h(t)\, dt\right)^{1/2} \right\}$$

for all $x, y \in X$, where $a \in [0, 1)$ and $h(t)$ as in Theorem 6. If

1) *the pair $\{A, S\}$ is S-subsequentially continuous and S-compatible of type (E), and*

2) *the pair $\{B, T\}$ is T-subsequentially continuous and T-compatible of type (E),*

then A, B, S and T have a unique common fixed point.

Proof. The proof it follows by Theorems 4, 7 and Example 6.   ∎

Corollary 3 (Theorem 3.5 [6]). *Let A, B, S and T be self mappings of a metric space (X, d) such that for all $x, y \in X$,*

$$\left[1 + a\left(\int_0^{d(Ax,By)} h(t)\, dt\right)^p\right] \left(\int_0^{d(Sx,Ty)} h(t)\, dt\right)^p$$

$$\leq \lambda \left(\begin{array}{c} \left(\int_0^{d(Sx,Ax)} h(t)\, dt\right)^p, \left(\int_0^{d(Ty,By)} h(t)\, dt\right)^p, \left(\int_0^{d(Sx,By)} h(t)\, dt\right)^p, \\ \left(\int_0^{d(Ax,Ty)} h(t)\, dt\right)^p, \left(\int_0^{d(Ax,By)} h(t)\, dt\right)^p \end{array} \right)$$

where $p \geq 1$, $h(t)$ as in Theorem 6 and $\lambda \in \Lambda$, where $\Lambda = \{\lambda : \mathbb{R}_+^5 \to \mathbb{R}_+, \lambda$ is continuous and $\lambda(0, 0, x, x, x) = kx$, $k \in (0, 1)\}$. If $\{A, S\}$ and $\{B, T\}$ are subsequentially continuous and compatible of type (E), then A, B, S and T have a unique common fixed point.

Proof. The proof it follows by Theorem 7 and Example 11. ∎

Corollary 4 (Corollary 3.6 [6])**.** *Let A, B, S and T be self mappings of a metric space (X, d) such that for all $x, y \in X$,*

$$(1 + a d^p(Ax, By)) d^p(Sx, Ty) \leq$$
$$\lambda (d^p(Sx, Ax), d^p(Ty, By), d^p(Sx, By), d^p(Ax, Ty), d^p(Ax, By))$$

where $a \in (0, \infty)$, $p \geq 1$, $\lambda \in \Lambda$, where $\Lambda = \{\lambda : \mathbb{R}_+^5 \to \mathbb{R}_+, \lambda \text{ is continuous}$ and $\lambda(0, 0, x, x, x) = kx, \ k \in (0, 1)\}$. If

1) the pair $\{A, S\}$ is S-subsequentially continuous and S-compatible of type (E), and

2) the pair $\{B, T\}$ is T-subsequentially continuous and T-compatible of type (E),

then A, B, S and T have a unique common fixed point.

Proof. The proof it follows by Theorem 5 and Example 12. ∎

5.2. Fixed points for mappings satisfying an φ-contractive condition

In [18], Matkowski initiated the study of fixed points for mappings satisfying contractive conditions of φ-type.

In the following, we denote by Φ be the set of all nondecreasing upper semi - continuous functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

1) $\varphi(0) = 0$,

2) $\varphi(t) < t, \forall t > 0$.

The following mappings are from $\mathcal{F}^*$.

Example 13. $F(t_1, ..., t_6) = t_1 - \varphi(\max\{t_2, t_3, ..., t_6\})$.

Example 14. $F(t_1, ..., t_6) = t_1 - \varphi\left(\max\{t_2, t_3, t_4, \dfrac{t_5 + t_6}{2}\}\right)$.

Example 15. $F(t_1, ..., t_6) = t_1 - \varphi\left(\max\{t_2, \dfrac{t_3 + t_4}{2}, \dfrac{t_5 + t_6}{2}\}\right)$.

Example 16. $F(t_1, ..., t_6) = t_1 - \varphi(\max\{t_2, \sqrt{t_3 t_4}, \sqrt{t_5 t_6}\})$.

Example 17. $F(t_1, ..., t_6) = t_1 - \varphi(at_2 + bt_3 + ct_4 + dt_5 + et_6)$, where $a, b, c, d, e \geq 0$ and $a + b + c + d + e \leq 1$.

Example 18. $F(t_1, ..., t_6) = t_1 - \varphi\left(at_2 + \dfrac{b\sqrt{t_5 t_6}}{1 + t_3 + t_4}\right)$, where $a, b \geq 0$ and $a + b \leq 1$.

Example 19. $F(t_1, ..., t_6) = t_1 - \varphi(a\max\{t_2, t_3\} + b\max\{t_3, t_4\} + c\max\{t_2, \dfrac{t_5 + t_6}{2}\})$, where $a, b, c \geq 0$ and $a + b + c \leq 1$.

Theorem 8. *Let A, B, S and T be self mappings of a metric space (X, d) such that for all $x, y \in X$*

$$\int_0^{d(Ax, By)} h(t)\, dt$$

$$\leq \varphi \left(\max \left\{ \begin{array}{c} \int_0^{d(Sx, Ty)} h(t)\, dt, \int_0^{d(Sx, Ax)} h(t)\, dt, \\ \int_0^{d(Ty, By)} h(t)\, dt, \int_0^{d(Sx, By)} h(t)\, dt, \int_0^{d(Ty, Ax)} h(t)\, dt \end{array} \right\} \right),$$

where $h(t)$ is as in Theorem 6 and $\varphi \in \Phi$.

If the pair $\{A, S\}$ is weakly subsequentially continuous and compatible of type (E), as well $\{B, T\}$, then A, B, S and T have a unique common fixed point.

Proof. The proof it follows by Theorem 7 and Example 13. ∎

Corollary 5 (Theorem 3 [4])**.** *Let A, B, S and T be self mappings of a metric space (X, d) such that for all $x, y \in X$*

$$d(Ax, By) \leq \varphi \left(\max \left\{ \begin{array}{c} d(Sx, Ty), d(Sx, Ax), d(Ty, By), \\ d(Sx, By), d(Ty, Ax) \end{array} \right\} \right),$$

where $\varphi \in \Phi$.

If the pair $\{A, S\}$ is weakly subsequentially continuous and compatible of type (E), as well $\{B, T\}$, then A, B, S and T have a unique common fixed point.

Proof. The proof it follows by Theorem 8 for $h(t) = 1$. ∎

5.3. Fixed points for (ψ, φ)-weakly contractive mappings

In 1997, Alber and Guerre-Delabriere [1] defined the concept of weak contraction and established the existence of fixed points for a self mapping in Hilbert space.

Rhoades [31] extended this concept in metric spaces.

Let Ψ be the set of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying:

a) ψ is continuous,

b) $\psi(0) = 0$ and $\psi(t) > 0$, $\forall t > 0$.

Let Φ be the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying:

a) φ is lower semi - continuous,

b) $\varphi(0) = 0$ and $\varphi(t) > 0$, $\forall t > 0$.

There exist a vast literature about the existence of fixed points for (ψ, φ) -weakly contractive mappings.

New results for (ψ, φ)-weakly contractive and compatible of type (E) mappings are obtained in [2] and [4].

The following functions are from $\mathcal{F}^*$.

Example 20. $F(t_1, ..., t_6) = \psi(t_1) - \psi\left(\max\left\{t_2, t_3, t_4, \dfrac{t_5 + t_6}{2}\right\}\right) + \varphi\left(\max\{t_2, t_3, t_4, \dfrac{t_5 + t_6}{2}\}\right).$

Example 21. $F(t_1, ..., t_6) = \psi(t_1) - \psi(\max\{t_2, t_3, ..., t_6\}) + \varphi(\max\{t_2, t_3, ..., t_6\}).$

Example 22. $F(t_1, ..., t_6) = \psi(t_1) - \psi\left(\max\left\{t_2, t_3, t_4, \dfrac{t_5 + t_6}{2}\right\}\right) + \varphi(\max\{t_2, t_3, ..., t_6\}).$

Example 23. $F(t_1, ..., t_6) = \psi(t_1) - \psi\left(\max\left\{t_2, \dfrac{t_3 + t_4}{2}, \dfrac{t_5 + t_6}{2}\right\}\right) + \varphi(\max\{t_2, t_3, ..., t_6\}).$

Example 24. $F(t_1, ..., t_6) = \psi(t_1) - \psi\left(\max\left\{\sqrt{t_1 t_2}, \sqrt{t_5 t_6}, \sqrt{t_3 t_4}\right\}\right) + \varphi\left(\max\left\{t_2, \dfrac{t_3 + t_4}{2}, \dfrac{t_5 + t_6}{2}\right\}\right).$

Example 25. $F(t_1, ..., t_6) = \psi(t_1) - \psi\left(\sqrt{t_2 t_3} + \sqrt{t_3 t_4} + \sqrt{t_4 t_5} + \sqrt{t_5 t_6}\right) + \varphi\left(\max\left\{\sqrt{t_1 t_2}, \sqrt{t_3 t_4}, \sqrt{t_5 t_6}\right\}\right).$

Example 26. $F(t_1, ..., t_6) = \psi(t_1) - \psi\left(\dfrac{\sqrt{t_2 t_4} + \sqrt{t_2 t_3} + \sqrt{t_5 t_6}}{1 + t_3 + t_4}\right) + \varphi(\max\{t_2, t_3, ..., t_6\}).$

Example 27. $F(t_1, ..., t_6) = \psi(t_1) - \psi\left(\max\left\{t_2, t_3, t_4, \dfrac{t_3 t_4}{1 + t_2}, \dfrac{t_5 t_6}{1 + t_1}\right\}\right) + \varphi\left(\max\left\{t_2, t_3, t_4, \dfrac{t_3 t_4}{1 + t_2}, \dfrac{t_5 t_6}{1 + t_1}\right\}\right).$

If $\psi(t) = t$ we obtain new examples 20' - 27'.
By Example 20 and Theorem 3 we obtain

Corollary 6 (Theorem 1 [4])**.** *Let (X, d) be a metric space and A, B, S and T be self mappings of X such that for all $x, y \in X$*

(4) $$\psi(d(Ax, By)) \leq \psi(M(x, y)) - \phi(M(x, y)),$$

where

$$M(x, y) = \max\left\{\begin{array}{c} d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), \\ \frac{1}{2}(d(Sx, By) + d(Ax, Ty)) \end{array}\right\},$$

$\psi \in \Psi$ and $\phi \in \Phi$.

If the pairs $\{A, S\}$ and $\{B, T\}$ are weakly subsequentially continuous and compatible of type (E), then A, B, S and T have a unique common fixed point.

By Example 20 and Theorem 5 we obtain

Corollary 7 (Theorem 2 [4])**.** *Let A, B, S and T be self mappings of a metric space (X, d) satisfying (4) for all $x, y \in X$, $\psi \in \Psi$ and $\phi \in \Phi$.*
If
1) $\{A, S\}$ *is weakly subsequentially continuous and S-compatible of type (E), and*
2) $\{B, T\}$ *is weakly subsequentially continuous and T-compatible of type (E),*
then A, B, S and T have a unique common fixed point.

By Theorem 3 and Example 27 we obtain

Corollary 8 (Theorem 3.1 [2])**.** *Let A, B, S and T be self mappings of a metric space (X, d) such that for all $x, y \in X$*

$$\psi \left(d \left(Ax, By \right) \right) \leq \psi \left(N \left(x, y \right) \right) - \phi \left(N \left(x, y \right) \right),$$

where

$$N \left(x, y \right) = \max \left\{ \begin{array}{c} d \left(Sx, Ty \right), d \left(Sx, Ax \right), d \left(Ty, By \right), \\ \dfrac{d \left(Sx, Ax \right) \cdot d \left(Ty, Ax \right)}{1 + d \left(Sx, Ty \right)}, \dfrac{d \left(Sx, By \right) \cdot d \left(Ax, By \right)}{1 + d \left(Ax, By \right)} \end{array} \right\},$$

$\psi \in \Psi$ *and* $\phi \in \Phi$.
If the pairs $\{A, S\}$ and $\{B, T\}$ are weakly subsequentially continuous and compatible of type (E), then A, B, S and T have a unique common fixed point.

By Theorem 3 and Example 20, for $\psi \left(t \right) = t$ we obtain

Corollary 9 (Corollary 1 [4])**.** *Let A, B, S and T be self mappings of a metric space (X, d) such that for all $x, y \in X$*

$$d \left(Ax, By \right) \leq M \left(x, y \right) - \phi \left(M \left(x, y \right) \right),$$

where $\phi \in \Phi$ *and*

$$M \left(x, y \right) = \max \left\{ \begin{array}{c} d \left(Sx, Ty \right), d \left(Sx, Ax \right), d \left(Ty, By \right), \\ d \left(Sx, By \right), d \left(Ty, Ax \right) \end{array} \right\}.$$

If the pairs $\{A, S\}$ and $\{B, T\}$ are weakly subsequentially continuous and compatible of type (E), then A, B, S and T have a unique common fixed point.

Remark 10. By Examples 21 - 26 and 21' - 26' we obtain new particular results.

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