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COMMON BEST PROXIMITY POINTS FOR PROXIMALLY F-DOMINATED MAPPINGS

Abstract. The principal aim of this work is to formulate an extension and improvement of the common best proximity point theorem for a pair of non-self mappings, one of which is dominated by the other as proved by Basha. The proposed extension discusses a common best proximity point theorem for a pair of non-self mappings, one of which is F-dominated by the other proximally, for a function F as defined by Wardowski.

Key words: global optimal approximate solution, common best proximity point, common fixed point, proximally F-dominated mappings.

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1. Introduction

Banach's fixed point theory deals with finding a solution to an equation $fx = x$, wherein f is a self map defined on a complete metric space X. The equation $fx = x$ may not have a solution if f is not a self mapping. In that case, one may focus on the problem of searching an element x that is in close proximity to fx in some sense. Best approximation theorems and best proximity point theorems have been developed in recent times to solve above mentioned problem. The following best approximation theorem was established by Fan [6] in 1969.

Theorem 1 ([6]). *For a nonempty convex compact subset K of a Hausdorff locally convex topological vector space X with a continuous semi-norm p and for a non self continuous map $f : K \to X$, there exists an element $x \in K$ called a best approximant in K such that $p(x - fx) = d_P(fx, K) = \inf\{p(fx - y) : y \in K\}$.*

The preceding best approximation theorem has been generalized in various directions by many authors.

A best proximity point theorem for contractive map has been proved by Sadiq Basha [9]. Anthony Eldred *et al.* [5] and thereafter Sankar Raj and Veeramani [11] have elicited a best proximity point theorem for relatively non expansive mappings. A best proximity point theorem for proximal pointwise contraction has been discussed by Anuradha and Veeramani [3]. Best proximity point theorems for many variants of contractions have been studied by many other authors also in [14, 13, 9, 10, 17, 18, 19, 21, 20, 22, 23, 24, 8, 12, 2, 1].

Sadiq Basha [25] mooted a common best proximity point theorem for a pair of non self mappings, one of which dominates the other proximally.

Wardowski [27] introduced a generalized contraction which he named as an F-contraction, in which he took F as a real valued function defined on the set of positive real numbers and meeting with certain requirements.

The main objective of this paper is to establish a common best proximity point theorem for a pair of non self mappings, one of which is F-dominated by the other proximally. Consequently, the common best proximity point theorem proved in this article guarantees a common optimal solution at which both the real valued multi objective functions $x \to d(x, fx)$ and $x \to d(x, gx)$ attain the global minimal value $d(A, B)$, thereby giving rise to a common optimal approximate solution to the fixed point equations $fx = x$ and $gx = x$, where the mapping $f : A \to B$ is proximally F-dominated by $g : A \to B$. Moreover, common best proximity point theorem due to Sadiq Basha for a pair of non self mappings (with an improvement of the statement in line with the actual proof of Basha's main theorem) is a special case of the aforementioned best proximity point theorem.

2. Preliminaries

Consider two nonempty subsets A and B of a metric space (X, d). Let $d(A, B) := \inf\{d(a, b) : a \in A \text{ and } b \in B\}$, $A_0 := \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\}$ and $B_0 := \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}$.

Definition 1 ([16]). *Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \phi$. Then the pair (A, B) is said to have the P-property if and only if*

$$\left.\begin{array}{l} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{array}\right\} \Rightarrow d(x_1, x_2) = d(y_1, y_2)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Example 1 ([16]). The pair (A, B) satisfies P-property for any two nonempty closed and convex subsets A and B of a Hilbert space X.

Example 2 ([16]). The pair (A, B) satisfies P-property for any two nonempty subsets A and B of a metric space (X, d) such that $A_0 \neq \phi$ and $d(A, B) = 0$.

Example 3 ([13]). The pair (A, B) satisfies P-property for any two nonempty, closed, bounded and convex subsets A and B of a uniformly convex Banach space X.

Definition 2. *Given non self mappings $f : A \to B$ and $g : A \to B$, an element x in A is called a common best proximity point of the mappings f and g if $d(x, fx) = d(x, gx) = d(A, B)$.*

It can be observed that a common best proximity point is the one at which both the functions $x \to d(x, fx)$ and $x \to d(x, gx)$ attain global minimum, as $d(x, fx) \geq d(A, B)$ and $d(x, gx) \geq d(A, B)$ for all $x \in A$.

Definition 3. *The mappings $f : A \to B$ and $g : A \to B$ are said to commute proximally if $d(u, fx) = d(v, gx) = d(A, B) \Rightarrow fv = gu$ for all $x, u, v \in A$.*

It can be observed that proximal commutativity of two self mappings means their commutativity and a common best proximity point of two self mappings is precisely a common fixed point of the two mappings.

Let $\mathcal{F}$ be the set of all mappings $F : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the following conditions:

(F1) F is strictly increasing, that is, $F(a) < F(b)$ whenever $a, b \in \mathbb{R}^+$ and $a < b$.

1. For any sequence of positive real numbers a_n we have $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.
2. There exists a real number k with $0 < k < 1$ and $\lim_{a\to 0^+} a^k F(a) = 0$.

These mappings have been used by Wardowski [27] to define an F-contraction which generalises the Banach's contraction [4].

Definition 4. *A mapping $g : A \to B$ is said to be dominating another mapping $f : A \to B$ proximally if there exists a number $k \in [0, 1)$ such that*

$$\left.\begin{array}{c} d(u_1, fx_1) = d(u_2, fx_2) = d(A, B) \\ d(v_1, gx_1) = d(v_2, gx_2) = d(A, B) \end{array}\right\} \Rightarrow d(u_1, u_2) \leq k d(v_1, v_2) \tag{1}$$

for all $u_1, u_2, v_1, v_2, x_1, x_2$ in A.

Definition 5. *For $F \in \mathcal{F}$, a mapping $f : X \to X$ is said to be F-dominated by another mapping $g : X \to X$ if there exists a number $\tau > 0$ such that for all x, y in X,*

$$fx \neq fy \Rightarrow \begin{cases} gx \neq gy \quad \text{and} \\ \tau + F(d(fx, fy)) \leq F(d(gx, gy)) \end{cases}$$

Definition 5 can be extended to cover non self-mappings $f : A \to B$ and $g : A \to B$ also.

Definition 6. *For $F \in \mathcal{F}$, a mapping $f : A \to B$ is said to be F-dominated by another mapping $g : A \to B$ if there exists a number $\tau > 0$ such that for all x, y in A,*

$$fx \neq fy \Rightarrow \begin{cases} gx \neq gy \quad \text{and} \\ \tau + F(d(fx, fy)) \leq F(d(gx, gy)) \end{cases}$$

Remark 1. For $F \in \mathcal{F}$, if f is F-dominated by g according to Definition 5 or Definition 6 then we have for all $x, y \in X$ (resp. for all $x, y \in A$) ,

$$d(fx, fy) \leq d(gx, gy).$$

Definition 7. *For a function $F \in \mathcal{F}$, a mapping $g : A \to B$ is said to be F-dominating another mapping $f : A \to B$ proximally if there exists a number $\tau > 0$ such that*

$$\left.\begin{array}{c} d(u_1, fx_1) = d(u_2, fx_2) = d(A, B) \\ d(v_1, gx_1) = d(v_2, gx_2) = d(A, B) \\ u_1 \neq u_2 \end{array}\right\} \Rightarrow \begin{cases} v_1 \neq v_2 \quad \text{and} \\ \tau + F(d(u_1, u_2)) \leq F(d(v_1, v_2)) \end{cases}$$

for all $u_1, u_2, v_1, v_2, x_1, x_2$ in A.

In the Definition 7, we obtain a variety of proximal dominations for various types of mapping F. Consider the following examples:

Example 4. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by $F(a) = \ln a$. Clearly F satisfies all the three conditions for being an F-contraction, specially $(F3)$ for any $k \in (0, 1)$. A mapping $g : A \to B$ is said to F-dominate another mapping $f : A \to B$ proximally if and only if there exists a number $\tau > 0$ such that

$$\left.\begin{array}{c} d(u_1, fx_1) = d(u_2, fx_2) = d(A, B) \\ d(v_1, gx_1) = d(v_2, gx_2) = d(A, B) \\ u_1 \neq u_2 \end{array}\right\} \Rightarrow d(u_1, u_2) \leq \exp(-\tau) d(v_1, v_2) \tag{2}$$

for all $u_1, u_2, v_1, v_2, x_1, x_2$ in A. In particular, if a mapping $f : A \to B$ is proximally dominated by another mapping $g : A \to B$, then by (1), we see that (2) holds for $\tau = \ln(1/k)$. So the mapping $f : A \to B$ is F-dominated by the mapping $g : A \to B$ proximally. Conversely, if a mapping $f : A \to B$ is F-dominated by another mapping $g : A \to B$ proximally, then by (2), we observe that (1) holds for $k = \exp(-\tau)$. So $f : A \to B$ becomes proximally dominated by $g : A \to B$.

Example 5. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by $F(a) = \ln a + a$. It is clear that F satisfies all the three conditions for being an F-contraction. A mapping $f : A \to B$ is said to be F-dominated by another mapping $g : A \to B$ proximally if and only if there exists a number $\tau > 0$ such that for all $u_1, u_2, v_1, v_2, x_1, x_2$ in A,

$$\tag{3} \begin{aligned} & d(u_1, fx_1) = d(u_2, fx_2) = d(A, B) \\ & d(v_1, gx_1) = d(v_2, gx_2) = d(A, B) \\ & u_1 \neq u_2 \\ \Rightarrow\ & d(u_1, u_2) \exp\left(d(u_1, u_2) - d(v_1, v_2)\right) \leq \exp(-\tau) d(v_1, v_2) \end{aligned}$$

In particular, if there exists a number $k \in [0, 1)$ such that for all $u_1, u_2, v_1, v_2, x_1, x_2$ in A,

$$\tag{4} \begin{aligned} & d(u_1, fx_1) = d(u_2, fx_2) = d(A, B) \\ & d(v_1, gx_1) = d(v_2, gx_2) = d(A, B) \\ \Rightarrow\ & d(u_1, u_2) \exp\left(d(u_1, u_2) - d(v_1, v_2)\right) \leq k d(v_1, v_2) \end{aligned}$$

then by (4), we see that (3) holds for $\tau = \ln(1/k)$. So the mapping $f : A \to B$ is F-dominated by the mapping $g : A \to B$ proximally. Conversely, if a mapping $f : A \to B$ is F-dominated by another mapping $g : A \to B$ proximally, then by (3), we observe that (4) holds for $k = \exp(-\tau)$.

Example 6. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by $F(a) = -1/\sqrt{a}$. Then F satisfies all the three conditions for being an F-contraction. A mapping $f : A \to B$ is said to be F-dominated by another mapping $g : A \to B$ proximally if and only if there exists a number $\tau > 0$ such that for all $u_1, u_2, v_1, v_2, x_1, x_2$ in A with $u_1 \neq u_2$, $d(u_1, fx_1) = d(u_2, fx_2) = d(A, B)$ and $d(v_1, gx_1) = d(v_2, gx_2) = d(A, B)$ we have

$$\tag{5} d(u_1, u_2) \leq (1 + \tau\sqrt{d(v_1, v_2)})^{-2} d(v_1, v_2).$$

In particular, if there exists a function $\alpha : A \times A \to [0, 1)$ such that for all $u_1, u_2, v_1, v_2, x_1, x_2$ in A with $d(u_1, fx_1) = d(u_2, fx_2) = d(A, B)$ and $d(v_1, gx_1) = d(v_2, gx_2) = d(A, B)$ we have

$$\tag{6} d(u_1, u_2) \leq \alpha(v_1, v_2) d(v_1, v_2)$$

then by (6), we see that (5) holds for $\tau = (1/\sqrt{d(v_1, v_2)})[(1/\sqrt{\alpha(v_1, v_2)}) - 1]$. So the mapping $f : A \to B$ is F-dominated by the mapping $g : A \to B$ proximally. Conversely, if a mapping $f : A \to B$ is F-dominated by another mapping $g : A \to B$ proximally, then by (5), we observe that (6) holds for $\alpha(v_1, v_2) = (1 + \tau\sqrt{d(v_1, v_2)})^{-2}$.

Example 7. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by $F(a) = \ln(a^2 + a)$. Then F satisfies all the three conditions for being an F-contraction. A mapping $f : A \to B$ is said to be F-dominated by another mapping $g : A \to B$ proximally if and only if there exists a number $\tau > 0$ such that for all $u_1, u_2, v_1, v_2, x_1, x_2$ in A with $u_1 \neq u_2$, $d(u_1, fx_1) = d(u_2, fx_2) = d(A, B)$ and $d(v_1, gx_1) = d(v_2, gx_2) = d(A, B)$ we have

$$d(u_1, u_2)(d(u_1, u_2) + 1) \leq \exp(-\tau) d(v_1, v_2)(d(v_1, v_2) + 1). \tag{7}$$

In particular, if there exists a number $k \in [0, 1)$ such that for all $u_1, u_2, v_1, v_2, x_1, x_2$ in A with $d(u_1, fx_1) = d(u_2, fx_2) = d(A, B)$ and $d(v_1, gx_1) = d(v_2, gx_2) = d(A, B)$ we have

$$d(u_1, u_2)(d(u_1, u_2) + 1) \leq k d(v_1, v_2)(d(v_1, v_2) + 1) \tag{8}$$

then by (8), we see that (7) holds for $\tau = \ln(1/k)$. So the mapping $f : A \to B$ is F-dominated by the mapping $g : A \to B$ proximally. Conversely, if a mapping $f : A \to B$ is F-dominated by another mapping $g : A \to B$ proximally, then by (7), we observe that (8) holds for $k = \exp(-\tau)$.

Remark 2. Let F_1, $F_2 \in \mathcal{F}$ be arbitrary. Suppose $F_1(a) \leq F_2(a)$ for all $a > 0$ and a mapping $G = F_2 - F_1$ is nondecreasing, then a mapping $f : A \to B$ is F_2-dominated by another mapping $g : A \to B$ proximally whenever $f : A \to B$ is F_1-dominated by $g : A \to B$.

Indeed, by Definition 7, there exists a number $\tau > 0$ such that

$$\left.\begin{array}{c} d(u_1, fx_1) = d(u_2, fx_2) = d(A, B) \\ d(v_1, gx_1) = d(v_2, gx_2) = d(A, B) \\ u_1 \neq u_2 \end{array}\right\} \Rightarrow \begin{cases} v_1 \neq v_2 \text{ and} \\ \tau + F_1(d(u_1, u_2)) \leq F_1(d(v_1, v_2)) \end{cases}$$

for all $u_1, u_2, v_1, v_2, x_1, x_2$ in A. This because F_1 is nondecreasing, gives $d(u_1, u_2) \leq d(v_1, v_2)$. Now

$$\begin{aligned} \tau + F_2(d(u_1, u_2)) &= \tau + F_1(d(u_1, u_2)) + G(d(u_1, u_2)) \\ &\leq F_1(d(v_1, v_2)) + G(d(v_1, v_2)) = F_2(d(v_1, v_2)). \end{aligned}$$

3. Main results

Theorem 2. *Let A and B be nonempty subsets of a complete metric space (X, d) such that A_0 is nonempty as well as closed. Let $F \in \mathcal{F}$ and the non self mappings $f : A \to B$ and $g : A \to B$ satisfy the following conditions:*

(a) f is proximally F-dominated by g.

(*b*) *f and g commute proximally.*
(*c*) *f and g are continuous.*
(*d*) $f(A_0) \subseteq B_0$.
(*e*) $f(A_0) \subseteq g(A_0)$.
Then, there exists a unique common best proximity point of f and g.

Proof. Choose an element $x_0 \in A_0$ arbitrarily. Since $f(A_0) \subseteq g(A_0)$, therefore, there exists an element $x_1 \in A_0$ such that $fx_0 = gx_1$. Again, since $f(A_0) \subseteq g(A_0)$, there exists an element $x_2 \in A_0$ such that $fx_1 = gx_2$. This process can be continued. Proceeding inductively, it can easily be asserted that there exists a sequence $\{x_n\}$ in A_0 satisfying

$$fx_{n-1} = gx_n$$

for all positive integers n. This is because of the fact that $f(A_0) \subseteq g(A_0)$.

On account of the fact that $f(A_0) \subseteq B_0$, there exists an element u_n in A_0 such that

$$d(fx_n, u_n) = d(A, B) \tag{9}$$

for all nonnegative integers n. Further, by the way we have chosen x_n and u_n, it follows that

$$d(fx_{n+1}, u_{n+1}) = d(A, B) \tag{10}$$

$$d(gx_n, u_{n-1}) = d(A, B) \tag{11}$$

$$d(gx_{n+1}, u_n) = d(A, B). \tag{12}$$

Let $d_n = d(u_n, u_{n+1})$ for all nonnegative integer values of n. If $u_n = u_{n+1} = u$ for some nonnegative integer n then by (10) and (12) and using proximal commutativity of f and g we obtain $fu = gu$ for some $u \in A$. Let us now assume that the sequence $\{u_n\}$ is such that any two consecutive terms are distinct. That is, $u_n \neq u_{n+1}$ for all nonnegative integers n. So $d_n > 0$ for all nonnegative integers n. Since f is F-dominated by g proximally, there exists a number $\tau > 0$ such that for every positive integer n,

$$F(d_n) \leq F(d_{n-1}) - \tau \leq F(d_{n-2}) - 2\tau \leq \ldots \leq F(d_0) - n\tau. \tag{13}$$

By (13), we get $\lim_{n\to\infty} F(d_n) = -\infty$ which together with ($F2$) gives

$$\lim_{n\to\infty} d_n = 0. \tag{14}$$

By using ($F3$), we can find a $k \in (0, 1)$ such that

$$\lim_{n\to\infty} d_n^k F(d_n) = 0. \tag{15}$$

By (13), the following holds for all positive integers n:

$$d_n^k F(d_n) - d_n^k F(d_0) \leq d_n^k (F(d_0) - n\tau) - d_n^k F(d_0) = -d_n^k n\tau \leq 0. \tag{16}$$

Letting $n \to \infty$ in (16), and using (14) and (15), we get

$$\lim_{n\to\infty} n d_n^k = 0 \tag{17}$$

Now, by (17), there exists a positive integer p such that $nd_n^k \leq 1$ for all $n \geq p$. Consequently we have for all $n \geq p$,

$$d_n \leq 1/n^{1/k} \tag{18}$$

Let us choose positive integers m and n such that $m > n \geq p$. By (18), we get

$$d(u_m, u_n) \leq d_{m-1} + d_{m-2} + \ldots + d_n < \sum_{i=n}^{\infty} d_i \leq \sum_{i=n}^{\infty} (1/i^{1/k}).$$

By the convergence of the series $\sum_{i=1}^{\infty}(1/i^{1/k})$, we obtain that $\{u_n\}$ is a Cauchy sequence in A_0. Since A_0 is a closed subset of a complete metric space, therefore, there exists an element $u \in A_0$ such that $u_n \to u$ as $n \to \infty$. Because of proximal commutativity of f and g and for all positive integers n, we have by (9) and (11),

$$gu_n = fu_{n-1}$$

Continuity of f and g now gives $fu = gu$.

Thus irrespective of the nature of the sequence $\{u_n\}$, we get an element $u \in A$ such that $fu = gu$.

In view of the fact that $f(A_0) \subseteq B_0$, there exists an element x in A such that

$$d(x, fu) = d(A, B) \tag{19}$$

$$d(x, gu) = d(A, B) \tag{20}$$

Since f and g commute proximally, by (19) and (20) we get $fx = gx$.

Again since $f(A_0) \subseteq B_0$, there exists an element z in A such that

$$d(z, fx) = d(A, B) \tag{21}$$

$$d(z, gx) = d(A, B) \tag{22}$$

If $x \neq z$ then by F-domination of f proximally by g and by (19), (20), (21) and (22) we obtain,

$$\tau + F(d(x, z)) \leq F(d(x, z)).$$

But this is not true. So we must have $x = z$. Thus, it follows that

$$d(x, gx) = d(z, gx) = d(A, B), \tag{23}$$

$$d(x, fx) = d(z, fx) = d(A, B). \tag{24}$$

Thus, x becomes a common best proximity point of non self mappings f and g.

To prove uniqueness, suppose that x^* is another common best proximity point of the mappings f and g. then we have

$$d(x^*, fx^*) = d(A, B) \tag{25}$$

$$d(x^*, gx^*) = d(A, B). \tag{26}$$

If $x \neq x^*$, then by F-domination of the mapping f proximally by the mapping g and by (23), (24), (25) and (26) we get,

$$\tau + F(d(x, x^*)) \leq F(d(x, x^*)).$$

But this is not true. So we must have $x = x^*$.

This completes the proof of the theorem. ∎

Remark 3. Conclusion of Theorem 2 remains valid even if the proximal F-domination of f by g is replaced by F-domination provided the pair (A, B) satisfies the P-property and rest of the conditions in Theorem 2 remain unchanged. This is simply because the P-property together with F-domination of $f : A \to B$ by $g : A \to B$ implies proximal F-domination. Further, in this situation, continuity of f is not to be explicitly stated as it is implied by continuity of g by Remark 1. Also the conditions that A, B and A_0 are nonempty may be omitted because these are implied by the condition that the pair (A, B) satisfies the P-property.

Thus we may state the discussion in Remark 3 in the form of the following theorem.

Theorem 3. *Let A and B be subsets of a complete metric space (X, d) such that the pair (A, B) satisfies the P-property and A_0 is closed. Let $F \in \mathcal{F}$ and the non self mappings $f : A \to B$ and $g : A \to B$ satisfy the following conditions:*

(a) f is F-dominated by g.
(b) f and g commute proximally.
(c) g is continuous.
(d) $f(A_0) \subseteq B_0$.
(e) $f(A_0) \subseteq g(A_0)$.

Then, there exists a unique common best proximity point of f and g.

If we take A=B=X in Theorem 2 or in Theorem 3, then we get the following result.

Corollary 1. *Let (X, d) be a complete metric space and $F \in \mathcal{F}$. Let $f : X \to X$ and $g : X \to X$ satisfy the following conditions:*
(a) *f is F-dominated by g.*
(b) *f and g are commuting.*
(c) *g is continuous.*
(d) *$f(X) \subseteq g(X)$.*
Then, there exists a unique common fixed point of f and g.

In Corollary 1 we have not assumed continuity of f as it follows by continuity of g by Remark 1.

Taking $F(a) = \ln a$ for all $a > 0$ in Theorem 2 we get the following result which is similar to the one in Sadiq Basha [25] with a little variation in the statement. Sadiq Basha has not assumed that A_0 is closed. Instead he has assumed that both A and B are closed. But in the proof, A_0 is required to be closed and A and B need not be closed. Further nonemptiness of B_0 is implied by nonemptiness of A_0 in his result, as $f(A_0) \subseteq B_0$.

Corollary 2. *Let A and B be nonempty subsets of a complete metric space (X, d) such that A_0 is nonempty and closed. Let $f : A \to B$ and $g : A \to B$ satisfy the following conditions:*
(a) *f is proximally dominated by g.*
(b) *f and g commute proximally.*
(c) *f and g are continuous.*
(d) *$f(A_0) \subseteq B_0$.*
(e) *$f(A_0) \subseteq g(A_0)$.*
Then, there exists a unique common best proximity point of f and g.

The following result due to Jungck [7] for common fixed point of two self mappings on a complete metric space can be proved by taking $F(a) = \ln a$ in Corollary 1.

Corollary 3. *Let (X, d) be a complete metric space and the mappings $f : X \to X$ and $g : X \to X$ satisfy the following conditions:*
(a) *There exists a number $\alpha \in [0, 1)$ satisfying $d(fx, fy) \leq \alpha d(gx, gy)$ for all $x, y \in X$.*
(b) *f and g are commuting.*
(c) *g is continuous.*
(d) *$f(X) \subseteq g(X)$.*
Then, there exists a unique common fixed point of f and g.

Example 8. Consider the sequence $\{s_n\}$ given by $s_n = 1+2+\ldots+n = n(n+1)/2$ for all $n \in \mathbb{N}$. Let $X = \mathbb{R} \times \{s_n : n \in \mathbb{N}\} = \{(x, s_n) : x \in \mathbb{R}, n \in$

$\mathbb{N}\}$. Let us consider Euclidean metric d on X. Then (X, d) is a complete metric space. Let $A := \{(x, s_n) : x \leq -1, n \in \mathbb{N}\}$ and $B := \{(x, s_n) : x \geq 1, n \in \mathbb{N}\}$. Clearly A and B are nonempty and closed subsets of X such that $d(A, B) = 2$. Also $A_0 = \{(-1, s_n) : n \in \mathbb{N}\}$ and $B_0 = \{(1, s_n) : n \in \mathbb{N}\}$.

Let $f : A \to B$ and $g : A \to B$ be defined by

$$f(x, s_n) = \begin{cases} (-x, s_1) & \text{if } n \text{ is odd,} \\ (-x, s_{n-1}) & \text{if } n \text{ is even} \end{cases}$$

and

$$g(x, s_n) = \begin{cases} (-x, s_1) & \text{if } n \text{ is odd,} \\ (-x, s_{n+1}) & \text{if } n \text{ is even.} \end{cases}$$

Then both f and g are continuous on A. Now, for any $(x, s_n), (u, s_m)$ and $(v, s_k) \in A$, $d(f(x, s_n), (u, s_m)) = d(A, B)$ and $d(g(x, s_n), (v, s_k)) = d(A, B)$, we have $d((-x, s_1), (u, s_m)) = d((-x, s_1), (v, s_k)) = d(A, B)$ if n is odd and $d((-x, s_{n-1}), (u, s_m)) = d((-x, s_{n+1}), (v, s_k)) = d(A, B)$ if n is even. Since d(A,B)=2, therefore, $(-x, s_1) \in B_0$ and $(u, s_m), (v, s_k) \in A_0$ if n is odd and also $(-x, s_{n-1}), (-x, s_{n+1}) \in B_0$ and $(u, s_m), (v, s_k) \in A_0$ if n is even. This gives $x = u = v = -1$, $s_1 = s_m = s_k$ if n is odd and $x = u = v = -1$, $s_{n-1} = s_m$, $s_{n+1} = s_k$ if n is even. Now $f(v, s_k) = f(-1, s_k) = f(-1, s_1) = (1, s_1) = g(-1, s_m) = g(u, s_m)$ if n is odd and $f(v, s_k) = f(-1, s_{n+1}) = (1, s_1) = g(-1, s_{n-1}) = g(u, s_m)$ if n is even. Thus $d(f(x, s_n), (u, s_m)) = d(A, B)$ and $d(g(x, s_n), (v, s_k)) = d(A, B) \Rightarrow f(v, s_k) = g(u, s_m)$ for all $n \in \mathbb{N}$. This implies that f and g are proximally commutative. Now,

$$f(-1, s_n) = \begin{cases} (1, s_1) & \text{if } n \text{ is odd and} \\ (1, s_{n-1}) & \text{if } n \text{ is even.} \end{cases}$$

So it is implied that $f(A_0) \subseteq B_0$. Further we have

$$f(A_0) = g(A_0) = \{(1, s_{2n-1}) : n \in \mathbb{N}\}.$$

Take $F_1 \in \mathcal{F}$ as in Example 4. The mapping f is not proximally F_1-dominated by g (which means f is not proximally dominated by g). Indeed if we take $(-1, s_n), (-1, s_m), (-1, s_p), (-1, s_q), (-1, s_k), (-1, s_l)$ in A_0 satisfying

$$\begin{aligned} &d((-1, s_n), f(-1, s_k)) = d((-1, s_m), f(-1, s_l)) = d(A, B) = 2 \\ &d((-1, s_p), g(-1, s_k)) = d((-1, s_q), g(-1, s_l)) = d(A, B) = 2 \\ &(-1, s_n) \neq (-1, s_m) \end{aligned}$$

then $f(-1, s_k) = (1, s_n)$ and $f(-1, s_l)) = (1, s_m)$. Also, we get $g(-1, s_k) = (1, s_p)$ and $g(-1, s_l)) = (1, s_q)$. If k is even but l is odd then we get

$$\begin{aligned}
&f(-1, s_k) = (1, s_{k-1}) = (1, s_n),\\
&g(-1, s_k) = (1, s_{k+1}) = (1, s_p),\\
&f(-1, s_l)) = (1, s_1) = (1, s_m) \text{ and}\\
&g(-1, s_l)) = (1, s_1) = (1, s_q).
\end{aligned}$$

This gives $k - 1 = n$, $k + 1 = p$, $m = 1$ and $q = 1$. So

$$\begin{aligned}
\frac{d((-1, s_n), (-1, s_m))}{d((-1, s_p), (-1, s_q)} &= \frac{|s_n - s_m|}{|s_p - s_q|} = \frac{s_{k-1} - s_1}{s_{k+1} - s_1}\\
&= \frac{k^2 - k - 2}{k^2 + 3k} \to 1 \text{ as } k \to \infty.
\end{aligned}$$

Thus f is not proximally F_1-dominated by g. Hence Theorem 3.1 of [25] can not be applied. Let $F_2 \in \mathcal{F}$ be taken as in Example 5. We obtain that f is F_2-dominated by g with $\tau = 2$. To see this, consider the following calculations: First, let us choose $(-1, s_n)$, $(-1, s_m)$, $(-1, s_p)$, $(-1, s_q)$, $(-1, s_k)$, $(-1, s_l)$ in A_0 satisfying

$$\begin{aligned}
&d((-1, s_n), f(-1, s_k)) = d((-1, s_m), f(-1, s_l)) = d(A, B) = 2\\
&d((-1, s_p), g(-1, s_k)) = d((-1, s_q), g(-1, s_l)) = d(A, B) = 2\\
&(-1, s_n) \neq (-1, s_m)
\end{aligned}$$

This implies that $f(-1, s_k) = (1, s_n)$ and $f(-1, s_l)) = (1, s_m)$. Also, we get $g(-1, s_k) = (1, s_p)$ and $g(-1, s_l)) = (1, s_q)$. If both k and l are odd then we get

$$\begin{aligned}
&f(-1, s_k) = (1, s_1) = (1, s_n),\\
&f(-1, s_l)) = (1, s_1) = (1, s_m),\\
&g(-1, s_k) = (1, s_1) = (1, s_p) \text{ and}\\
&g(-1, s_l)) = (1, s_1) = (1, s_q).
\end{aligned}$$

This gives $n = m = p = q = 1$, which is not possible as $n \neq m$. If k is even but l is odd then we get

$$\begin{aligned}
&f(-1, s_k) = (1, s_{k-1}) = (1, s_n),\\
&g(-1, s_k) = (1, s_{k+1}) = (1, s_p),\\
&f(-1, s_l)) = (1, s_1) = (1, s_m) \text{ and}\\
&g(-1, s_l)) = (1, s_1) = (1, s_q).
\end{aligned}$$

This gives $k-1=n$, $k+1=p$, $m=1$ and $q=1$. So

$$\begin{aligned}
&\frac{d((-1,s_n),(-1,s_m))}{d((-1,s_p),(-1,s_q)))}e^{d((-1,s_n),(-1,s_m))-d((-1,s_p),(-1,s_q))}\\
&\quad=\frac{|s_n-s_m|}{|s_p-s_q|}e^{|s_n-s_m|-|s_p-s_q|}=\frac{s_{k-1}-s_1}{s_{k+1}-s_1}e^{sk-1-s_{k+1}}\\
&\quad=\frac{k^2-k-2}{k^2+3k}e^{-2k-1}<e^{-2}.
\end{aligned}$$

Now if k is odd but l is even then we get

$$\begin{aligned}
&f(-1,s_k)=(1,s_1)=(1,s_n),\\
&g(-1,s_k)=(1,s_1)=(1,s_p),\\
&f(-1,s_l))=(1,s_{l-1})=(1,s_m) \text{ and}\\
&g(-1,s_l))=(1,s_{l+1})=(1,s_q).
\end{aligned}$$

This gives $l-1=m$, $l+1=q$, $n=1$ and $p=1$. So

$$\begin{aligned}
&\frac{d((-1,s_n),(-1,s_m))}{d((-1,s_p),(-1,s_q))}e^{d((-1,s_n),(-1,s_m))-d((-1,s_p),(-1,s_q))}\\
&\quad=\frac{|s_n-s_m|}{|s_p-s_q|}e^{|s_n-s_m|-|s_p-s_q|}=\frac{s_{l-1}-s_1}{s_{l+1}-s_1}e^{s_{l-1}-s_{l+1}}\\
&\quad=\frac{l^2-l-2}{l^2+3l}e^{-2l-1}<e^{-2}.
\end{aligned}$$

Now assume that both k and l are even. Then we obtain

$$\begin{aligned}
&f(-1,s_k)=(1,s_{k-1})=(1,s_n),\\
&g(-1,s_k)=(1,s_{k+1})=(1,s_p),\\
&f(-1,s_l))=(1,s_{l-1})=(1,s_m) \text{ and}\\
&g(-1,s_l))=(1,s_{l+1})=(1,s_q).
\end{aligned}$$

This gives $k-1=n$, $k+1=p$, $l-1=m$, $l+1=q$. So in case $k>l$, we have

$$\begin{aligned}
&\frac{d((-1,s_n),(-1,s_m))}{d((-1,s_p),(-1,s_q))}e^{(d((-1,s_n),(-1,s_m))-d((-1,s_p),(-1,s_q)))}\\
&\quad=\frac{|s_n-s_m|}{|s_p-s_q|}e^{(|s_n-s_m|-|s_p-s_q|)}=\frac{s_{k-1}-s_{l-1}}{s_{k+1}-s_{l+1}}e^{s_{k-1}-s_{l-1}-s_{k+1}+s_{l+1}}\\
&\quad=\frac{k+l-1}{k+l+3}e^{-2(k-l)}<e^{-2}.
\end{aligned}$$

That inequality holds for $l>k$ also and can be proved easily by similar method as for $k>l$. Thus f is F_2-dominated by g. Hence all the conditions of Theorem 2 are satisfied. We observe that $(-1,s_1)$ is a unique common best proximity point of f and g.

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