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Raghavendra G. Kulkarni

TSCHIRNHAUS TRANSFORMATION METHOD SANS FALSE SOLUTIONS FOR SOLVING CUBICS

Abstract. This paper presents a new Tschirnhaus method for solving cubic equations, which produces only genuine solutions, in contrast to the earlier methods, which produce genuine as well as false solutions.

Key words: Tschirnhaus transformation, cubic equation, genuine solution, false solution.

AMS Mathematics Subject Classification: 12E12.

1. Introduction

It is well-known that the methods for solving polynomial equations using Tschirnhaus transformation generate genuine as well as false solutions, and the genuine solutions have to be picked by trial and error only [1, 2, 3]. In a recent paper, a method is described to identify the genuine solutions of cubic equations; however, the method identifies all the three genuine solutions only when the discriminant of the cubic is positive, otherwise it identifies only one genuine solution [4]. Thus the task of identifying all the three genuine solutions of cubics all the time (i.e., irrespective of the sign of the dicriminant) is yet to be addressed. In this paper, we take up this issue, and present a new Tschirnhaus transformation method for solving cubic equations, which produces only the genuine solutions.

2. The proposed method

Consider the cubic equation,

$$x^3 + ax + b = 0, \tag{1}$$

where a and b are coefficients in (1). We define a special quadratic Tschirnhaus transformation,

$$x^2 = 2cx + d^2 - c^2 + y, \tag{2}$$

where c and d are two unknown numbers in (2) and y is a new variable. Expressing (2) as,

$$x = c \pm \sqrt{d^2 + y}. \tag{3}$$

and using it in (1) to eliminate x yields,

$$3cy + c^3 + (3d^2 + a)c + b = \mp\sqrt{d^2 + y}\,(y + 3c^2 + d^2 + a). \tag{4}$$

Notice that squaring the two expressions in (4) results in a single expression, and we rearrange it in descending powers of y as below.

$$\begin{aligned} y^3 - [3(c^2 - d^2) - 2a]y^2 + [3(c^2 - d^2)^2 + 4ad^2 - 6bc + a^2]y \\ + d^2(3c^2 + d^2 + a)^2 - [c^3 + (3d^2 + a)c + b]^2 = 0 \end{aligned} \tag{5}$$

In the traditional method described in [1], the coefficients of y and y^2 in the transformed cubic equation (5) are set to zero, rendering (5) to be a binomial cubic equation in y, so that it can be solved. However this method ends up in generating false solutions along with genuine solutions, without offering a way to identify the genuine solutions. The reason for generation of false solutions is that—each of the three solutions of y [see (5)] churns out two values of x from the transformation (3), in which only one is the genuine solution of x.

It is now clear that we need to follow a path, which is different from the traditional one in order to get only the genuine solutions. Notice that in the traditional method, both the unknowns (c and d) are determined in one go; however in the proposed method given here, we plan to determine one unknown (c) now, and reserve the determination of second unknown (d) for an appropriate stage. So, for this purpose we further rearrange (5) as,

$$\begin{aligned} \{y - [c^2 - d^2 - (2a/3)]\}^3 + [4ac^2 - 6bc - (a^2/3)]y \\ = (c^2 - d^2)[4ac^2 - (a^2/3)] + 2bc(c^2 + 3d^2 + a) + b^2 + (8a^3/27). \end{aligned} \tag{6}$$

Observe that equating the second term, $4ac^2 - 6bc - (a^2/3)$, in (6) to zero serves two purposes; first, it results in a quadratic equation in the unknown c: $4ac^2 - 6bc - (a^2/3) = 0$, leading to the determination of c,

$$c = \frac{1}{4a}\left[3b \pm \sqrt{(4a^3 + 27b^2)/3}\,\right], \tag{7}$$

and the second, (6) becomes a perfect cube as,

$$\{y - [c^2 - d^2 - (2a/3)]\}^3 = g^3, \tag{8}$$

where g^3 is given by,

$$g^3 = 8bc^3 + 2abc + b^2 + (8a^3/27). \tag{9}$$

Note that the cube-root of (8) yields the three solutions of y as follows,

$$\begin{aligned} y &= [c^2 - d^2 - (2a/3)] + g, \\ y &= [c^2 - d^2 - (2a/3)] + gw, \\ y &= [c^2 - d^2 - (2a/3)] + gw^2. \end{aligned} \tag{10}$$

where $w = (-1 + \sqrt{3}i)/2$.

Notice that d appearing in the expressions given in (10) is still an unknown, and to determine d we need to assign a value to y in any one of the expressions in (10). The value assigned to y should be such that it yields only genuine solutions of cubic equation (1). It is observed that when y is assigned a zero value ($y = 0$) and used in (5), it results in,

$$d^2(3c^2 + d^2 + a)^2 - [c^3 + (3d^2 + a)c + b]^2 = 0,$$

which after simplification yields the following factored expression,

$$[(c + d)^3 + a(c + d) + b][(c - d)^3 + a(c - d) + b] = 0. \tag{11}$$

Also, when $y = 0$ is used in the transformation (3), we obtain two expressions, $x = c \pm d$, as probable solutions of cubic equation (1), but only one of them is the genuine solution. A comparison of (1) and (11) reveals that if $x = c + d$ has to be a genuine solution of (1), then the factor, $(c + d)^3 + a(c + d) + b$, in (11) has to be zero; and if $x = c - d$ has to be genuine solution of (1), then the factor, $(c - d)^3 + a(c - d) + b$, given in (11) has to vanish. Let us choose that $x = c + d$ be genuine solution of (1), which implies:

$$(c + d)^3 + a(c + d) + b = 0. \tag{12}$$

Observe that assigning $y = 0$ in the first expression in (10) results in,

$$d^2 = c^2 - (2a/3) + g, \tag{13}$$

from which two values of d are obtained,

$$d = \pm\sqrt{c^2 - (2a/3) + g}. \tag{14}$$

Our next task is to select one of the two values of d, which has to satisfy the cubic equation (12). This is accomplished by eliminating d^2 and d^3 terms from (12) using (13), leading to a single value for d as,

$$d = -\frac{4c^3 + (3g - a)c + b}{4c^2 + g + (a/3)}. \tag{15}$$

We determine the desired value of d from (15). Thus one solution of cubic equation (1) is determined as: $x = c + d$, which is a genuine solution. To obtain the second genuine solution, we use $y = 0$ in the second expression in (10), determine d [by replacing g with gw in (15)] and use it in $x = c+d$. To get the third genuine solution, use $y = 0$ in the third expression in (10), determine d [by replacing g with gw^2 in (15)] and use it in $x = c + d$.

3. Numerical examples

Let us solve some numerical examples; in the first instance, consider the cubic equation,

$$x^3 - 6x + 9 = 0.$$

From (7) we get two values of c as: -2 and -0.25. Let $c = -2$; from (9) we determine g^3 as, -343, and subsequently g, gw, and gw^2 as:

$$-7, \;\; 3.5 - 6.06217782649...i, \;\; \text{and} \;\; 3.5 + 6.06217782649...i.$$

Using (14) we get two values of d as: ± 1. However the desired value of d is determined from (15) as: -1. The other two desired values of d corresponding to gw and gw^2 are obtained [by replacing g in (15) with gw and gw^2 respectively] as: $(7 \mp \sqrt{3}i)/2$. Using the expression $x = c + d$, we determine the three genuine solutions of $x^3 - 6x + 9 = 0$ as: -3, $(3 \mp \sqrt{3}i)/2$. The interested reader may use the other value of c and verify the results.

Let us take another cubic equation, $x^3 - (2/3)x + (1/3) = 0$, for solving using the method proposed here. Using (7), we determine two values of c as: $-2/3$ and $-1/12$. Letting $c = -2/3$, we determine g^3 as: $-0.47050754458...$, from which g, gw, and gw^2 are obtained as:

$$-0.77777777777..., \;\; 0.3888888888... - 0.6735753140...i,$$

$$\text{and} \;\; 0.3888888888... + 0.6735753140...i.$$

The three desired values of d corresponding to g, gw, and gw^2 are determined from (15) as: $-0.3333333333...$, $1.1666666666... - 0.2886751345...i$, and $1.1666666666... + 0.2886751345...i$. The three genuine solutions of cubic equation, $x^3 - (2/3)x + (1/3) = 0$, are then determined from $x = c + d$ as: -1, $0.5 - 0.2886751345...i$, and $0.5 + 0.2886751345...i$.

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RAGHAVENDRA G. KULKARNI
DEPARTMENT OF ELECTRONICS
AND COMMUNICATION ENGINEERING
PES UNIVERSITY
100 FEET RING ROAD, BSK III STAGE
BENGALURU-560085, INDIA
e-mail: dr_ rgkulkarni@yahoo.com *or* raghavendrakulkarni@pes.edu

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