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ON THE QUALITATIVE BEHAVIOR OF THE SOLUTIONS TO SECOND-ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

Abstract. In this paper, we study the qualitative behavior of the solutions to second-order neutral delay differential equations of the form

$$\Big(r(t)\big(\big(x(t)+p(t)x(\tau(t))\big)'\big)^{\gamma}\Big)' + q(t)f\left(x(\sigma(t))\right) = 0.$$

Our main tool is Lebesgue's dominated convergence theorem. Examples illustrating the applicability of the results are also given.

Key words: oscillation, non-oscillation, neutral, delay, nonlinear, Lebesgue's dominated convergence theorem..

AMS Mathematics Subject Classification: 34C10, 34K11.

1. Introduction

Consider the class of nonlinear neutral delay differential equations of the form:

$$\big(r(z'^{\gamma}\big)'(t) + q(t)f\left(x(\sigma(t))\right) = 0, \tag{1}$$

where $z(t) = x(t)+p(t)x(\tau(t))$ and γ is the ratio of two odd positive integers. We assume the following conditions hold.

(C1) $r, q, \tau, \sigma \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that $\tau(t) \le t$, $\sigma(t) \le t$ for $t \ge t_0$, $\tau(t) \to \infty$, $\sigma(t) \to \infty$ as $t \to \infty$;

(C2) $f \in C(\mathbb{R}, \mathbb{R})$ is non-decreasing with $vf(v) > 0$ for $v \neq 0$;

(C3) $r(t) > 0$ and $\int_0^{\infty} \big(r(\eta)\big)^{-1/\gamma} d\eta = \infty$. By letting $\Pi(t) = \int_0^t \big(r(\eta)\big)^{-1/\gamma} d\eta$, we have $\lim_{t\to\infty} \Pi(t) = \infty$;

(C4) $p \in C(\mathbb{R}_+, \mathbb{R}_-)$ with $-1 + (2/3)^{1/\gamma} \le -a \le p(t) \le 0$ for $t \in \mathbb{R}_+$;

(C5) $p \in C(\mathbb{R}_+, \mathbb{R}_-)$ with $-1 < -a \le p(t) \le 0$ for $t \in \mathbb{R}_+$.

As examples, the functions $f(u) = u^{\gamma}$ with γ being the ratio of two positive integers and $r(t) = e^{-t}$ or $r(t) = 1$ satisfy (C2) and (C3), respectively.

In 1978, Brands [2] showed that for bounded delays, the solutions to

$$x''(t) + q(t)x(t - \sigma(t)) = 0$$

are oscillatory, if and only if, the solutions to $x''(t) + q(t)x(t) = 0$ are oscillatory. Baculikova *et al.* [3] have studied the linear counterpart of (1) for $0 \leq p(t) \leq p_0 < \infty$ and (C3). They have obtained sufficient conditions for the oscillation of the solutions of the linear counterpart of (1), using comparison techniques. Recently, Chatzarakis *et al.* [7] have established sufficient conditions for the oscillation and asymptotic behavior of all solutions of second-order half-linear differential equations of the form

$$\big(r(x')^\alpha\big)'(t) + q(t)x^\alpha(\sigma(t)) = 0. \tag{2}$$

In an another paper, Chatzarakis *et al.* [8] have considered (2) and established new oscillation criteria. Džurina [9] has studied the linear counterpart of (1) when $0 \leq p(t) \leq p_0 < \infty$ and (C3) and has established sufficient conditions for the oscillation of the solutions of the linear counterpart of (1) by comparison techniques. Karpuz and Santra [12] have obtained several sufficient conditions for the oscillatory and asymptotic behavior of the solutions of (1), for different ranges of p. Pinelas and Santra [15] have studied necessary and sufficient conditions for the solutions of

$$\big(x(t) + p(t)x(t - \tau)\big)' + \sum_{i=1}^{m} q_i(t) f\big(x(t - \sigma_i)\big) = 0.$$

Wong [20] established necessary and sufficient conditions for the oscillation of the solutions to

$$\big(x(t) + px(t - \tau)\big)'' + q(t)f(t - \sigma) = 0,$$

where the constant p satisfies $-1 < p < 0$. Grace et al. [10] have studied (1) and established sufficient conditions for $0 \leq p(t) < 1$. For further work on the oscillation of the solutions to this type of equations, we refer the readers to [1, 4, 5, 14, 16, 21, 22] and the references cited therein. Note that the majority of publications consider only sufficient conditions, and merely a few consider necessary and sufficient conditions. Hence, the objective in this work is to establish both necessary and sufficient conditions for the oscillatory and asymptotic behavior of solutions of (1) without using comparison techniques.

In this paper, we restrict our attention to the study of (1), which includes the class of nonlinear functional differential equations of neutral type.

By a solution to equation (1), we mean a function $x \in \mathrm{C}([T_x, \infty), \mathbb{R})$, where $T_x \geq t_0$, such that $rz' \in \mathrm{C}^1([T_x, \infty), \mathbb{R})$, and satisfies (1) on the interval $[T_x,\infty)$. A solution x of (1) is said to be proper if x is not identically zero eventually, i.e., $\sup\{|x(t)| :\ t\ge T\}>0$ for all $T\ge T_x$. We assume that (1) possesses such solutions. A solution of (1) is called *oscillatory* if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise, it is said to be *non-oscillatory.* (1) itself is said to be oscillatory if all of its solutions are oscillatory.

Remark 1. When the domain is not specified explicitly, all functional inequalities considered in this paper are assumed to hold eventually, i.e., they are satisfied for all t large enough.

2. Preliminaries

Lemma 1. *Assume that (C1)–(C3) and (C4) or (C5) hold, and x is an eventually positive solution of (1). Then we have*

(i) $z(t)<0$ $z'(t)>0$ *and* $(r(z')^{\gamma})'(t)<0$;
(ii) $z(t)>0$ $z'(t)>0$ *and* $(r(z')^{\gamma})'(t)<0$

for sufficiently large t.

Proof. Suppose that there exists a $t_1\ge t_0$ such that $x(t)>0$, $x(\tau(t))$, and $x(\sigma(t))>0$ for $t\ge t_1$. From (1) and (C2), we have

$$(r(z')^{\gamma})'(t)=-q(t)f\big(x(\sigma(t))\big)<0 \quad \text{for } t\ge t_1, \tag{3}$$

which means that $\big(r(z')^{\gamma}\big)(t)$ is non increasing on $[t_1,\infty)$. Since $r(t)>0$, and thus either $z'(t)<0$ or $z'(t)>0$ for $t\ge t_2$, where $t_2\ge t_1$.

If $z'(t)>0$ for $t\ge t_2$, then we have (i) and (ii). We prove now that $z'(t)<0$ can not occur.

If $z'(t)<0$ for $t\ge t_2$, then there exists $\kappa_1>0$ such that $\big(r(z')^{\gamma}\big)(t)\le-\kappa_1$ for $t\ge t_2$, which yields upon integration over $[t_2,t)\subset[t_2,\infty)$ after dividing through by r that

$$z(t)\le z(t_2)-\kappa_1^{1/\gamma}\int_{t_2}^{t}\big(r(\eta)\big)^{-1/\gamma}d\eta \quad \text{for } t\ge t_2. \tag{4}$$

By virtue of condition (C3), $\lim_{t\to\infty}z(t)=-\infty$. We consider now the following possibilities separately.

If x is unbounded, then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty}t_k=\infty$ and $\lim_{k\to\infty}x(t_k)=\infty$, where $x(t_k)=\max\{x(\eta);t_0\le\eta\le t_k\}$. Since $\lim_{t\to\infty}\tau(t)=\infty$, $\tau(t_k)>t_0$ for all sufficiently large k. By $\tau(t)\le t$,

$$x\big(\tau(t_k)\big)=\max\{x(\eta);t_0\le\eta\le\tau(t_k)\}\le\max\{x(\eta);t_0\le\eta\le t_k\}=x(t_k).$$

Therefore, for all large k,

$$z(t_k) = x(t_k) + p(t_k)x\big(\tau(t_k)\big) \geq (1 + p(t_k))x(t_k) > 0,$$

If x is bounded, then z is also bounded, which contradicts $\lim_{t\to\infty} z(t) = -\infty$. Hence, z satisfies one of the cases (i) or (ii).

This completes the proof. ■

Lemma 2. *Assume that (C1)–(C3), (C4) or (C5) and (i) hold, and x is an eventually positive solution of* (1)*. Then,* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t))$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. Then, Lemma 1 holds and z satisfies one of the cases (i) or (ii) for $t_2 \geq t_1$, where $t \geq t_2$. Let z satisfies (i) for $t \geq t_2$. Therefore,

$$\begin{aligned} 0 \geq \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty} z(t) \geq \limsup_{t\to\infty}\big(x(t) - ax(\tau(t))\big) \\ &\geq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}\big(-ax(\tau(t))\big) = (1-a)\limsup_{t\to\infty} x(t) \end{aligned}$$

which implies that $\limsup_{t\to\infty} x(t) = 0$ and hence $\lim_{t\to\infty} x(t) = 0$. ■

Remark 2. In view of (ii) of Lemma 1, it is obvious that $\lim_{t\to\infty} z(t) > 0$, i.e., there exists $\kappa > 0$ such that $z(t) \geq \kappa$ for all large t.

3. Main results

3.1. Non-increasing $f(v)/v^\beta$

We assume that there exists a constant β such that $0 < \beta < \gamma$ and

$$\frac{f(v)}{v^\beta} \geq \frac{f(u)}{u^\beta}, \quad \text{for} \quad 0 < v \leq u\,. \tag{5}$$

Theorem 1. *Assume that (C1)–(C4) and* (5) *hold. Then every unbounded solution of* (1) *oscillates if and only if*

$$\int_T^\infty q(\eta) f\big(\kappa^{1/\gamma}\Pi(\sigma(\eta))\big)d\eta = +\infty \quad \forall T > 0 \quad \textit{and} \quad \kappa > 0. \tag{6}$$

Proof. To prove sufficiency assume, for the sake of contradiction, that there exists a non-oscillatory unbounded solution $x(t)$ of (1). Suppose that $x(t)$ is eventually positive. Then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 1, we see $\big(r(z')^\gamma\big)(t)$ is non-increasing, and z satisfies one of the cases (i) or (ii) on $[t_2, \infty)$, where $t_2 \geq t_1$. Then, there exist two possible cases.

Case 1. Let z satisfies (i) for $t \geq t_2$. As x is unbounded, there exists $T \geq t_2$ such that $x(T) = \max\{x(\eta) : \ t_2 \leq \eta \leq T\}$. Since $z(t) = x(t) + p(t)x(\tau(t))$, we have $x(T) \leq z(T) + \{1 - (2/3)^{1/\gamma}\}x(\tau(T)) < x(T)$, which is a contradiction.

Case 2. Let z satisfies (ii) for $t \geq t_2$. Note that $\lim_{t\to\infty} \big(r(z')^\gamma\big)(t)$ exists. Using $z(t) \leq x(t)$ in (1) and integrating the final inequality from t to $+\infty$, we obtain

$$\int_t^\infty q(\eta) f\big(z(\sigma(\eta))\big) d\eta \leq \big(r(z')^\gamma\big)(t) .$$

That is

$$z'(t) \geq \Big[\frac{1}{r(t)} \int_t^\infty q(\eta) f\big(z(\sigma(\eta))\big) d\eta\Big]^{1/\gamma} \tag{7}$$

for $t \geq t_3$. Let $t_4 > t_3$ be a point such that

$$\Pi(t) - \Pi(t_3) \geq \frac{1}{2}\Pi(t), \ t \geq t_4.$$

Then integrating (7) from t_3 to t, we get

$$\begin{aligned} z(t) - z(t_3) &\geq \int_{t_3}^t \Big[\frac{1}{r(\eta)} \int_\eta^\infty q(\zeta) f\big(z(\sigma(\zeta))\big) d\zeta\Big]^{1/\gamma} d\eta \\ &\geq \int_{t_3}^t \Big[\frac{1}{r(\eta)} \int_t^\infty q(\zeta) f\big(z(\sigma(\zeta))\big) d\zeta\Big]^{1/\gamma} d\eta, \end{aligned} \tag{8}$$

i.e.,

$$\begin{aligned} z(t) &\geq \big(\Pi(t) - \Pi(t_3)\big)\Big[\int_t^\infty q(\zeta) f\big(z(\sigma(\zeta))\big) d\zeta\Big]^{1/\gamma} \\ &\geq \frac{1}{2}\Pi(t)\Big[\int_t^\infty q(\zeta) f\big(z(\sigma(\zeta))\big) d\zeta\Big]^{1/\gamma}. \end{aligned} \tag{9}$$

Using the fact that $\big(r(z')^\gamma\big)(t)$ is non-increasing on $[t_4, \infty)$, we can find a constant $\kappa > 0$ and $t_5 > t_4$ such that $\big(r(z')^\gamma\big)(t) \leq \kappa$ for $t \geq t_5$. Integrating the inequality $z'(t) \leq (\kappa/r(t))^{1/\gamma}$, we have

$$z(t) \leq z(t_5) + \kappa^{1/\gamma}\big(\Pi(t) - \Pi(t_5)\big).$$

Since $\lim_{t\to\infty} \Pi(t) = \infty$, the last inequality becomes

$$z(t) \leq \kappa^{1/\gamma}\Pi(t) \quad \text{for } t \geq t_5 .$$

On the other hand, (5) implies that

$$f\big(z(\sigma(\zeta))\big)=\frac{f\big(z(\sigma(\zeta))\big)}{z^{\beta}\big(\sigma(\zeta)\big)}z^{\beta}\big(\sigma(\zeta)\big)\geq\frac{f\big(\kappa^{1/\gamma}\Pi(\sigma(\zeta))\big)}{\big(\kappa^{1/\gamma}\Pi(\sigma(\zeta))\big)^{\beta}}z^{\beta}\big(\sigma(\zeta)\big).$$

Consequently, (9) becomes

$$z(t)\geq\frac{\Pi(t)}{2}\Big[\int_t^{\infty}\frac{q(\zeta)f\big(\kappa^{1/\gamma}\Pi(\sigma(\zeta))\big)z^{\beta}(\sigma(\zeta))}{\big(\kappa^{1/\gamma}\Pi(\sigma(\zeta))\big)^{\beta}}d\zeta\Big]^{1/\gamma}.$$

If we define

$$w(t)=\int_t^{\infty}\frac{q(\zeta)f\big(\kappa^{1/\gamma}\Pi(\sigma(\zeta))\big)z^{\beta}(\sigma(\zeta))}{\big(\kappa^{1/\gamma}\Pi(\sigma(\zeta))\big)^{\beta}}d\zeta,$$

then $z^{\beta}/\big(\kappa^{1/\gamma}R\big)^{\beta}\geq w^{\beta/\gamma}/\big(2\kappa^{1/\gamma}\big)^{\beta}$. Taking the derivative of w we have

$$\begin{aligned}w'(t)&\leq-\frac{q(t)f\big(\kappa^{1/\gamma}\Pi(\sigma(t))\big)z^{\beta}(\sigma(t))}{\big(\kappa^{1/\gamma}\Pi(\sigma(t))\big)^{\beta}}\\&\leq-\frac{q(t)f\big(\kappa^{1/\gamma}\Pi(\sigma(t))\big)}{\big(2\kappa^{1/\gamma}\big)^{\beta}}w^{\beta/\gamma}(\sigma(t))\leq 0.\end{aligned}$$

Therefore, $w(t)$ is non-increasing on $[t_5,\infty)$ so $w^{\beta/\gamma}(\sigma(t))/w^{\beta/\gamma}(t)\geq 1$, and

$$\begin{aligned}\Big(w^{1-\beta/\gamma}(t)\Big)'&\leq-(1-\beta/\gamma)w^{-\beta/\gamma}(t)\frac{q(t)f\big(\kappa^{1/\gamma}\Pi(\sigma(t))\big)}{\big(2\kappa^{1/\gamma}\big)^{\beta}}w^{\beta/\gamma}\big(\sigma(t)\big)\\&\leq-(1-\beta/\gamma)\frac{q(t)f\big(\kappa^{1/\gamma}\Pi(\sigma(t))\big)}{\big(2\kappa^{1/\gamma}\big)^{\beta}}.\end{aligned}$$

Since $\beta/\gamma<1$ and $w(t)$ is positive and nonincresing. Integrating the last inequality, from t_5 to t, we have

$$\begin{aligned}\frac{(1-\beta/\gamma)}{(2\kappa^{1/\gamma})^{\beta}}\int_{t^5}^{t}q(\eta)f\big(\kappa^{1/\gamma}\Pi(\sigma(\eta))\big)d\eta&\leq-\Big[w^{1-\beta/\gamma}(\eta)\Big]_{t_5}^{t}\\&<w^{1-\beta/\gamma}(t_5)<\infty,\end{aligned}$$

which contradicts (6).

If $x(t)<0$ for $t\geq t_1$, then we set $y(t):=-x(t)$ for $t\geq t_1$ in (1). Using (C2), we find

$$\big(r(t)(\overline{z}'(t))^{\gamma}\big)+q(t)\overline{f}\,(y(\sigma(t)))=0\quad\text{for}\;\;t\geq t_1,$$

where $\overline{z}(t) = y(t)+p(t)y(\tau(t))$ and $\overline{f}(u) := -f(-u)$ for $u \in \mathbb{R}$. Clearly, $\overline{f}$ also satisfies (C2). Then, proceeding as above, we reach the same contradiction. This proves the oscillation of all unbounded solutions of (1).

Next, we show that (6) is a necessary condition. Suppose that (6) does not hold; so for some $\kappa > 0$ the integral in (6) is finite. Then there exists $T \geq t_0$ such that

$$\int_T^\infty q(\eta)f\big(\kappa^{1/\gamma}\Pi(\sigma(\eta))\big)d\eta \leq \frac{\kappa}{3}.$$

Let us consider the closed subset M of continuous functions

$$\begin{aligned}M = \{x : x \in C([t_0,\infty),\mathbb{R}),\ x(t) = 0 \text{ for } t \in [t_0,\ T] \text{ and}\\ (\frac{\kappa}{3})^{1/\gamma}[\Pi(t)-\Pi(T)] \leq x(t) \leq \kappa^{1/\gamma}[\Pi(t)-\Pi(T)] \text{ for } t \geq t_0\}.\end{aligned}$$

We define the operator $\Omega : M \to C([t_0,+\infty),\mathbb{R})$ by

$$(\Omega x)(t) = \begin{cases} 0, \quad t \in [t_0,T] \\ -p(t)x\big(\tau(t)\big) \\ \quad + \int_T^t \left[\frac{1}{r(\eta)}\left[\frac{\kappa}{3} + \int_\eta^\infty q(\zeta)f\big(x(\sigma(\zeta))\big)d\zeta\right]\right]^{1/\gamma} d\eta, \quad t \geq T. \end{cases}$$

For every $x \in M$ and $t \geq T$, we have

$$\begin{aligned}(\Omega x)(t) &\geq \int_T^t \left[\frac{1}{r(\eta)}\left[\frac{\kappa}{3} + \int_\eta^\infty q(\zeta)f\big(x(\sigma(\zeta))\big)d\zeta\right]\right]^{1/\gamma} d\eta\\ &\geq \int_T^t \left[\frac{1}{r(\eta)}\frac{\kappa}{3}\right]^{1/\gamma} d\eta = \left(\frac{\kappa}{3}\right)^{1/\gamma}[\Pi(t)-\Pi(T)].\end{aligned}$$

For every $x \in M$ and $t \geq T$, we have $x(t) \leq \kappa^{1/\gamma}\Pi(t)$ and $f(x(t)) \leq f(\kappa^{1/\gamma}\Pi(t))$. Then

$$\begin{aligned}(\Omega x)(t) &\leq -p(t)x\big(\tau(t)\big) + \int_T^t \left[\frac{1}{r(\eta)}\left(\frac{\kappa}{3}+\frac{\kappa}{3}\right)\right]^{1/\gamma} d\eta\\ &\leq a\kappa^{1/\gamma}\big[\Pi(\tau(t))-\Pi(T)\big]+(2\kappa/3)^{1/\gamma}\big[\Pi(t)-\Pi(T)\big]\\ &\leq a\kappa^{1/\gamma}\big[\Pi(t)-\Pi(T)\big]+(2\kappa/3)^{1/\gamma}\big[\Pi(t)-\Pi(T)\big]\\ &= \big(a+(2/3)^{1/\gamma}\big)\kappa^{1/\gamma}\big[\Pi(t)-\Pi(T)\big] \leq \kappa^{1/\gamma}\big[\Pi(t)-\Pi(T)\big]\end{aligned}$$

which implies that $(\Omega x)(t) \in M$. Let us define now a sequence of continuous function $v_n : [t_0,+\infty) \to \mathbb{R}$ by the recursive formula

$$v_0(t) = \begin{cases} 0, \quad t \in [t_0,T] \\ \frac{\kappa}{3}[\Pi(t)-\Pi(T)], \quad t \geq T. \end{cases}$$

$$v_n(t) = \big(\Omega v_{n-1}\big)(t), \quad n \geq 1$$

Since f is non-decreasing, it is easy to verify that for $n > 1$,

$$\Big(\frac{\kappa}{3}\Big)^{1/\gamma}\big[\Pi(t) - \Pi(T)\big] \leq v_{n-1}(t) \leq v_n(t) \leq \kappa^{1/\gamma}\big[\Pi(t) - \Pi(T)\big].$$

Therefore the point-wise limit of the sequence exists. Let $\lim_{t\to\infty} v_n(t) = v(t)$ for $t \geq t_0$. By Lebesgue's dominated convergence theorem, $u \in M$ and $(\Omega v)(t) = v(t)$, where $v(t)$ is a solution of (1) on $[T, \infty)$ such that $v(t) > 0$. Hence, (6) is necessary.

The proof of the theorem is complete. ■

Example 1. Consider the delay differential equation

$$\text{(10)} \qquad \Big(e^{-t}\big((x(t) - e^{-t}x(t-1))'\big)^{3/5}\Big)'^{1/3} + t\,\big(x(t-2)\big)^{\frac{1}{3}} = 0, \quad t \geq 0\,.$$

Here $\gamma = 3/5$, $r(t) = e^{-t}$, $-1 < p(t) = -e^{-t} \leq 0$, $\tau(t) = t - 1$, $\sigma(t) = t - 2$, $\Pi(t) = \int_0^t e^{5\eta/3}\,d\eta = \frac{3}{5}\big(e^{5t/3} - 1\big)$, $f(v) = v^{1/3}$. For $\beta = 1/2$, we have $f(v)/v^{\beta} = v^{-1/6}$ which is a decreasing function. To check (6) we have

$$\int_0^{\infty} q(\eta) f\big(\kappa^{1/\gamma}\Pi(\sigma(\eta))\big)d\eta = \int_0^{\infty} \eta\Big(\kappa^{5/3}\frac{3}{5}\big(e^{5(\eta-2)/3} - 1\big)\Big)^{1/3} d\eta$$
$$= \infty \quad \forall \kappa > 0,$$

since the integral approaches $+\infty$ as $\eta \to +\infty$. So, all the conditions of Theorem 1 hold, and therefore every unbounded solution of (10) are oscillatory.

Theorem 2. *Assume that* $(C1)$*–*$(C4)$ *hold. Then every unbounded solution of* (1) *oscillates if and only if* (6) *holds for every* $\kappa > 0$.

Proof. To prove sufficiency by contradiction, assume that x is an eventually positive unbounded solution of (1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x\big(\tau(t)\big) > 0$ and $x\big(\sigma(t)\big) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 1, we see that $\big(r(z')^{\gamma}\big)(t)$ is non-increasing, z satisfies one of the cases (i) or (ii) on $[t_2, \infty)$, where $t_2 \geq t_1$. We have the following two possible cases.

Case 1. Let z satisfies (i) for $t \geq t_2$. This case is the same as in the proof of Theorem 1.

Case 2. Let z satisfies (ii) for $t \geq t_2$. Since $z(t)$ is unbounded and monotonically increasing, it follows that

$$\lim_{t\to\infty} \frac{z^{\gamma}(t)}{\Pi^{\gamma}(t)} = \lim_{t\to\infty} \frac{(z'(t))^{\gamma}}{(\Pi'(t))^{\gamma}} = \lim_{t\to\infty} \big(r(z')^{\gamma}\big)(t) = c < \infty.$$

If $c = 0$, then $\lim_{t\to\infty} \Pi(t) = +\infty$ implies that $\lim_{t\to\infty} z(t) < +\infty$, which is invalid ($\because$ $z(t)$ is unbounded). Hence $c \neq 0$. Therefore, there exists a constant $\kappa > 0$ and a $t_2 > t_1$ such that $z(t) \geq \kappa^{1/\gamma}\Pi(t)$ for $t \geq t_2$. Consequently, $x(t) \geq z(t) \geq \kappa^{1/\gamma}\Pi(t)$ for $t \geq t_2$. Using $x(t) \geq \kappa^{1/\gamma}\Pi(t)$ in (1) and then integrating from t_2 to $+\infty$, we obtain a contradiction to (6) for every $\kappa > 0$.

The case where x is an eventually unbounded negative solution is very similar and we omit it here. This proves the oscillation of all unbounded solutions. The necessary part is same as in Theorem 1.

This completes the proof. ■

Theorem 3. *Assume that* (*C1*)–(*C4*) *and* (5) *hold. Then every solution of* (1) *oscillates or* $\lim_{t\to\infty} x(t) = 0$ *if and only if* (6) *holds for every* $\kappa > 0$.

Proof. We show sufficiency by contradiction. Assume that x is an eventually positive solution of (1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 1, we see $\big(r(z')^\gamma\big)(t)$ is non-increasing, and z satisfies one of the cases (i) or (ii) on $[t_2, \infty)$, where $t_2 \geq t_1$. Thus, there exist two possible cases.

Case 1. Let z satisfies (i) for $t \geq t_2$. Then, by Lemma 2, we have $\lim_{t\to\infty} x(t) = 0$.

Case 2. Let z satisfies (ii) for $t \geq t_2$. The case directly follows from Theorem 1.

The case where x is a negative solution is similar and we omit it here.

The necessary part is the same as in Theorem 1.

This completes the proof. ■

3.2. Non-increasing $f(u)/u^\beta$

Let $\beta > \gamma$ such that

$$\frac{f(v)}{v^\beta} \leq \frac{f(u)}{u^\beta}, \quad \text{for } 0 < v \leq u\,. \tag{11}$$

Theorem 4. *Assume that* (*C1*)–(*C3*), (*C5*) *and* (11) *hold,* $\sigma'(t) \geq 1$, *for* $t \in \mathbb{R}_+$. *Then every solution of* (1) *oscillates or* $\lim_{t\to\infty} x(t) = 0$ *if and only if*

$$\int_T^\infty \Big[\frac{1}{r(\eta)}\Big[\int_\eta^\infty q(\zeta)d\zeta\Big]\Big]^{1/\gamma} d\eta = +\infty \quad \forall T > 0. \tag{12}$$

Proof. To prove sufficiency by contradiction, we use a similar argument as in the proof of Theorem 3, for the first case when z satisfies (i). Let's consider Case 2, for $t \geq t_2$. By Remark 2, there exists a constant $\kappa > 0$ and $t_2 > t_1$ such that $z\big(\sigma(t)\big) \geq \kappa$ for $t \geq t_2$. Consequently,

$$f\big(z(\sigma(t))\big) = \frac{f\big(z(\sigma(t))\big)}{z^\beta\big(\sigma(t)\big)} z^\beta\big(\sigma(t)\big) \geq \frac{f(\kappa)}{\kappa^\beta} z^\beta\big(\sigma(t)\big) \tag{13}$$

for $t \geq t_2$. Using $z(t) \leq x(x)$ and (13) in (1), and then integrating the final inequality we have

$$\lim_{A\to\infty} \big[\big(r(z'^\gamma)'\big)(\eta)\big]_t^A + \frac{f(\kappa)}{\kappa^\beta} \int_t^\infty q(\eta) z^\beta\big(\sigma(\eta)\big) d\eta \leq 0 .$$

Using $\big(r(z'^\gamma)'\big)(t)$ is positive and non-increasing, we have

$$\begin{aligned}\frac{f(\kappa)}{\kappa^\beta} \int_t^\infty q(\eta) z^\beta\big(\sigma(\eta)\big) d\eta &\leq \big(r(z')^\gamma\big)(t) \\ &\leq \big(r(z')^\gamma\big)(\sigma(t)) \leq r(t)\big((z')^\gamma\big)(\sigma(t))\end{aligned}$$

for all $t \geq t_2$. Therefore,

$$\Big(\frac{f(\kappa)}{\kappa^\beta}\Big)^{1/\gamma} \Big[\frac{1}{r(t)} \Big[\int_t^\infty q(\eta) z^\beta\big(\sigma(\eta)\big) d\eta\Big]\Big]^{1/\gamma} \leq z'\big(\sigma(t)\big)$$

implies that

$$\Big(\frac{f(\kappa)}{\kappa^\beta}\Big)^{1/\gamma} \Big[\frac{1}{r(t)} \Big[\int_t^\infty q(\eta) d\eta\Big]\Big]^{1/\gamma} \leq \frac{z'\big(\sigma(t)\big)}{z^{\beta/\gamma}\big(\sigma(t)\big)} \leq \frac{z'\big(\sigma(t)\big)\sigma'(t)}{z^{\beta/\gamma}\big(\sigma(t)\big)}$$

Integrating the last inequality from t_2 to $+\infty$, we have

$$\begin{aligned}\Big(\frac{f(\kappa)}{\kappa^\beta}\Big)^{1/\gamma} \int_{t_2}^\infty \Big[\frac{1}{r(\eta)} \Big[\int_\eta^\infty q(\zeta) d\zeta\Big]\Big]^{1/\gamma} d\eta &< \int_{t_2}^\infty \frac{z'\big(\sigma(\eta)\big)\sigma'(\eta)}{z^{\beta/\gamma}\big(\sigma(\eta)\big)} d\eta \\ &\leq \frac{z^{1-\beta/\gamma}(\sigma(t_2))}{\beta/\gamma - 1} < \infty ,\end{aligned}$$

which contradicts (12).

The case where x is an eventually negative solution is omitted since it can be dealt similarly.

Next, we show that (12) is necessary. Assume that (12) does not hold and let there exists $T \geq t_0$ such that

$$\int_T^t \Big[\frac{1}{r(\eta)} \Big[\int_\eta^\infty q(\zeta) d\zeta\Big]\Big]^{1/\gamma} d\eta \leq \frac{1-a}{5\big(f(1)\big)^{1/\gamma}},$$

where $\kappa > 0$ is a constant. Let us consider the closed subset M of continuous functions

$$M = \Big\{x \in C([t_0,\infty),\mathbb{R}) : x(t) = \frac{1-a}{5},\ t \in [t_0,T];\\ \frac{1-a}{5} \le x(t) \le 1 \text{ for } t \ge T\Big\}.$$

We define the operator $\Omega : M \to C([t_0,\infty),\mathbb{R})$ by

$$(\Omega x)(t) = \begin{cases} \frac{1-a}{5}, \quad t \in [t_0,T] \\ -p(t)x\big(\tau(t)\big) + \frac{1-a}{5} \\ +\int_T^t \Big[\frac{1}{r(\eta)}\big[\int_\eta^\infty q(\zeta)f\big(x(\sigma(\zeta))\big)d\zeta\big]\Big]^{1/\gamma} d\eta, \quad t \ge T. \end{cases}$$

For every $x \in M$ and $t \ge T$, $(\Omega x)(t) \ge \frac{1-a}{5}$ and

$$\begin{aligned}(\Omega x)(t) &\le a + \frac{1-a}{5} + \big(f(1)\big)^{1/\gamma}\int_T^t \Big[\frac{1}{r(\eta)}\big[\int_\eta^\infty q(\zeta)d\zeta\big]\Big]^{1/\gamma} d\eta \\ &\le a + \frac{1-a}{5} + \frac{1-a}{5} = \frac{3a+2}{5} < 1\end{aligned}$$

which implies that $\Omega x \in M$. The rest of the proof follows from Theorem 1. The proof of the theorem is complete. ■

Example 2. Consider the delay differential equation

$$\Big(\Big(\big(x(t) - e^{-t}x(\tau(t))\big)'\Big)^{1/5}\Big)'^{\frac{7}{3}} + (t+1)\big(x(t-2)\big)^{\frac{7}{3}} = 0, \quad t \ge 0. \tag{14}$$

Here $\gamma = 1/5$, $r(t) = 1$, $\sigma(t) = t-2$, $f(v) = v^{\frac{7}{3}}$. For $\beta = 4/3$, we have $f(v)/v^\beta = v$ which is an increasing function. To check (12) we have

$$\int_2^\infty \Big[\int_\eta^\infty (\zeta+1)d\zeta\Big]^5 d\eta = \infty.$$

So, all the conditions of Theorem 4 hold, and therefore every solution of (14) oscillates or $\lim_{t\to\infty} x(t) = 0$.

4. Comments

It is worth noting that the necessary and sufficient conditions we have established, hold when $-1 < p(t) \le 0$. These conditions do not hold in other ranges of $p(t)$. Therefore, the undertaken problem is incomplete for all range of $p(t)$.

At this point, we will be giving one remarks and two examples to conclude the paper.

Remark 3. The results in this paper also hold for equations of the form

$$\Big(r(t)\big(\big(x(t)+p(t)x(\tau(t))\big)'\big)^{\gamma}\Big)' + \sum_{i=1}^{m} q_i(t) f_i\big(x(\sigma_i(t))\big) = 0\,,$$

where p, r, q_i, f_i, σ_i $(i = 1, 2, \ldots, m)$ satisfy assumptions (C1)–(C5). In order to extend Theorems 1–4, there exists an index i such that q_i, f_i, σ_i fulfill (6) and (12).

We conclude the paper by presenting two examples, which show how Remark 3 can be applied.

Example 3. Consider the delay differential equation

$$\begin{aligned}(15)\qquad \Big(e^{-t}\big(\big(x(t)-e^{-t}x(\tau(t))\big)'\big)^{3/5}\Big)' &+ \frac{1}{t+1}(x(t-2))^{1/3} \\ &+ \frac{1}{t+2}(x(t-1))^{1/5} = 0, \quad t \geq 0.\end{aligned}$$

Here $\gamma = 3/5$, $r(t) = e^{-t}$, $p(t) = -e^{-t}$, $\sigma_1(t) = t-2$, $\sigma_2(t) = t-1$, $\Pi(t) = \int_0^t e^{5s/3}\, ds = \frac{3}{5}\big(e^{5t/3}-1\big)$, $f_1(v) = v^{1/3}$ and $f_2(v) = v^{1/5}$. For $\beta = 1/2$, we have $f_1(v)/v^{\beta} = v^{-1/6}$ and $f_2(v)/v^{\beta} = v^{-3/10}$ which both are decreasing functions. To check (6) we have

$$\begin{aligned}\int_0^{\infty}\sum_{i=1}^{m} q_i(\eta) f_i\big(\kappa^{1/\gamma}\Pi(\sigma_i(\eta))\big)d\eta &\geq \int_0^{\infty} q_1(\eta) f_1\big(\kappa^{1/\gamma}\Pi(\sigma_1(\eta))\big)d\eta \\ &= \int_0^{\infty}\frac{1}{\eta+1}\Big(\kappa^{5/3}\frac{3}{5}\big(e^{5(\eta-2)/3}-1\big)\Big)^{1/3} d\eta = \infty \quad \forall \kappa > 0,\end{aligned}$$

since the integral approaches $+\infty$ as $\eta \to +\infty$. So, all the conditions of Theorem 1 hold, and therefore, every unbounded solution of (15) are oscillatory.

Example 4. Consider the delay differential equation

$$(16)\qquad \Big(\big(\big(x(t)-e^{-t}x(\tau(t))\big)'\big)^{5/7}\Big)'^{5/3} + (t+1)(x(t-1))^3 = 0, \quad t \geq 0\,.$$

Here $\gamma = 5/7$, $r(t) = 1$, $\sigma_1(t) = t-2$, $\sigma_2(t) = t-1$, $f_1(v) = v^{5/3}$ and $f_2(v) = v^3$. For $\beta = 4/3$, we have $f_1(v)/v^{\beta} = v^{1/3}$ and $f_2(v)/v^{\beta} = v^{5/3}$ which both are increasing functions. Clearly, all the conditions of Theorem 4 hold. Thus, all solution of (16) oscillates or $\lim_{t\to\infty} x(t) = 0$.

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