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COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED METRIC SPACES

Abstract. This paper deals with some coupled fixed point theorems for a mapping with mixed monotone property and satisfying certain generalized rational contraction in a partially ordered metric space. Also, the result for the existence and uniqueness of a coupled fixed point to the map is given under ordered relation in a space. These results generalize and extend some existing results in the literature.

Key words: partially ordered metric space, rational contractions, coupled fixed point, mixed monotone property.

AMS Mathematics Subject Classification: 47H10, 55M20, 54H25, 26A42.

1. Introduction

In 1922, Banach has proved a fixed point theorem for a contraction mapping in a complete metric space. It plays an important role in analysis to find a unique solution of many results. It is very popular tool in many branches of mathematics for solving existing problems. Since then there are numerous generalizations [1, 2, 3, 4, 5, 6, 7] of this result by weakening its hypotheses while retaining the convergence property of successive iterates for a unique fixed point of mappings. The concept of metric space has also been generalized in different directions during past decades. Some important generalization are on rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi-metric spaces, probabilistic metric spaces, D-metric spaces, G-metric spaces, F-metric spaces, cone metric spaces, and so on.

The discussion on the extended Banach contraction principle over partially ordered sets can be found from the papers of Wolk [8] and Monjardet [9]. The existence of fixed points in partially ordered metric spaces with some applications to matrix equations was studied by Ran and Reurings [10]. Further the extended results of [10] over partially order sets and the applications of these results for a first order ordinary differential equations with periodic boundary conditions were discussed by Nieto et. al.[11, 12, 13]. The results presented therein papers [14, 15, 16, 17, 18, 19, 20, 21, 22, 23] will provide a vast information about the existence and uniqueness of fixed points in cone metric spaces, partially ordered metric spaces and other spaces too.

First the concept of coupled fixed points in ordered spaces was introduced by Bhaskar and Lakshmikantham [24] and applied their results to boundary value problems for the unique solution. Also, Lakshmikanthm and Cirić [25] introduced the concept of coupled coincidence, common fixed points to nonlinear contractions in ordered metric spaces. More results on coupled coincidence, common fixed points in ordered metric spaces, one can see [26, 27, 28, 29, 30, 31, 32, 33, 34, 35].

The purpose of this paper is to prove some coupled fixed point results in the frame work of a partially ordered metric space. These results generalize a result of Singh and Chatterjee [5] in a partially ordered metric space with mixed monotone property for the mapping.

2. Preliminaries

Definition 1. *Let $(X, \preceq)$ be a partially ordered set. A self-mapping $f : X \to X$ is said to be strictly increasing if $f(x) \prec f(y)$, for all $x, y \in X$ with $x \prec y$ and is also said to be strictly decreasing if $f(x) \succ f(y)$, for all $x, y \in X$ with $x \prec y$.*

Definition 2. *Let $(X, \preceq)$ be a partially ordered set and f is a self mapping defined over X is said to be strict mixed monotone property, if $f(x, y)$ is strictly increasing in x and strictly decreasing in y as well.*

$$i.e., \text{ for any } x_1, x_2 \in X \text{ with } x_1 \prec x_2 \Rightarrow f(x_1, y) \prec f(x_2, y) \text{ and}$$

$$\text{for any } y_1, y_2 \in X, \text{ with } y_1 \prec y_2 \Rightarrow f(x, y_1) \succ f(x, y_2).$$

Definition 3. *Let $(X, \preceq)$ be a partially ordered set and $f : X \times X \to X$ be a mapping. A point $(x, y) \in X \times X$ is said to be a coupled fixed point to f, if $f(x, y) = x$ and $f(y, x) = y$.*

Definition 4. *The triple $(X, d, \preceq)$ is called a partially ordered metric space, if $(X, \preceq)$ is a partially ordered set together with (X, d) is a metric space.*

Definition 5. *If (X, d) is a complete metric space, then the triple $(X, d, \preceq)$ is called a partially ordered complete metric space.*

Definition 6. *A partially ordered metric space $(X, d, \preceq)$ is called an ordered complete (OC), for any convergent sequence $\{x_n\}_{n=0}^{+\infty} \subset X$ such that one the following hold: either*

- *if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$, for all $n \in \mathbb{N}$ that is, $x = \sup\{x_n\}$ or*
- *if $\{x_n\}$ is a non-increasing sequence in X such that $x_n \to x$, then $x \preceq x_n$, for all $n \in \mathbb{N}$ that is, $x = \inf\{x_n\}$.*

3. Main results

In this section, we prove some coupled fixed point theorems for a self mapping in the context of ordered metric space.

Theorem 1. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that a self mapping $f : X \times X \to X$ has a strict mixed monotone property on X satisfying the following condition*

$$\begin{aligned} (1)\quad d(f(x,y), f(\mu,\upsilon)) \leq\ & \alpha \frac{d(x, f(x,y))\,[1 + d(\mu, f(\mu,\upsilon))]}{1 + d(x,\mu)} \\ & + \beta[d(x, f(\mu,\upsilon)) + d(\mu, f(x,y))] + \gamma d(x,\mu), \end{aligned}$$

for all x, y, μ, υ in X and $\alpha, \beta, \gamma \in [0,1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$. Suppose that either f is continuous or X has an ordered complete property (OC) then f has a coupled fixed point $(x, y) \in X \times X$, if there exists two points $x_0, y_0 \in X$ with $x_0 \prec f(x_0, y_0)$ and $y_0 \succ f(y_0, x_0)$.

Proof. Suppose f is continuous and let $x_0, y_0 \in X$ such that $x_0 \prec f(x_0, y_0)$ and $y_0 \succ f(y_0, x_0)$. Now define two sequences $\{x_n\}, \{y_n\}$ in X as follows

$$(2)\qquad x_{n+1} = f(x_n, y_n) \text{ and } y_{n+1} = f(y_n, x_n), \text{ for all } n \geq 0.$$

Now, we have to show that for all $n \geq 0$,

$$(3)\qquad x_n \prec x_{n+1}$$

and

$$(4)\qquad y_n \succ y_{n+1},$$

for this, we use the method of mathematical induction. Suppose $n = 0$, since $x_0 \prec f(x_0, y_0)$ and $y_0 \succ f(y_0, x_0)$, and from (2), we have $x_0 \prec f(x_0, y_0) = x_1$ and $y_0 \succ f(y_0, x_0) = y_1$. Hence, the inequalities (3) and (4) hold for $n = 0$.

Suppose that the inequalities (3) and (4) hold for all $n > 0$ and by use of the strict mixed monotone property of f, we get

$$x_{n+1} = f(x_n, y_n) \prec f(x_{n+1}, y_n) \prec f(x_{n+1}, y_{n+1}) = x_{n+2} \tag{5}$$

and

$$y_{n+1} = f(y_n, x_n) \succ f(y_{n+1}, x_n) \succ f(y_{n+1}, x_{n+1}) = y_{n+2}. \tag{6}$$

Therefore, the inequalities (3) and (4) hold for all $n \geq 0$ and finally we obtain that

$$x_0 \prec x_1 \prec x_2 \prec x_3 \prec \ldots\ldots\ldots \prec x_n \prec x_{n+1} \prec \ldots\ldots\ldots\ldots \tag{7}$$

and

$$y_0 \succ y_1 \succ y_2 \succ y_3 \succ \ldots\ldots\ldots \succ y_n \succ y_{n+1} \succ \ldots\ldots\ldots\ldots \tag{8}$$

Since $x_n \prec x_{n+1}$, $y_n \succ y_{n+1}$ and from (2), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n, y_n), f(x_{n-1}, y_{n-1})) \\ &\leq \alpha \frac{d(x_n, f(x_n, y_n))\,[1 + d(x_{n-1}, f(x_{n-1}, y_{n-1}))]}{1 + d(x_n, x_{n-1})} \\ &\quad + \beta[d(x_n, f(x_{n-1}, y_{n-1})) \\ &\quad + d(x_{n-1}, f(x_n, y_n))] + \gamma d(x_n, x_{n-1}), \end{aligned}$$

which implies that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \alpha\ \frac{d(x_n, x_{n+1})\,[1 + d(x_{n-1}, x_n)]}{1 + d(x_n, x_{n-1})} + \beta[d(x_n, x_n) \\ &\quad + d(x_{n-1}, x_{n+1})] + \gamma d(x_n, x_{n-1}). \end{aligned}$$

Finally, we arrive at

$$d(x_{n+1}, x_n) \leq \left(\frac{\beta + \gamma}{1 - \alpha - \beta}\right) d(x_n, x_{n-1}). \tag{9}$$

Similarly by following above procedure, we get

$$d(y_{n+1}, y_n) \leq \left(\frac{\beta + \gamma}{1 - \alpha - \beta}\right) d(y_n, y_{n-1}). \tag{10}$$

Therefore, from (9) and(10), we get

$$d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \leq \left(\frac{\beta + \gamma}{1 - \alpha - \beta}\right) [d(x_n, x_{n-1}) + d(y_n, y_{n-1})]\,.$$

Now, let us define a sequence $\{S_n\}$ by $\{S_n\} = \{d(x_{n+1}, x_n) + d(y_{n+1}, y_n)\}$. Then by induction we get

$$0 \leq S_n \leq kS_{n-1} \leq k^2 S_{n-2} \leq k^3 S_{n-3} \leq .... \leq k^n S_0\,,$$

where $k = \frac{\beta+\gamma}{1-\alpha-\beta} < 1$ and hence,

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] = 0.$$

Consequently, we obtain that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$ and $\lim_{n\to+\infty} d(y_n, y_{n+1}) = 0$. Using triangular inequality for $m \geq n$, we have

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + ... + d(x_{n+1}, x_n)$$

and

$$d(y_m, y_n) \leq d(y_m, y_{m-1}) + d(y_{m-1}, y_{m-2}) + ... + d(y_{n+1}, y_n).$$

Therefore

$$\begin{aligned} d(x_m, x_n) + d(y_m, y_n) &\leq S_{m-1} + S_{m-2} + ... + S_n \\ &\leq \left(k^{m-1} + k^{m-2} + ... + k^n\right) S_0 \\ &\leq \frac{k^n}{1-k} S_0\,, \end{aligned}$$

as $m, n \to \infty$, $d(x_m, x_n) + d(y_m, y_n) \to 0$, which shows that both the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. So, by the completeness of X, there exists a point $(x, y) \in X \times X$ such that $x_n \to x$ and $y_n \to y$. Further by the continuity of f, we have

$$x = \lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} f(x_n, y_n) = f(\lim_{n\to+\infty} x_n, \lim_{n\to+\infty} y_n) = f(x, y)$$

and

$$y = \lim_{n\to+\infty} y_{n+1} = \lim_{n\to+\infty} f(y_n, x_n) = f(\lim_{n\to+\infty} y_n, \lim_{n\to+\infty} x_n) = f(y, x).$$

Therefore, f has has a coupled fixed point in X.

Another way suppose that X has an ordered complete (OC) property. From above argument there is an increasing Cauchy sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$. Hence, $x = \sup\{x_n\}$, i.e., $x_n \preceq x$, for all $n \in \mathbb{N}$. We can also prove that $x_n \prec x$, for all n otherwise there is some $n_0 \in \mathbb{N}$ with $x_{n_0} = x$, and thence $x \prec x_{n_0} \preceq x_{n_0+1} = x$ which is wrong. So, by the strict monotone increasing nature of f over the first variable, we get

$$f(x_n, y_n) \prec f(x, y_n). \tag{11}$$

Similarly, there is another decreasing Cauchy sequence $\{y_n\}$ in X such that $y_n \to y \in X$. So, by similar argument above we have $y_n \succeq y$ for all $n \in \mathbb{N}$. Also, the strict monotone decreasing nature of f over the second variable, we get

$$f(x, y_n) \prec f(x, y). \tag{12}$$

Hence, from equations (11) and (12), we get

$$f(x_n, y_n) \prec f(x, y) \Rightarrow x_{n+1} \prec f(x, y), \text{ for all } n \in \mathbb{N}. \tag{13}$$

But $x_n \prec x_{n+1} \prec f(x, y)$, for all $n \in \mathbb{N}$ and $x = \sup\{x_n\}$ then we get $x \preceq f(x, y)$. Now let $z_0 = x$ and $z_{n+1} = f(z_n, y_n)$. Then from similar argument above, the sequence $\{z_n\}$ is a nondecreasing Cauchy sequence in X and converges to a point say z in X. Hence, we have $z = \sup\{z_n\}$.

But for all $n \in \mathbb{N}, x_n \preceq x = z_0 \preceq f(z_0, y_0) \preceq z_n \preceq z$ then from (1), we have

$$\begin{aligned} d(x_{n+1}, z_{n+1}) &= d(f(x_n, y_n), f(z_n, y_n)) \\ &\leq \alpha \frac{d(x_n, f(x_n, y_n))\,[1 + d(z_n, f(z_n, y_n))]}{1 + d(x_n, z_n)} \\ &\quad + \beta[d(x_n, f(z_n, y_n)) + d(z_n, f(x_n, y_n))] + \gamma d(x_n, z_n). \end{aligned}$$

Taking limit as $n \to \infty$ on both sides of the above equation, we get

$$d(x, z) \leq (2\beta + \gamma) d(x, z).$$

As $2\beta + \gamma < 1$ then we get $d(x, z) = 0$. Hence, $x = z = \sup\{x_n\}$ which implies that $x \preceq f(x, y) \preceq x$. Thus, $x = f(x, y)$. Again by following the similar argument above, we can obtain $y = f(y, x)$. Therefore, f has a coupled fixed point in $X \times X$. ■

For the existence and uniqueness of a coupled fixed point of f over a complete partial ordered metric space X, we furnish the following partial order relation.

$$(\mu, \upsilon) \preceq (x, y) \Leftrightarrow x \succeq \mu,\ y \preceq \upsilon, \text{ for any } (x, y), (\mu, \upsilon) \in X \times X.$$

Theorem 2. *Along the hypotheses stated in Theorem 1 and suppose that for every $(x, y), (r, s) \in X \times X$, there exists $(u, v) \in X \times X$ such that $(f(u, v), f(v, u))$ is comparable to $(f(x, y), f(y, x))$ and $(f(r, s), f(s, r))$ then f has a unique coupled fixed point in $X \times X$.*

Proof. As we know from Theorem 1, the set of coupled fixed points of f is non empty. Suppose that (x, y) and (r, s) are two coupled fixed points of the mapping f, then $x = f(x,y), y = f(y,x)$ and $r = f(r,s)$, $s = f(s,r)$. Now, we have to show that $x = r$, $y = s$ for the uniqueness of a coupled fixed point to f.

From hypotheses, there exists $(u,v) \in X \times X$ such that $(f(u,v), f(v,u))$ is comparable to $(f(x,y), f(y,x))$ and $(f(r,s), f(s,r))$. Put $u = u_0$ and $v = v_0$ and let $u_1, v_1 \in X$, then $u_1 = f(u_0, v_0)$ and $v_1 = f(v_0, u_0)$. Similarly by induction from Theorem 1, we can define two sequences $\{u_n\}$ and $\{v_n\}$ from $u_{n+1} = f(u_n, v_n)$ and $v_{n+1} = f(v_n, u_n)$ for all $n \in \mathbb{N}$. Also, define the sequences $\{x_n\}$, $\{y_n\}$ and $\{r_n\}$, $\{s_n\}$ by setting $x = x_0$, $y = y_0$, $r = r_0$ and $s = s_0$. Then from the proof of Theorem 1, we have $x_n \to x = f(x,y)$, $y_n \to y = f(y,x)$, $r_n \to r = f(r,s)$ and $s_n \to s = f(s,r)$ for all $n \geq 1$. But $(f(x,y), f(y,x)) = (x,y)$ and $(f(u_0,v_0), f(v_0,u_0)) = (u_1,v_1)$ are comparable then we have $x \succeq u_1$ and $y \preceq v_1$. Next to show that (x,y) and (u_n, v_n) are comparable, i.e., to show that $x \succeq u_n$ and $y \preceq v_n$ for all $n \in \mathbb{N}$. Suppose the inequalities hold for some $n \geq 0$, then from the strict mixed monotone property of f, we have $u_{n+1} = f(u_n, v_n) \preceq f(x,y) = x$ and $v_{n+1} = f(v_n, u_n) \succeq f(y,x) = y$ and hence $x \succeq u_n$ and $y \preceq v_n$ for all $n \in \mathbb{N}$.

Now, from contraction condition (1), we have

$$\begin{aligned} d(x, u_{n+1}) &= d(f(x,y), f(u_n, v_n)) \\ &\leq \alpha \frac{d(x, f(x,y))\,[1 + d(u_n, f(u_n, v_n))]}{1 + d(x, u_n)} \\ &\quad + \beta[d(x, f(u_n, v_n)) + d(u_n, f(x,y))] + \gamma d(x, u_n), \end{aligned}$$

which implies that

$$d(x, u_{n+1}) \leq \left(\frac{\beta + \gamma}{1 - \beta}\right) d(x, u_n).$$

Similarly, we can get

$$d(y, v_{n+1}) \leq \left(\frac{\beta + \gamma}{1 - \beta}\right) d(y, v_n).$$

Suppose $D = \frac{\beta+\gamma}{1-\beta} < 1$, then by adding above two equations, we get

$$\begin{aligned} d(x, u_{n+1}) + d(y, v_{n+1}) &\leq D\,[d(x, u_n) + d(y, v_n)] \\ &\leq D^2\,[d(x, u_{n-1}) + d(y, v_{n-1})] \\ &\qquad \dots\dots\dots\dots\dots\dots \\ &\leq D^n\,[d(x, u_0) + d(y, v_0)]\,. \end{aligned}$$

On taking limit as $n \to \infty$ to the above inequality, we get $\lim_{n\to+\infty} d(x, u_{n+1}) + d(y, v_{n+1}) = 0$. Thus, we have $\lim_{n\to+\infty} d(x, u_{n+1}) = 0$ and $\lim_{n\to+\infty} d(y, v_{n+1}) = 0$.

Similarly, we can also prove that $\lim_{n\to+\infty} d(r,u_n)=0$ and $\lim_{n\to+\infty} d(s,v_n)=0$. Further from triangular inequality, we have

$$d(x,r)\leq d(x,u_n)+d(u_n,r) \text{ and } d(y,s)\leq d(y,v_n)+d(v_n,s).$$

Letting $n\to\infty$ to the above inequalities, we obtain $d(x,r)=0=d(y,s)$. Hence $x=r$ and $y=s$, this shows the uniqueness of f.

This completes the proof. ∎

Theorem 3. *Along the hypotheses stated in Theorem 1 and if x_0, y_0 are comparable then f has a coupled fixed point in $X\times X$.*

Proof. Suppose (x,y) is a coupled fixed point of f, then from the proof of Theorem 1, we have two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_n\to x$ and $y_n\to y$.

Suppose that $x_0\preceq y_0$, we have to show that $x_n\preceq y_n$, for all $n\geq 0$. Suppose it holds for some $n\geq 0$ and from the strict mixed monotone property of f, we obtained that $x_{n+1}=f(x_n,y_n)\preceq f(y_n,x_n)=y_{n+1}$. Therefore, from the contraction condition (1), we have

$$\begin{aligned} d(x_{n+1},y_{n+1}) &= d(f(x_n,y_n),f(y_n,x_n)) \\ &\leq \alpha\frac{d(x_n,f(x_n,y_n))\,[1+d(y_n,f(y_n,x_n))]}{1+d(x_n,y_n)} \\ &\quad + \beta[d(x_n,f(y_n,x_n))+d(y_n,f(x_n,y_n))]+\gamma d(x_n,y_n). \end{aligned}$$

On taking limit as $n\to\infty$, we get

$$d(x,y)\leq(2\beta+\gamma)d(x,y).$$

Thus, we have $d(x,y)=0$, since $2\beta+\gamma<1$. Therefore, $f(x,y)=x=y=f(y,x)$.

Similarly, we can also show that $f(x,y)=x=y=f(y,x)$ by considering $y_0\leq x_0$. Hence, (x,y) is a coupled fixed point of f in $X\times X$. ∎

4. Applications

In this section, we state some applications to the main result of a self mapping which is involved in an integral type contraction.

Let us denote a set τ of all functions χ defined on $[0,+\infty)$ satisfying the following properties:

(i). Each χ is Lebesgue integrable mapping on every compact subset of $[0,+\infty)$,

(ii). For any $\epsilon>0$, we have $\int_0^\epsilon \chi(t)dt>0$.

Theorem 4. *Let* $(X, d, \preceq)$ *be a complete partially ordered metric space. Suppose that a self mapping* $f : X \times X \to X$ *has a strict mixed monotone property on* X *satisfying the following condition*

$$\int_0^{d(f(x,y),f(\mu,\upsilon))} \varphi(t)dt \le \alpha \int_0^{\frac{d(x,f(x,y))[1+d(\mu,f(\mu,\upsilon))]}{1+d(x,\mu)}} \varphi(t)dt + \beta \int_0^{d(x,f(\mu,\upsilon))+d(\mu,f(x,y))} \varphi(t)dt + \gamma \int_0^{d(x,\mu)} \varphi(t)dt\,, \tag{14}$$

for all $x, y, \mu, \upsilon \in X$ *with* $x \succeq \mu$ *and* $y \preceq \upsilon$, $\varphi(t) \in \tau$ *and where* $\alpha, \beta, \gamma \in [0, 1)$ *such that* $0 \le \alpha + 2\beta + \gamma < 1$. *Suppose that either* f *is continuous or* X *has an ordered complete* (OC) *property then* f *has a coupled fixed point* $(x, y) \in X \times X$, *if there exists two points* $x_0, y_0 \in X$ *with* $x_0 \prec f(x_0, y_0)$ *and* $y_0 \succ f(y_0, x_0)$.

Similarly, we can obtain the following coupled fixed point results in partially ordered metric space by taking $\beta = 0$; $\gamma = 0$; $\alpha = 0$ and $\alpha = \beta = 0$ in Theorem 4.

Theorem 5. *Let* $(X, d, \preceq)$ *be a complete partially ordered metric space. Suppose that a self mapping* $f : X \times X \to X$ *has a strict mixed monotone property on* X *satisfying the following condition*

$$\int_0^{d(f(x,y),f(\mu,\upsilon))} \varphi(t)dt \le \alpha \int_0^{\frac{d(x,f(x,y))[1+d(\mu,f(\mu,\upsilon))]}{1+d(x,\mu)}} \varphi(t)dt + \gamma \int_0^{d(x,\mu)} \varphi(t)dt\,, \tag{15}$$

for all $x, y, \mu, \upsilon \in X$ *with* $x \succeq \mu$ *and* $y \preceq \upsilon$, $\varphi(t) \in \tau$ *and* $\alpha, \gamma \in [0, 1)$ *with* $0 \le \alpha + \gamma < 1$. *Suppose that either* f *is continuous or* X *has (OC) property then* f *has a coupled fixed point* $(x, y) \in X \times X$, *if there exists two points* $x_0, y_0 \in X$ *with* $x_0 \prec f(x_0, y_0)$ *and* $y_0 \succ f(y_0, x_0)$.

Theorem 6. *Let* $(X, d, \preceq)$ *be a complete partially ordered metric space. Suppose that a self mapping* $f : X \times X \to X$ *has a strict mixed monotone property on* X *satisfying the following condition*

$$\int_0^{d(f(x,y),f(\mu,\upsilon))} \varphi(t)dt \le \alpha \int_0^{\frac{d(x,f(x,y))[1+d(\mu,f(\mu,\upsilon))]}{1+d(x,\mu)}} \varphi(t)dt + \beta \int_0^{d(x,f(\mu,\upsilon))+d(\mu,f(x,y))} \varphi(t)dt\,, \tag{16}$$

for all $x, y, \mu, \upsilon \in X$ *with* $x \succeq \mu$ *and* $y \preceq \upsilon$, $\varphi(t) \in \tau$ *and* $\alpha, \beta \in [0, 1)$ *with* $0 \leq \alpha + 2\beta < 1$. *Suppose that either* f *is continuous or* X *has* (OC) *property then* f *has a coupled fixed point* $(x, y) \in X \times X$, *if there exists two points* $x_0, y_0 \in X$ *with* $x_0 \prec f(x_0, y_0)$ *and* $y_0 \succ f(y_0, x_0)$.

Theorem 7. *Let* $(X, d, \preceq)$ *be a complete partially ordered metric space. Suppose that a self mapping* $f : X \times X \to X$ *has a strict mixed monotone property on* X *satisfying the following condition*

$$\int_0^{d(f(x,y),f(\mu,\upsilon))} \varphi(t)dt \leq \beta \int_0^{d(x,f(\mu,\upsilon))+d(\mu,f(x,y))} \varphi(t)dt + \gamma \int_0^{d(x,\mu)} \varphi(t)dt\,, \tag{17}$$

for all $x, y, \mu, \upsilon \in X$ *with* $x \succeq \mu$ *and* $y \preceq \upsilon$, $\varphi(t) \in \tau$ *and* $\beta, \gamma \in [0, 1)$ *with* $0 \leq 2\beta + \gamma < 1$. *Suppose that either* f *is continuous or* X *has* (OC) *property then* f *has a coupled fixed point* $(x, y) \in X \times X$, *if there exists two points* $x_0, y_0 \in X$ *with* $x_0 \prec f(x_0, y_0)$ *and* $y_0 \succ f(y_0, x_0)$.

Theorem 8. *Let* $(X, d, \preceq)$ *be a complete partially ordered metric space. Suppose that a self mapping* $f : X \times X \to X$ *has a strict mixed monotone property on* X *satisfying the following condition*

$$\int_0^{d(f(x,y),f(\mu,\upsilon))} \varphi(t)dt \leq \gamma \int_0^{d(x,\mu)} \varphi(t)dt\,, \tag{18}$$

for all $x, y, \mu, \upsilon \in X$ *with* $x \succeq \mu$ *and* $y \preceq \upsilon$, $\varphi(t) \in \tau$ *and* $\gamma \in [0, 1)$ *with* $0 \leq \gamma < 1$. *Suppose that either* f *is continuous or* X *has* (OC) *property then* f *has a coupled fixed point* $(x, y) \in X \times X$, *if there exists two points* $x_0, y_0 \in X$ *with* $x_0 \prec f(x_0, y_0)$ *and* $y_0 \succ f(y_0, x_0)$.

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