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# SOME RESULTS ON GRAPH ASSOCIATED TO CO-IDEALS OF COMMUTATIVE SEMIRINGS

Abstract. For a commutative semiring $R$ with non-zero identity, the graph $\Omega(R)$ of $R$, is the graph whose vertices are all elements of $R$ and two distinct vertices $x$ and $y$ are adjacent if and only if the product of the co-ideals generated by $x$ and $y$ is $R$. In this paper, we study some properties of this graph such as planarity, domination number and connectivity.

Key words: semiring, graph, maximal co-ideal.

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## 1. Introduction

Throughout this paper, every semiring $R$ is assumed to be commutative with non-zero identity. For a semiring $R$, we denote by $Co - Max(R)$, $UM(R)$ and $IM(R)$, the set of maximal co-ideals, the union of all the maximal co-ideals and the intersection of all the maximal co-ideals of $R$, respectively. Also, if $R$ is a ring, then $R$ has no proper co-ideals, thus in this paper we consider the semiring which is not a ring.

The study of algebraic structures using the properties of graphs is a research topic in the recent years. There are many papers on assigning a graph to a ring and semiring, for instance see [1, 4, 8, 9]. In [10], for a semiring $R$, the authors defined a graph on $R$, $\Omega(R)$, with vertices as elements of $R$ and two distinct vertices $x$ and $y$ are adjacent if and only if the product of the co-ideals generated by $x$ and $y$ is $R$ (i.e. $F(x)F(y) = R$). Moreover, we considered the subgraphs $\Omega_1(R)$ and $\Omega_2(R)$ of $\Omega(R)$ with vertex-set $r(ZS(R))$ (radical of $ZS(R)$) and $UM(R)$, respectively. We showed that $\Omega_1(R)$ is a complete graph and since for each $x \in IM(R)$, $deg_{\Omega_2(R)}(x) = 0$ so we study the properties of the graph $\Omega_2(R) \setminus IM(R)$ with vertex-set $UM(R) \setminus IM(R)$. Also, we investigated some properties of these graphs such as diameter, radius, girth, clique number and chromatic number. In this paper, we continue our study over these graphs and investigate some graph-theoretic properties such as planarity, domination number and connectivity.

It is useful to recall the following definitions and notations of the graphs. Let $G$ be an undirected graph with the vertex-set $V(G)$ and the edge-set $E(G)$. A graph $G$ is *connected* if there exists a path between every two distinct vertices and we say that $G$ is *totally disconnected* if no two vertices of $G$ are adjacent. The *components* of a graph $G$ are its maximal connected subgraphs. For a given vertex $x$, the number of all vertices which is adjacent to it, is called *degree* of the vertex $x$, denoted by $deg(x)$. An *isolated* vertex is a vertex of degree 0. A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph.* A graph $G$ is said to be *bipartite* if $V(G)$ can be partitioned into two disjoint sets $V_1$ and $V_2$ such that no two vertices within any $V_1$ or $V_2$ are adjacent. If for every $x \in V_1$ and $y \in V_2$, $x$ and $y$ are adjacent, then we call $G$ is *complete bipartite* graph. We denote the complete graph on $n$ vertices by $K_n$ and the complete bipartite graph on $m$ and $n$ vertices by $K_{m,n}$ (we allow $m$ and $n$ to be infinite cardinals). We will sometimes call a $K_{1,n}$ a *star graph.* A *clique* in a graph $G$ is a set of pairwise adjacent vertices. In other word, a clique is a complete subgraph of $G$.

A *semiring* $R$ is an algebraic system $(R, +, \cdot)$ such that $(R, +)$ is a commutative monoid with identity element 0 and $(R, \cdot)$ is a semigroup. In addition, operations $+$ and $\cdot$ are connected by distributivity and 0 annihilates $R$. A semiring $R$ is said to be *commutative* if $(R, \cdot)$ is a commutative semigroup and $R$ is said to have an *identity* if there exists $1 \in R$ such that $1x = x1 = x$. If $I$ is an ideal of $R$, the *radical* of $I$ is $r(I) = \{x \in R : x^n \in I \text{ for some } n > 0\}$.

A non-empty subset $I$ of $R$ is called a *co-ideal* of $R$ and denoted by $I \trianglelefteq^c R$ if and only if it is closed under multiplication and satisfies the condition that $a + r \in I$ for all $a \in I$ and $r \in R$. According to this definition, $0 \in I$ if and only if $I = R$. Also, it is obvious that if $R$ is a ring, then $R$ has no proper co-ideal. A co-ideal $I$ of a semiring $R$ is called *subtractive* if $x \in I$ and $xy \in I$, then $y \in I$ for $x, y \in R$. A proper co-ideal $P$ of $R$ is called *prime* if $a + b \in P$, then $a \in P$ or $b \in P$ for $a, b \in R$. A co-ideal $I$ of $R$ is *maximal* if $I \neq R$ and there exists no co-ideal $J$ such that $I \subset J \subset R$. If the semiring $R$ has exactly one maximal co-ideal, then we say that the semiring $R$ is *c-local* and $R$ is said to be a *c-semilocal* semiring if $R$ has only a finite number of maximal co-ideals. If $A$ is a non-empty subset of a semiring $R$, then the set $F(A)$ of all elements of $R$ of the form $a_1a_2...a_n + r$, where $a_i \in A$ for all $1 \leq i \leq n$ and $r \in R$, is a co-ideal of $R$ containing $A$. In fact, $F(A)$ is the unique smallest co-ideal of $R$ containing $A$. If $a \in R$, then $F(\{a\}) = F(a) = \{a^n + r : r \in R \text{ and } n \in \mathbf{N}\}$. It is obvious, if $a \in I$, then $F(a) \subseteq I$. An element $x$ of a semiring $R$ is called a *zero-sum* of $R$ if there exists an element $y \in R$ such that $x + y = 0$ [7]. We will denote the set of all zero-sums of $R$ by $ZS(R)$. Indeed, $ZS(R) = \{x \in R : x + y =$

0, for some $y \in R\}$. Note that $ZS(R) \neq \emptyset$, since $0 \in ZS(R)$. Let $I$ and $J$ be two co-ideals of a semiring $R$. In [10], we defined the product of $I$ and $J$ as follows:

$$IJ = \{xy + r : \ x \in I, \ y \in J \text{ and } r \in R\}$$

Similarly, for any co-ideal $I$, we have $I^n = \{a_1...a_n + r : \ a_i \in I \text{ and } r \in R\}$.

In the following we give a proposition that is used to prove the next theorems.

**Proposition 1.** *Let $R$ be a commutative semiring with non-zero identity.*

1). *If $R$ is not a ring, then it must have a maximal co-ideal. Moreover, every maximal co-ideal contains* 1 *[11].*

2). *If $I$ is a proper co-ideal of $R$, then $I$ is contained in a maximal co-ideal of $R$. In particular, $Co - Max(R) \neq \emptyset$ [5].*

3). *(Prime Avoidance Theorem) Let $I_1, ..., I_n$ be subtractive co-ideals of $R$ such that at most two of the $I_i$ are not prime. If $I$ is a co-ideal of $R$ such that $I \subseteq \cup_{i=1}^{n} I_i$, then $I \subseteq I_i$ for some $i$ [3].*

4). *Let $I_1, ..., I_n$ be co-ideals of a semiring $R$ and $P$ be a prime co-ideal containing $\bigcap_{i=1}^{n} I_i$. Then $I_i \subseteq P$ for some $i = 1, ..., n$. Moreover, if $P = \bigcap_{i=1}^{n} I_i$, then $P = I_i$ for some $i$ [5].*

5). *If $m$ is a maximal co-ideal of a semiring $R$, then $m$ is subtractive [6].*

6). *If $m$ is a maximal co-ideal of a semiring $R$, then $m$ is a prime co-ideal [5].*

**Remark 1.** By Proposition 1, if $m$ is a maximal co-ideal of a semiring $R$, then $m$ is a subtractive and prime co-ideal. So we can conclude, Prime Avoidance Theorem also holds for the case where co-ideals are maximal.

## 2. Planarity and domination number

In this section, first, we are going to find a necessary condition for the planarity of $\Omega_2(R) \setminus IM(R)$ when $R$ is a c-semilocal semiring. Next, we investigate the domination number of this graph.

A *subdivision* of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A *planar* graph is a graph that can be drawn in the plane without crossings of the edges. We need the following lemma which is proved in [12, p.246].

**Lemma 1.** *A graph is planar if and only if it does not contain a subdivision of $K_5$ or $K_{3,3}$.*

**Theorem 1.** *Let $R$ be a c-semilocal semiring with $|Co-Max(R)| \geq 2$. If $\Omega_2(R) \setminus IM(R)$ is planar, then $|Co-Max(R)| = 2$ such that $|m_i \setminus IM(R)| \leq 2$ for some $m_i \in Co - Max(R)$, or $|Co - Max(R)| = 3$ or 4.*

**Proof.** Suppose that $\Omega_2(R) \setminus IM(R)$ is planar. If $|Co - Max(R)| \geq 5$, then by [10, Theorem 3.6], $\Omega_2(R) \setminus IM(R)$ contains $K_5$ as a subgraph and so $\Omega_2(R) \setminus IM(R)$ can not be a planar by Lemma 1. Hence we must have $|Co - Max(R)| \leq 4$. Now, if $|Co - Max(R)| = 2$, then we must have $|m_i \setminus IM(R)| \leq 2$ for some $i$, in order that $\Omega_2(R) \setminus IM(R)$ does not contain $K_{3,3}$ as a subgraph because $\Omega_2(R) \setminus IM(R)$ is a complete bipartite graph by [10, Theorem 3.4]. ■

In a graph $G$, a set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V(G)$, is either in $S$ or is adjacent to a vertex in $S$. The *domination number* $\gamma(G)$ of a graph $G$ is the minimum size of a dominating set in $G$. A dominating set $S$ is said to be a *total dominating set* if every vertex in $V(G)$ is adjacent to a vertex in $S$. The minimum cardinality among the total dominating sets of $G$ is called *total domination number* and denoted by $\gamma_t(G)$. Also, a dominating set $S$ is called an *independent dominating set* if no two vertices of $S$ are adjacent. The minimum cardinality of an independent dominating set of $G$ is the *independent domination number* $\gamma_i(G)$.

In the following results, we characterize domination number for the graph $\Omega_2(R) \setminus IM(R)$ for the case $|Co - Max(R)| = 2$ and we give a general result about domination number of $\Omega_2(R) \setminus IM(R)$ when $R$ is a c-semilocal.

**Remark 2.** By definition of the domination number, it is clear that if $\Omega_2(R) \setminus IM(R)$ is a star graph, then $\gamma(\Omega_2(R) \setminus IM(R)) = 1$. Also, $\gamma_t(\Omega_2(R) \setminus IM(R)) = 2$ and $\gamma_i(\Omega_2(R) \setminus IM(R)) = n$.

**Theorem 2.** Let $R$ be a semiring with $Co - Max(R) = \{m_1, m_2\}$ such that $|m_1 \setminus IM(R)| \geq |m_2 \setminus IM(R)|$. If $\Omega_2(R) \setminus IM(R)$ is not a star graph, then $\gamma(\Omega_2(R) \setminus IM(R)) = \gamma_t(\Omega_2(R) \setminus IM(R)) = 2$ and $\gamma_i(\Omega_2(R) \setminus IM(R)) = |m_2 \setminus IM(R)|$.

**Proof.** Let $Co - Max(R) = \{m_1, m_2\}$. By [10, Theorem 3.4], $\Omega_2(R) \setminus IM(R))$ is complete bipartite graph with two vertex-set $V_1 = m_1 \setminus IM(R)$ and $V_2 = m_2 \setminus IM(R)$. Let $x \in V_1$ and $y \in V_2$. Clearly that $S = \{x, y\}$ dominates all the vertices of $\Omega_2(R) \setminus IM(R)$. Also, $\gamma(\Omega_2(R) \setminus IM(R)) \neq 1$ because $\Omega_2(R) \setminus IM(R)$ can not be a star graph by assumption. Hence $\gamma(\Omega_2(R) \setminus IM(R)) = 2$. Since $x$ and $y$ are adjacent, so $S$ is a total dominating set and therefore $\gamma_t(\Omega_2(R) \setminus IM(R)) = 2$. Now, we will compute the independent domination number. Let $S$ be an independent dominating set for the graph $\Omega_2(R) \setminus IM(R)$. Thus $S \subseteq V_i$ for some $i$, because the elements of $S$ are not adjacent. Also, as $V_i$ is an independent set, so $S = V_i$ for some $i$. By our assumption $V_2 = m_2 \setminus IM(R)$ has minimum cardinality, hence $\gamma_i(\Omega_2(R) \setminus IM(R)) = |m_2 \setminus IM(R)|$. ■

**Theorem 3.** *Let $R$ be a c-semilocal semiring with $|Co - Max(R)| = n$. If $n \geq 3$, then $2 \leq \gamma(\Omega_2(R) \setminus IM(R)) \leq n$. In particular, $2 \leq \gamma_t(\Omega_2(R) \setminus IM(R)) \leq n$.*

**Proof.** Let $Co - Max(R) = \{m_1, ..., m_n\}$. By [10, Theorem 3.6], there is a clique $S = \{x_1, ..., x_n\}$ in $\Omega_2(R) \setminus IM(R)$ where $x_i \in m_i \setminus \bigcup_{\substack{j=1 \\ j\neq i}}^{n} m_j$ for each $i$. We show that $S$ is a dominating set of $\Omega_2(R) \setminus IM(R)$. Clearly that $S$ dominates all the elements of $\bigcup_{i=1}^{n}(m_i \setminus \bigcup_{\substack{j=1 \\ j\neq i}}^{n} m_j)$. For each other vertex $x$ of $\Omega_2(R) \setminus IM(R)$, if no vertex of $S$ dominates $x$, then $F(x)F(x_k) \neq R$ for each $x_k \in S$ and so we have $F(x)F(x_k) \subseteq m_i$ for some $1 \leq i \leq n$. Hence $x$ and $x_k$ belong to $m_i$ and since $x_k \in m_k \setminus \bigcup_{\substack{j=1 \\ j\neq k}}^{n} m_j$, we have $x \in m_k$ for each $k$. This implies $x \in IM(R)$, that is impossible. Thus $S$ is a dominating set for $\Omega_2(R) \setminus IM(R)$. On the other hand, $\gamma(\Omega_2(R) \setminus IM(R)) \neq 1$ because $\Omega_2(R) \setminus IM(R)$ can not be a star graph since $n \geq 3$ by [10, Theorem 3.10]. Therefore $2 \leq \gamma(\Omega_2(R) \setminus IM(R)) \leq n$. Now, since the dominating set $S$ is a total, thus we can conclude that $2 \leq \gamma_t(\Omega_2(R) \setminus IM(R)) \leq n$. ■

**Corollary 1.** *Let $R = R_1 \times R_2 \times ... \times R_n$ be the product of c-local semirings with unique maximal co-ideals $m_i$ for each $1 \leq i \leq n$ ($n \geq 3$). Then $2 \leq \gamma(\Omega_2(R) \setminus IM(R)) \leq n$.*

**Proof.** Clearly that $m_i = R_1 \times ... \times R_{i-1} \times m_i \times R_{i+1} ... \times R_n$ is only maximal co-ideal of $R$ for each $1 \leq i \leq n$. Since $n \geq 3$, by Theorem 3 we have $2 \leq \gamma(\Omega_2(R) \setminus IM(R)) \leq n$. ■

**Theorem 4.** *Let $R$ be a semiring with $|Co - Max(R)| = 3$. Then $\gamma(\Omega_2(R) \setminus IM(R)) = \gamma_t(\Omega_2(R) \setminus IM(R)) = 3$.*

**Proof.** Suppose that $|Co - Max(R)| = 3$. By Theorem 3, we have $\gamma(\Omega_2(R) \setminus IM(R)) = 2$ or 3. We show that $\gamma(\Omega_2(R) \setminus IM(R)) = 2$ can not be true. Assume that $T = \{x, y\}$ be a dominating set of $\Omega_2(R) \setminus IM(R)$. Let $x \in m$ and $y \in m'$ for some $m, m' \in Co - Max(R)$. Also, $|m \setminus IM(R)| \geq 3$ for each $m \in Co - Max(R)$, since $|Co - Max(R)| = 3$. If $m = m'$, then $x$ and $y$ are not adjacent to $z \in m \setminus IM(R)$ where $z \neq x, y$. Now, if $m \neq m'$, then $x$ and $y$ are not adjacent to none of elements of $m \cap m' \setminus IM(R)$, so we see that $T$ can not be dominate the vertex-set of $\Omega_2(R) \setminus IM(R)$. Hence $\gamma(\Omega_2(R) \setminus IM(R)) = 3$ and by Theorem 3, $\gamma_t(\Omega_2(R) \setminus IM(R)) = 3$. ■

**Example 1.** A semiring $S$ is said to be idempotent if it is both additively and multiplicatively idempotent. Consider the idempotent semiring $S = \{0, 1, a\}$ in which $a + 1 = 1 + a = a$. It is clearly that $S$ is a c-local semiring with maximal co-ideal $m = \{1, a\}$. Let $R = S \times S \times S$ be the direct product of the semiring $S$. The maximal co-ideals of the semiring $R = (S\times S\times S, +, \cdot)$ are as follows:

$$m_1 = m \times S \times S$$
$$m_2 = S \times m \times S$$
$$m_3 = S \times S \times m.$$

It can be shown that $T = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a dominating set for $\Omega_2(R)\setminus IM(R)$. Also, $T$ is a total dominating set and so $\gamma(\Omega_2(R)\setminus IM(R)) = \gamma_t(\Omega_2(R) \setminus IM(R)) = 3$.

## 3. Connectivity

A *cut-vertex* of a graph $G$ is a vertex whose deletion increases the number of components. Thus for a connected graph $G$, $x$ is a cut-vertex of $G$ if $G \setminus \{x\}$ is not connected. Equivalently, a vertex $x$ of a connected graph $G$ is a cut-vertex of $G$, if there exist vertices $y, z \in G$, such that $x \neq y, x \neq z$ and $x$ lies on every path from $y$ to $z$.

A *separating set* of a graph $G$ is a set $S \subseteq V(G)$ such that $G \setminus S$ has more than one component. The *connectivity* of $G$, denoted by $\kappa(G)$, is the minimum size of a vertex-set $S$ such that $G \setminus S$ is disconnected or has only one vertex. Thus if $x$ is a cut-vertex of $G$, then $\kappa(G) = 1$. A *block* of a graph G is a maximal connected subgraph of $G$ that has no cut-vertex. If $G$ itself is connected and has no cut-vertex, then $G$ is a block.

In this section, we characterize commutative semiring whose $\Omega_2(R) \setminus IM(R)$ has a cut-vertex or its graph is a block. Also we determine the connectivity of $\Omega_2(R) \setminus IM(R)$ for different cases.

**Theorem 5.** *Let $R$ be a semiring with $Co-Max(R) = \{m_1, m_2\}$. Then*
*(i) $\Omega_2(R) \setminus IM(R)$ has a cut-vertex if and only if $\Omega_2(R) \setminus IM(R)$ is a star graph.*
*(ii) If $|m_i \setminus IM(R)| \geq 2$ for each $i$, then $\Omega_2(R) \setminus IM(R)$ is a block.*

**Proof.** (*i*) The sufficiency is obvious, so we need to prove the necessity. Suppose that $Co-Max(R) = \{m_1, m_2\}$ and $\Omega_2(R)\setminus IM(R)$ has a cut-vertex. By [10, Theorem 3.4] $\Omega_2(R) \setminus IM(R)$ is a complete bipartite graph with vertex set $m_1 \setminus m_2$ and $m_2 \setminus m_1$. Now, let $|m_1 \setminus m_2| = \alpha$ and $|m_2 \setminus m_1| = \beta$, hence $\Omega_2(R) \setminus IM(R)$ is a graph of the form $K_{\alpha,\beta}$. By [2, p. 50], we have $\kappa(K_{l,k}) = l$ when $l \leq k$. So if $\alpha \geq 2$ and $\beta \geq 2$, then $\kappa(\Omega_2(R) \setminus IM(R)) \geq 2$, this means that $\Omega_2(R) \setminus IM(R)$ has no cut-vertex, a contradiction. Hence $\Omega_2(R) \setminus IM(R)$ is a star graph.

(*ii*) Let $|m_i \setminus IM(R)| \geq 2$ for each $i = 1, 2$. Thus by part (i), $\Omega_2(R) \setminus IM(R)$ has no cut-vertex. On the other hand, by [10, Theorem 4.1], $\Omega_2(R)\setminus$

$IM(R)$ is a connected graph. Hence we can conclude that $\Omega_2(R) \setminus IM(R)$ is a block. ■

**Theorem 6.** *Let $R$ be a c-semilocal semiring with maximal co-ideal $m_1, ..., m_n$. If $|m_1 \setminus \bigcup_{\substack{j=1 \\ j\neq 1}}^{n} m_j| \geq |m_2 \setminus \bigcup_{\substack{j=1 \\ j\neq 2}}^{n} m_j| \geq ... \geq |m_n \setminus \bigcup_{j=1}^{n-1} m_j|$, then $\kappa(\Omega_2(R) \setminus IM(R)) = |m_n \setminus \bigcup_{j=1}^{n-1} m_j|$.*

**Proof.** Let $x \in \bigcap_{j=1}^{n-1} m_j \setminus m_n$. By [10, Lemma 3.1], $F(x)F(y) = R$ for some $y \in m_n$. It is obvious that $y \in m_n \setminus \bigcup_{j=1}^{n-1} m_j$. This means that $x$ is not adjacent to none of elements of $m_j$ for $1 \leq j \leq n-1$ and it is adjacent to all elements of $m_n \setminus \bigcup_{j=1}^{n-1} m_j$. Indeed, if we delete the elements of $m_n \setminus \bigcup_{j=1}^{n-1} m_j$ whose adjacent to $x$, then $x$ becomes an isolated vertex and $\Omega_2(R) \setminus IM(R)$ is a disconnected graph. On the other hand, $x$ is a vertex with minimum degree for $\Omega_2(R) \setminus IM(R)$ and thus by definition of connectivity for a graph, we have $\kappa(\Omega_2(R) \setminus IM(R)) = |m_n \setminus \bigcup_{\substack{j=1 \\ j\neq n}}^{n} m_j|$. ■

**Corollary 2.** *Let $R$ be a c-semilocal semiring with $Co - Max(R) = \{m_1, ..., m_n\}$. If $|m_i \setminus \bigcup_{\substack{j=1 \\ j\neq i}}^{n} m_j| = 1$ for some $m_i \in Co - Max(R)$, then $\Omega_2(R) \setminus IM(R)$ has a cut-vertex.*

**Proof.** By Theorem 6, we have $\kappa(\Omega_2(R) \setminus IM(R)) = 1$. Thus $\Omega_2(R) \setminus IM(R)$ has a cut-vertex. ■

A *cut-edge* of a graph $G$ is an edge whose deletion increases the number of components. This implies an edge of a connected graph $G$ is a cut-edge if its deletion disconnects the graph. It has been proven that an edge is a cut-edge if and only if it belongs to no cycle. A *disconnecting set* of edges is a set $F \subseteq E(G)$ such that $G \setminus F$ has more than one component. The *edge-connectivity* of $G$, written $\kappa'(G)$, is the minimum size of a disconnecting set.

In the following we obtain a necessary and sufficient condition for the semiring $R$ that $\Omega_2(R) \setminus IM(R)$ includes a cut-edge. Next, we determine edge-connectivity for c-semilocal semirings. We need the following lemma which is proved in [12, p.23]

**Lemma 2.** *An edge is a cut-edge if and only if it belongs to no cycle.*

**Theorem 7.** *Let $R$ be a c-semilocal semiring with maximal co-ideals $m_1, ..., m_n$. If $|m_1 \setminus \bigcup_{\substack{j=1 \\ j\neq 1}}^{n} m_j| \geq |m_2 \setminus \bigcup_{\substack{j=1 \\ j\neq 2}}^{n} m_j| \geq ... \geq |m_n \setminus \bigcup_{j=1}^{n-1} m_j|$, then $\kappa'(\Omega_2(R) \setminus IM(R)) = |m_n \setminus \bigcup_{j=1}^{n-1} m_j|$.*

**Proof.** Let $x \in \bigcap_{j=1}^{n-1} m_j \setminus m_n$. By the proof of Therorem 6, $x$ is only adjacent to the elements of $m_n \setminus \bigcup_{j=1}^{n-1} m_j$ and it is a vertex with minimum degree for $\Omega_2(R) \setminus IM(R)$. However, as $\kappa^{'}(\Omega_2(R) \setminus IM(R))$ is the minimum size of a disconnecting set, so $\kappa'(\Omega_2(R) \setminus IM(R)) = |m_n \setminus \bigcup_{j=1}^{n-1} m_j|$. ■

**Corollary 3.** *Let $R$ be a c-semilocal semiring with $Co - Max(R) = \{m_1, ..., m_n\}$. If $|m_i \setminus \bigcup_{\substack{j=1 \\ j\neq i}}^{n} m_j| = 1$ for some $m_i \in Co - Max(R)$, then $\Omega_2(R) \setminus IM(R)$ has a cut-edge.*

**Proof.** Suppose that $|m_k \setminus \bigcup_{\substack{j=1 \\ j\neq k}}^{n} m_j| = 1$ for some $m_k \in Co - Max(R)$. Let $x \in \bigcap_{\substack{j=1 \\ j\neq k}}^{n} m_j \setminus m_k$ and $m_k \setminus \bigcup_{\substack{j=1 \\ j\neq k}}^{n} m_j = \{y\}$. Since $x$ is only adjacent to $y$, thus, if we delete the edge $x - y$, then $x$ becomes an isolated vertex and $\Omega_2(R) \setminus IM(R)$ will be disconnected. Hence $x - y$ is a cut-edge. ■

In the following we give an example that clarifies the previous results:

**Example 2.** $(i)$ Let $S$ be a semiring as defined in Example 1 and let $R = (S \times S, +, \cdot)$. The maximal co-ideals of $R$ are as follows:

$$m_1 = \{(0,1), (0,a), (1,a), (a,1), (1,1), (a,a)\}$$
$$m_2 = \{(1,0), (a,0), (1,a), (a,1), (1,1), (a,a)\}$$

The graph $\Omega_2(R)\setminus IM(R)$ is complete bipartite with vertex-sets $m_1 \setminus IM(R) = \{(0,1), (0,a)\}$ and $m_2 \setminus IM(R) = \{(1,0), (a,0)\}$. Indeed, $\Omega_2(R) \setminus IM(R)$ forms $K_{2,2}$ and so this graph has no cut-vertex. Also, every edges of this graph belong to a cycle and this implies that $\Omega_2(R)\setminus IM(R)$ has no cut-edge.

$(ii)$ Let $X = \{a, b, c\}$ and $R = (P(X), \cup, \cap)$ be a semiring, where $P(X)$ is power set of $X$. In this case, the maximal co-ideals of the semiring $R$ are as follows:

$$m_1 = \{\{a\}, \{a,b\}, \{a,c\}, X\}$$
$$m_2 = \{\{b\}, \{a,b\}, \{b,c\}, X\}$$
$$m_3 = \{\{c\}, \{a,c\}, \{b,c\}, X\}.$$

In the graph $\Omega_2(R) \setminus IM(R)$ for the semiring $R$, the vertex $\{a,b\}$ is only adjacent to $\{c\}$ and so $\{a,b\} - \{c\}$ is a cut-edge. Also, $\{a,c\} - \{b\}$ and $\{b,c\} - \{a\}$ are cut-edges. Thus $\Omega_2(R) \setminus IM(R)$ has three cut-edges. Since $\{a,b\}$, $\{a,c\}$ and $\{b,c\}$ are vertices of degree 1, thus $\{a\}$, $\{b\}$ and $\{c\}$ are cut-vertex for $\Omega_2(R) \setminus IM(R)$.

A *unicyclic* graph is a connected graph with a unique cycle. To this end, we characterize all semirings whose $\Omega_2(R) \setminus IM(R)$ is unicyclic.

**Theorem 8.** *Let $R$ be a c-semilocal semiring. If $\Omega_2(R) \setminus IM(R)$ is a unicyclic graph, then $|Co - Max(R)| \leq 3$. Moreover, if $Co - Max(R) = \{m_1, m_2, m_3\}$, then $|m_i \setminus \bigcup_{\substack{j=1 \\ j \neq i}}^{3} m_j| = 1$ for each $1 \leq i \leq 3$.*

**Proof.** Assume contrary that $|Co - Max(R)| \geq 4$. By [10, Theorem 3.6], $\Omega_2(R) \setminus IM(R)$ contains $K_4$ as a subgraph and so it contains more than one cycle, which is a contradiction by our assumption. Now, let $Co - Max(R) = \{m_1, m_2, m_3\}$. Without loss of generality, we may assume that $|m_1 \setminus m_2 \cup m_3| \geq 2$. Let $x, y \in m_1 \setminus m_2 \cup m_3$, $z \in m_2 \setminus m_1 \cup m_3$ and $s \in m_3 \setminus m_1 \cup m_2$. Hence $x - z - s - x$ and $y - z - s - y$ are two cycles in $\Omega_2(R) \setminus IM(R)$, which is a contradiction. Hence $|m_i \setminus \bigcup_{\substack{j=1 \\ j \neq i}}^{3} m_j| = 1$ for each $1 \leq i \leq 3$. ■

**Proposition 2.** *Let $R$ be a semiring with $Co - Max(R) = \{m_1, m_2\}$. Then $\Omega_2(R) \setminus IM(R)$ is a unicyclic graph if and only if $|m_i \setminus IM(R)| = 2$ for each $i = 1, 2$.*

**Proof.** Assume that $\Omega_2(R) \setminus IM(R)$ is a unicyclic graph. By [10, Theorem 3.4], $\Omega_2(R) \setminus IM(R)$ is a complete bipartite graph. Thus, if $|m_i \setminus IM(R)| = 1$ for some $i$, then $\Omega_2(R) \setminus IM(R)$ is a star graph and contains no cycle. Also, if $|m_i \setminus IM(R)| \geq 3$ for each $i$, then $\Omega_2(R) \setminus IM(R)$ will include more than two distinct cycles, a contradiction. Thus we must have $|m_i \setminus IM(R)| = 2$ for each $i$.

Conversely, if $|m_i \setminus IM(R)| = 2$ for each $i$, then $\Omega_2(R) \setminus IM(R)$ is of the form $K_{2,2}$. Thus, it is clear that $\Omega_2(R) \setminus IM(R)$ is a unicyclic graph. ■

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