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Achyutanand Baruah and Amar Jyoti Dutta

ON THE CLASS OF p-BOUNDED VARIATION SEQUENCE OF INTERVAL NUMBERS

Abstract. In this paper we have introduced the class of p-bounded variation sequences of interval numbers bv_p^I $(1 \leq p < \infty)$ and studied some algebric and topological properties like Solid, Symmetric and Convergence free etc.

Key words: p-bounded variation, Interval number, complete, solid, symmetric, convergence free.

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1. Introduction

Most of the mathematical structures have constructed with real or complex numbers. In recent years it was further extended by interval arithmetic which finds many applications in different field of science and technology. The concept of interval number was first suggested by Dwyer ([4], [5]) in 1951. It has been further developed as a computational device by Moore [17] in 1959 and Moore and Yang [18]. The concepts have been studied by many authors and some important developments have been found in ([8], [9], [10], [12], [13], [14]). Chiao [2] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Şengönül and Eryilmaz [20] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space. Recently Dutta and Tripathy [3] introduced the p - absolutely summable sequence of interval numbers and some Dutta [6] studied on the class of sequence on other aspect. Recently Esi [8], Baruah and Dutta [1] introduced and studied some important properties of different classes of sequence of interval numbers.

2. Preliminaries

A set of closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set and we denote the set of all real valued closed intervals by R. Any elements of R is called closed interval and denoted by $\bar{x}$ that is $\bar{x} = \{x \in R : a \leq x \leq b\}$. Thus an interval number $\bar{x}$ is a closed subset of real numbers. Let x_l and x_r be first and last points of interval number $\bar{x}$, respectively then we have for $x_1, x_2 \in R$,

$$\bar{x_1} = \bar{x_2} \Leftrightarrow x_{1_\ell} = x_{2_\ell}, x_{1_r} = x_{2_r}$$

The addition and scalar multiplication is defined by

$$\bar{x_1} + \bar{x_2} = \{x \in R : x_{1_\ell} + x_{2_\ell} \leq x \leq x_{1_r} + x_{2_r}\}$$

$$\alpha\bar{x} = \{x \in R : \alpha x_{1_\ell} \leq x \leq \alpha x_{1_r}\}, \text{ for } \alpha \geq 0.$$

$$\text{and } \alpha x = \{x \in R : \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\}, \text{ for } \alpha < 0.$$

$$\begin{aligned} \bar{x_1}\bar{x_2} &= \{x \in R : \min\{x_{1_\ell}x_{2_\ell}, x_{1_\ell}x_{2_r}, x_{1_r}x_{2_l}, x_{1_r}x_{2_r}\} \\ &\quad \leq x \leq \max\{x_{1_\ell}x_{2_\ell}, x_{1_\ell}x_{2_r}, x_{1_r}x_{2_\ell}, x_{1_r}x_{2_r}\}\} \end{aligned}$$

The set of all interval numbers R is a complete metric space defined by

$$d(\bar{x_1}, \bar{x_2}) = \max\{|x_{1_\ell} - x_{2_\ell}|, |x_{1_r} - x_{2_r}|\}.$$

In the special case, $\bar{x}_1 = [a, a]$ and $\bar{x}_2 = [b, b]$, we obtain usual metric of R.

Consider the function $f : N \rightarrow R$, by $k \rightarrow f(k) = \bar{x}$, where $\bar{x} = (\bar{x}_k)$, then $\bar{x} = (\bar{x}_k)$ is called sequence of interval numbers. The term $\bar{x}_k$ is called the k^{th} term of sequence $\bar{x} = (\bar{x}_k)$.

By w^i we denotes the set of all interval numbers with real terms. We give the following definitions of convergence of interval numbers.

A sequence $\bar{x} = (\bar{x}_k)$ of interval numbers is said to be convergent to the interval number $\bar{x}_0$ if for each $\varepsilon > 0$ there exists a positive integer k_0 such that $d(\bar{x}_k, \bar{x}_0) < \varepsilon$ for all $k \geq k_0$, denoted by $\lim_k \bar{x}_k = \bar{x}_0$. This imply that

$$\lim_k \bar{x}_k = \bar{x}_0 \Leftrightarrow \lim_k x_{k_\ell} = x_{0_\ell} \text{ and } \lim_k x_{k_r} = x_{0_r}.$$

An interval valued sequence space $\bar{E}$ is said to be solid if $\bar{y} = (\bar{y}_k) \in \bar{E}$ whenever $|\bar{y}_k| \leq |\bar{x}_k|$, for all $k \in N$ and $\bar{x} = (\bar{x}_k) \in \bar{E}$.

An interval valued sequence space $\bar{E}$ is said to be monotone if $\bar{E}$ contains the canonical pre- image of all its step spaces.

An interval valued sequence space $\bar{E}$ is said to be convergence free if $\bar{y} = (\bar{y}_k) \in \bar{E}$ whenever $\bar{x} = (\bar{x}_k) \in \bar{E}$ and $\bar{x}_k = \bar{0}$ implies $\bar{y}_k = \bar{0}$.

Throughout the paper, $p = (p_k)$ is a sequence of bounded strictly positive numbers. Dutta and Tripathy [3] define the following interval valued sequence space

$$\ell^i(p) = \left\{\bar{x} = (\bar{x}_k) : \sum_{k=1}^{\infty} [d(\bar{x}_k, \bar{0})]^{p_k} < \infty\right\},$$

where $\bar{x}_k = [x_{kl}, x_{kr}]$ and $p = (p_k)$ is a bounded sequence of positive numbers so that $0 < p_k \leq \sup p_k < \infty$

If $p_k = 1$ for all $k \in N$, then we have

$$\bar{\ell} = \left\{ \bar{x} = (\bar{x}_k) : \sum_{k=1}^{\infty} [d(\bar{x}_k, \bar{0})] < \infty \right\}.$$

We introduce the class of p-bounded variation sequences of interval number bv_p^I for $(1 \leq p < \infty)$ as follows

$$bv_p^I = \left\{ \bar{x} = (\bar{x}_k) \in w^i : \sum_{k=1}^{\infty} [d(\Delta \bar{x}_k, \bar{0})]^p < \infty \right\}$$

where $\Delta \bar{x}_k = \bar{x}_k - \bar{x}_{k+1}$ for all $k \in N$.

3. Main results

Theorem 1. *The class of sequences $bv_p^I, (1 \leq p < \infty)$ is a complete metric space with the metric*

$$\rho(\bar{x}, \bar{y}) = d(\bar{x}_1, \bar{y}_1) + \left[\sum_{k=1}^{\infty} \{ d(\Delta \bar{x}_k, \Delta \bar{y}_k) \}^p \right]^{\frac{1}{p}}$$

where $\bar{x} = (\bar{x}_k), \bar{y} = (\bar{y}_k) \in bv_p^I$

Proof. Let $(\bar{x}^i)$ be a Cauchy sequence in bv_p^I such that $(\bar{x}^i) = (\bar{x}_1^i, \bar{x}_2^i, \bar{x}_3^i, \ldots) \in bv_p^I$, for each $i \in N$. Then for given $\varepsilon > 0$, there exists $n_0 \in N$, such that

$$(1) \quad \rho(\bar{x}^i, \bar{x}^j) = d(\bar{x}_1^i, \bar{x}_1^j) + \left[\sum_{k=1}^{\infty} \left\{ d(\Delta \bar{x}_k^i, \Delta \bar{x}_k^j) \right\}^p \right]^{\frac{1}{p}} < \varepsilon, \text{ for all } i, j \geq n_0$$

Now we have

$$(2) \qquad d(\bar{x}_1^i, \bar{x}_1^j) < \varepsilon, \text{ for all } i, j \geq n_0$$

and

$$(3) \qquad \left[\sum_{k=1}^{\infty} \left\{ d(\Delta \bar{x}_k^i, \Delta \bar{x}_k^j) \right\}^p \right]^{\frac{1}{p}} < \varepsilon, \text{ for all } i, j \geq n_0$$

$$\Rightarrow d(\Delta \bar{x}_k^i, \Delta \bar{x}_k^j) < \varepsilon, \qquad \text{for all } i, j \geq n_0$$

Thus $(\bar{x}_1^i)$ and $(\Delta\bar{x}_k^i)$ for $k \in N$ are Cauchy sequence in R. Since R is complete, therefore $(\bar{x}_1^i)$ and $(\Delta\bar{x}_k^i)$ for $k \in N$ are convergent in R.
Let

$$\lim_{i\to\infty} \bar{x}_1^i = \bar{x}_1 \tag{4}$$

and

$$\lim_{i\to\infty} \Delta\bar{x}_k^i = z_k \tag{5}$$

for all $k \in N$.
From (4) and (5), We have

$$\lim_{i\to\infty} \bar{x}_k^i = \bar{x}_k$$

for all $k \in N$.
Now fix $i \geq n_0$ and taking $j \to \infty$ in (2) and (3)

$$d(\bar{x}_1^i, \bar{x}_1) < \varepsilon$$

and

$$\left[\sum_{k=1}^{\infty} \left\{d(\Delta\bar{x}_k^i, \Delta\bar{x}_k)\right\}^p\right]^{\frac{1}{p}} < \varepsilon \quad \text{for all} \quad i \geq n_0. \tag{6}$$

Which gives

$$\rho(\bar{x}^i, \bar{x}) < \varepsilon \quad \text{for all} \quad i \geq n_0$$

i.e. $\bar{x}^i \to \bar{x}$, as $i \to \infty$. Now we shall show that $\bar{x} \in bv_p^I$.
From (6) we have for all $i \geq n_0$.

$$\sum_{k=1}^{\infty} \left\{d(\Delta\bar{x}_k^i, \Delta\bar{x}_k)\right\}^p < \varepsilon$$

Again for all $i \in N$

$$(\bar{x}) = (\bar{x}_k^i) \in bv_P^I$$

$$\Rightarrow \sum_{k=1}^{\infty} \left\{d(\Delta\bar{x}_k^i, \bar{0})\right\}^p < \infty$$

Now for all $i \geq n_0$ we have

$$\sum_{k=1}^{\infty} \{d(\Delta\bar{x}, \bar{0})\}^p < \sum_{k=1}^{\infty} \left\{d(\Delta\bar{x}, \Delta\bar{x}_k^i)\right\}^p + \sum_{k=1}^{\infty} \left\{d(\Delta\bar{x}_k^i, \bar{0})\right\}^p < \infty$$

Hence $\bar{x} \in bv_p^I$.
This proves the completeness of bv_p^I. ∎

Theorem 2. *The class of the sequences bv_p^I is a solid space.*

Proof. Let $\bar{x} = (\bar{x}_k) \in bv_p^I$ and $\bar{y} = (\bar{y}_k) \in bv_p^I$ be a interval valued sequence such that $|\bar{y}_k| \leq |\bar{x}_k|$ for all $k \in N$.
Then for all $k \in N$

$$\sum_{k=1}^{\infty} \{d(\Delta \bar{x}_k, \bar{0})\}^p < \infty$$

and

$$\sum_{k=1}^{\infty} \{d(\Delta \bar{y}_k, \bar{0})\}^p \leq \sum_{k=1}^{\infty} \{d(\Delta \bar{x}_k, \bar{0})\}^p < \infty$$

Thus $\bar{y} = (\bar{y}_k) \in bv_p^I$ and hence bv_p^I is a solid. ■

Theorem 3. *The class of the sequences bv_p^I is not symmetric space.*

Proof. The result follows from following example.

Example 1. For each $k \in N$ consider the sequence $\bar{x} = (\bar{x}_k) = (\bar{k}) = (\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, ...)$.
It is clear that

$$\bar{x} = (\bar{x}_k) \in bv_p^I$$

Now consider the sequence $(\bar{y}_k)$ be the rearrangement of $(\bar{x}_k)$ defined by $(\bar{y}_k) = (\bar{1}, \bar{5}, \bar{3}, \bar{9}, ...) \notin bv_p^I$.

Hence the sequence space bv_p^I is not symmetric space.
This completes the proof. ■

Theorem 4. *The class of the sequences bv_p^I is not sequence algebra.*

Proof. The result follows from following example

Example 2. For each $k \in N$ consider the sequence $(\bar{x}_k)$, $(\bar{y}_k) \in bv_p^I$ defined by $(\bar{x}_k) = (\bar{k})$ and $(\bar{y}_k) = (\bar{k}) \in bv_p^I$.
Then clearly $(\bar{x}_k) \otimes (\bar{y}_k) \notin bv_p^I$.
Hence bv_p^I is not sequence algebra.

■

Theorem 5. *The class of the sequences bv_p^I is not convergence free.*

Proof. The result follows from following example

Example 3. Consider the sequence $(\bar{x}_k)$

$$\bar{x}_k = \left[\frac{-1}{k^2}, 0\right], \Delta\bar{x} = \left[\frac{-1}{k^2}, \frac{1}{(k+1)^2}\right], \text{ for all } k \in N.$$

Then for $p = 1$

$$\sum_{k=1}^{\infty}[d(\Delta\bar{x}_k, \bar{0})] < \sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right) < \infty.$$

Clearly $\bar{x} = (\bar{x}_k) \in bv_p^I$.
Now let us define $(\bar{y}_k)$ as follows

$$\bar{y}_k = [-k^2, 0], \text{ then } \Delta\bar{y}_k = [-k^2, (k+1)^2], \text{ for all } k \in N.$$

Then

$$\sum_{k=1}^{\infty}[d(\Delta\bar{y}_k, \bar{0})] \leq \sum_{k=1}^{\infty}(k+1)^2 = \infty.$$

Thus $\bar{y} = (\bar{y}_k) \notin bv_p^I$. Hence bv_p^I is not convergence free.

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References

[1] BARUAH A.,DUTTA A.J., A new type of difference class of interval numbers, *Proyec. Jour. Math.,* 37(3)(2018), 581-589.
[2] CHIAO K.P., Fundamental properties of interval vector max-norm, *Tamsui Oxford Jour. Math.,* 18(2)(2002), 219-233.
[3] DUTTA A.J., TRIPATHY B.C., On the class of p-absolutely summable sequence $\bar{\ell}^i(p)$ numbers, *Songk. Jour. Sci. Tech.,* 38(2)(2016), 1-4.
[4] DWYER P.S., *Linear Computation,* New York, Wiley 1951.
[5] DWYER P.S., Errors of matrix computation, simultaneous equations and eigenvalues, *National Bureu Stand., Appl. Math. Series,* 29(1953), 49-58.
[6] DUTTA A., Lacunary p-Absolutely Summable Sequence of Fuzzy Real Numbers, *Fasc. Math.,* 46(2011), 57-64.
[7] ESI A., Strongly almost Λ-convergence and statistically almost Λ-convergence of interval numbers, *Scientia Magna,* 7(2)(2011), 117-122.
[8] ESI A., A new class of interval numbers, *Qafqaz University Mathematics and Computer Sciences,* 31(2011), 98-102.
[9] ESI A., A new class of double interval numbers, *Scientia Magna,* 7(4)(2011), 54- 59.
[10] ESI A., Double lacunary sequence spaces of double sequence of interval numbers, *Proyec. Jour. Math.,* 31(1)(2012), 297-306.
[11] ESI A., Lacunary sequence spaces of interval numbers, *Thai Jour. Math.,* 10(2)(2012), 445-451.
[12] ESI A., ESI A., Asymptotically lacunary statistically equivalent sequences of interval numbers, *Int. Jour. Math. Appl.,* 1(1)(2013), 43-48.

[13] Esi A., Double sequences of interval numbers defined by Orlicz functions, *Acta Et Comm. Univ. Tar. de Math.,* 17(1)(2013), 57-64.

[14] Esi A., λ-convergence of interval numbers, *Appl. Math. Inf. Sci.,* 8(3)(2014), 1099-1102.

[15] Esi A, Braha N., On asymptotically Λ-statistical equivalent sequences of interval numbers, *Acta Sc. Tech.,* 35(3)(2013), 515 - 520.

[16] Fischer P.S., *Automatic propagated and round off error analysis,* 13th National Meeting of the Association of Computing Machinary, June 1958.

[17] Kizmaz H., On certain sequence spaces, *Canad. Math. Bull.,* 24(2)(1981), 168-176.

[18] Moore R.E., *Automatic Error Analysis in Digital Computation,* Lockheed Missiles and Space Company, LSMD-48421 1959.

[19] Tripathy B.C. and Das P.C., Moore R.E., Yang C.T., *Interval Analysis I, Lockheed Missiles and Space Company,* LMSD 285875, 1962..

[20] Şengönül M., Eryilmaz A., On the sequence spaces of interval numbers, *Thai Jour. Math.,* 8(3)(2010), 503-510.

[21] Tripathy B.C., Esi A., A new type of difference sequence spaces, *Inter. Jour. Sci. Tech.,* 1(1)(2006), 11-14.

[22] Tripathy B.C., Baruah A., Et. M., Gungor M., On almost statistical convergence of new type of generalized difference sequence of fuzzy numbers, *Iranian Jour. Sc. Tech., Trans. A : Sci.,* 36(2) (2012), 147-155.

[23] Tripathy B.C. and Das P.C., On convergence of series of fuzzy real numbers, *Kuwait Jour. Sci. Eng.,* 39(1A)(2012), 57-70.

Achyutananda Baruah
Department of Mathematics
North Gauhati College
Guwahati, Assam, India
e-mail: achyutanandabaruah@gmail.com

Amar Jyoti Dutta
Department of Mathematics
Pragjyotish College
Guwahati, Assam, India
e-mail: amar_iasst@yahoo.co.in

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