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ON SIMULTANEOUS STRONG PROXIMINALITY

Abstract. In this paper, we extend the notions of simultaneous strong proximinality and simultaneous strong Chebyshevity available in Banach spaces to metric spaces and prove that if W is a simultaneously approximatively compact subset of a metric space (X, d) then W is simultaneously strongly proximinal. The converse holds if the set of all best simultaneous approximations to every bounded subset S of X from W is compact. We show that simultaneously strongly Chebyshev sets are precisely the sets which are simultaneously strongly proximinal and simultaneously Chebyshev. How simultaneous strong proximinality is transmitted to and from quotient spaces has also been discussed when the underlying spaces are metric linear spaces.

Key words: strongly proximinal, simultaneously strongly proximinal, strongly Chebyshev, simultaneously strongly Chebyshev, approximatively compact, simultaneously approximatively compact.

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1. Introduction

Let W be a non-empty closed subset of a metric space (X, d) and $x \in X$. An element $w_0 \in W$ is said to be a *best approximation* to x from W if

$$d(x, w_0) = \inf_{w \in W} d(x, w) \equiv d(x, W).$$

The set of all best approximations to x from W is denoted by $P_W(x)$. The set W is called *proximinal* if $P_W(x) \neq \phi$ for every $x \in X$. If for each $x \in X$, $P_W(x)$ is a singleton then the set W is called *Chebyshev*.

A proximinal subset W of a metric space (X, d) is said to be *strongly proximinal* (see [1], [6], [11]) if for any $x \in X$, and for every minimizing sequence $\{y_n\} \subseteq W$ for x, i.e., $\lim_{n\to\infty} d(x, y_n) = d(x, W)$, there is a subsequence $\{y_{n_k}\}$ and a sequence $\{z_k\} \subseteq P_W(x)$ such that $d(y_{n_k}, z_k) \to 0$.

A subset W of a metric space (X, d) is said to be *approximatively compact* if for any $x \in X$, every minimizing sequence $\{y_n\} \subseteq W$ for x, i.e., $d(x, y_n) \to d(x, W)$ has a convergent subsequence in W.

A subset W of a metric space (X, d) is said to be *strongly Chebyshev* (see [1]) if for any $x \in X$, every minimizing sequence $\{y_n\} \subseteq W$ for x is convergent in W.

Sometimes, it may happen that an element to be approximated is not known exactly but it is known to lie in a bounded subset S of a metric space (X, d). In that case, it is reasonable to approximate simultaneously all $s \in S$ by a single element of W by solving

$$\inf_{w \in W} \sup_{s \in S} d(s, w) \equiv d(S, W).$$

An element $w_0 \in W$ is said to be a *best simultaneous approximation* (see [5]) to S from W if

$$\sup_{s \in S} d(s, w_0) = d(S, W).$$

The set of all best simultaneous approximations to S from W is denoted by $L_W(S)$. The set W is called *simultaneously proximinal* if for each bounded subset S of X, $L_W(S) \neq \phi$. If for each bounded subset S of X, $L_W(S)$ is a singleton then the set W is called *simultaneously Chebyshev*. For any $\delta > 0$, the set $\{y \in W : \sup_{s \in S} d(s, y) < \sup_{s \in S} d(s, w) + \delta \ for \ all \ w \in W\}$ is denoted by $L_W(S, \delta)$.

Motivated by the notions of best simultaneous approximation and strong proximinality, we defined in [7] the notion of simultaneous strong proximinality in Banach spaces. We can extend this notion to metric spaces as under:

A simultaneously proximinal subset W of a metric space (X, d) is said to be *simultaneously strongly proximinal* if for each bounded subset S of X and for any minimizing sequence $\{y_n\} \subseteq W$ for S, i.e., $\lim_{n \to \infty} \sup_{s \in S} d(s, y_n) = d(S, W)$, there is a subsequence $\{y_{n_k}\}$ and a sequence $\{z_k\} \subseteq L_W(S)$ such that $d(y_{n_k}, z_k) \to 0$.

A subset W of a metric space (X, d) is said to be *simultaneously approximatively compact* (see also [4]) if for any bounded subset S of X, every minimizing sequence $\{y_n\} \subseteq W$ for S has a convergent subsequence in W.

A subset W of a metric space (X, d) is said to be *simultaneously strongly Chebyshev* if for any bounded subset S of X, every minimizing sequence $\{y_n\} \subseteq W$ for S is convergent in W.

Several researchers have discussed strongly proximinal and strongly Chebyshev sets in Banach spaces (see e.g. [1], [2], [9], [11], [14] and references cited therein). In this paper, we extend the notion of simultaneous strong proximinality and simultaneous strong Chebyshevity to metric spaces and prove that if W is simultaneously approximatively compact subset of a metric space (X, d) then W is simultaneously strongly proximinal. The converse holds if $L_W(S)$ is compact for every bounded subset S of X. We show that simultaneously strongly Chebyshev sets are precisely the sets which are simultaneously strongly proximinal and simultaneously Chebyshev. We also prove that if F is a simultaneously proximinal subspace and W a subspace of a metric linear space (X, d) then $(W + F)/F$ is simultaneously strongly proximinal in X/F if $W + F$ is simultaneously strongly proximinal in X.

2. Simultaneous strong proximinality and simultaneous approximative compactness

In this section, we give some relationships between simultaneous strong proximinality, simultaneous approximative compactness and simultaneous strong Chebyshevity in metric spaces. We start with the following result:

Theorem 1. *A nonempty subset W of a metric space (X, d) is simultaneously approximatively compact if and only if W is simultaneously strongly proximinal and $L_W(S)$ is compact for every bounded subset S of X.*

Proof. Suppose W is simultaneously approximatively compact and S a bounded subset of X. Let $\{y_n\} \subseteq W$ be a minimizing sequence for S, i.e.,

$$\lim_{n\to\infty} \sup_{s\in S} d(s, y_n) = d(S, W). \tag{1}$$

Since W is simultaneously approximatively compact, $\{y_n\}$ has a subsequence $\{y_{n_k}\} \to y_0 \in W$. From (1), we have $\sup_{s\in S} d(s, y_0) = d(S, W)$, i.e., $y_0 \in L_W(S)$ and so W is simultaneously proximinal for S. Then for the constant sequence $\{y_0\} \subseteq L_W(S)$, we have $d(y_{n_k}, y_0) \to 0$. Hence W is simultaneously strongly proximinal for S.

Now suppose that $\{z_n\}$ is any sequence in $L_W(S)$, i.e., $\sup_{s\in S} d(s, z_n) = d(S, W)$ for every $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \sup_{s\in S} d(s, z_n) = d(S, W)$ and so $\{z_n\}$ is a minimizing sequence for S. Since W is simultaneously approximatively compact, $\{z_n\}$ has a converngent subsequence $\{z_{n_k}\}$. Hence $L_W(S)$ is compact.

Conversely, suppose that W is simultaneously strongly proximinal and $L_W(S)$ is compact for every bounded subset S of X. Let $\{y_n\} \subseteq W$ be a minimizing sequence for S. Since W is simultaneously strongly proximinal, there exist a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and sequence $\{z_k\} \subseteq L_W(S)$ such that $d(y_{n_k}, z_k) \to 0$. Since $L_W(S)$ is compact, $\{z_k\}$ has a subsequence $\{z_{k_l}\} \to z_0 \in W$. This gives $\{y_{n_k}\} \to z_0$ and so W is simultaneously approximatively compact. ■

A closed subset W of a metric space (X, d) is called *simultaneously quasi-Chebyshev* (see [8]) if $L_W(S)$ is non-empty and compact for every bounded subset S of X. Therefore, we have

Corollary 1. *A nonempty subset W of a metric space (X, d) is simultaneously approximatively compact if and only if W is simultaneously strongly proximinal and simultaneously quasi-Chebyshev.*

The following theorem gives relationships between simultaneous approximative compactness, simultaneous strong Chebyshevity and simultaneous strong proximinality.

Theorem 2. *For a nonempty subset W of a metric space (X, d), the following statements are equivalent:*

(i) W is simultaneously strongly Chebyshev.

(ii) W is simultaneously strongly proximinal and simultaneously Chebyshev.

(iii) W is simultaneously approximatively compact and simultaneously Chebyshev.

Proof. $(i) \Rightarrow (ii)$. Since W is simultaneously strongly Chebyshev, it is simultaneously approximatively compact and so by Theorem 1, W is simultaneously strongly proximinal. Now, suppose S is any bounded subset of X and $w_1, w_2 \in L_W(S),\ w_1 \neq w_2$. Then

$$\sup_{s \in S} d(s, w_1) = d(S, W) = \sup_{s \in S} d(s, w_2).$$

Consider the sequence $\{y_n\}$ in W such that $y_{2n} = w_1$ and $y_{2n+1} = w_2$. Then $\{y_n\}$ is a minimizing sequence for S in W. Since $w_1 \neq w_2$, $\{y_n\}$ is not convergent, a contradiction to simultaneous strong Chebyshevity of W. Thus $w_1 = w_2$ and hence W is simultaneously Chebyshev.

$(ii) \Rightarrow (iii)$ follows from Theorem 1.

$(iii) \Rightarrow (i)$. Let $\{y_n\} \subseteq W$ be a minimizing sequence for a bounded subset S of X, i.e., $\lim_{n \to \infty} \sup_{s \in S} d(s, y_n) = d(S, W)$. Since W is simultaneously approximatively compact, $\{y_n\}$ has a subsequence $\{y_{n_k}\} \to y_0$. Then $\sup_{s \in S} d(s, y_0) = d(S, W)$, i.e., $y_0 \in L_W(S)$. We claim that every subsequence of $\{y_n\}$ also converges to y_0. Suppose $\{y_n\}$ has a subsequence $\{y_{n_i}\}$ such that $\{y_{n_i}\} \to z_0,\ z_0 \neq y_0$. Then $\sup_{s \in S} d(s, z_0) = d(S, W)$, i.e., z_0 is also a best simultaneous approximation to S from W. But W is simultaneously Chebyshev and so $y_0 = z_0$, a contradiction. Therefore, every subsequence of $\{y_n\}$ converges to y_0 and hence $\{y_n\} \to y_0$. ∎

The following example shows that simultaneously approximatively compact set need not be simultaneously strongly Chebyshev even in Banach spaces.

Example 1. Let $X = (\mathbb{R}^2, \|.\|)$, where $\|(x,y)\| = \max(|x|,|y|)$, $W = \{(x,0) : x \in \mathbb{R}\}$ and $S = \{(1,1),(1,2),(2,1),(2,2)\}$. Then W being finite dimensional subspace of X, is simultaneously approximatively compact. But W is not simultaneously strongly Chebyshev for S. Consider the sequence $\{y_n\}$ such that $y_{2n} = (2.5,0)$ and $y_{2n+1} = (1.5,0)$. Then $\{y_n\}$ is a minimizing sequence for S which is not convergent.

Concerning strong Chebyshevity, we also have the following result.

Theorem 3. *Let W be a closed subset of a metric space (X,d) and S a bounded subset of X. Then W is simultaneously strongly Chebyshev for S if and only if diam $L_W(S,\delta) < \varepsilon$.*

Proof. If W is strongly Chebyshev for S then $L_W(S) = \{y_0\}$. Therefore, every minimizing sequence $\{y_n\} \subseteq W$ for S converges to y_0. Suppose that the given condition does not hold. Then there exists an $\varepsilon > 0$ and $z_n \in L_W(S, \frac{1}{n})$ such that $d(z_n, y_0) \geq \varepsilon$. This implies that $\{z_n\}$ is a minimizing sequence for S that does not converge to y_0, a contradiction. Hence *diam* $L_W(S,\delta) < \varepsilon$.

Conversely, suppose that $\{y_n\} \subseteq W$ is any minimizing sequence for S, i.e., $\lim_{n\to\infty} \sup_{s\in S} d(x,y_n) = d(S,W)$. Then for any $\delta > 0$, $y_n \in L_W(x,\delta)$ after some stage. This implies that for any $\delta > 0$, $d(y_n, y_{n+p}) \leq$ *diam* $L_W(S,\delta) < \varepsilon$ after some stage. This implies that the sequence $\{y_n\}$ is Cauchy. Since W is a closed subset of a complete metric space, W is complete. Therefore, $\{y_n\} \to y_0 \in W$, i.e., every minimizing sequence for x is convergent. Hence W is strongly Chebyshev. ■

Remarks. Theorems 1, 2 and 3 extend the corresponding Theorems 2.2, 2.3 and Proposition 2.7 of [1] from Banach spaces to metric spaces respectively.

Remark 1. Let $X = \{x = (\xi_1, \xi_2) : 0 \leq |\xi_1| \leq 1, \xi_2 = 0\} \cup \{(-1,1),(1,1)\}$ be a subset of the real Euclidian plane endowed with the induced metric and let $W = \{(1,1),(-1,1)\}$. Then W being a compact subset of metric space X, is simultaneously approximatively compact.

Remark 2. Whereas a simultaneously strongly Chebyshev subset of a metric space is simultaneously approximatively compact, a simultaneously approximatively compact subset of a metric space need not be simultaneously strongly Chebyshev.

Let $X = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ with usual metric and $W = \{(x,y) \in X : x^2 + y^2 = 1\}$. Then W, being a compact subset of the metric space X, is simultaneously approximatively compact. But for $S = \{(0,0)\}$, $L_W(S) = W$, i.e., W is not simultaneously Chebyshev. Therefore, it follows from Theorem 2 that W is not simultaneously strongly Chebyshev.

Remark 3. Whereas an approximatively compact subset of a Banach space is strongly proximinal (see [1]), a strongly proximinal subset of a Banach space need not be approximatively compact.

Let $X = l_\infty$, $W = c_0$. Then W, being an M-ideal, is strongly proximinal in X. But, for $x = (1, 1, 1, ...) \in l_\infty$, the sequence $y_n = (1, 1, ..., 1, 0, 0, ...) \in W$ is minimizing sequence for x but $\{y_n\}$ has no convergent subsequence (see [1]).

Remark 4. A proximinal subset of a Banach space need not be strongly proximinal. Even a proximinal convex subset of a Banach space need not be strongly proximinal.

Let $X = (l_1, \|.\|_H)$, as constructed by Smith (see [13], Example 5). Then the unit ball $B(X_H)$ is proximinal but not strongly proximinal in X (see [14]).

Remark 5. The results proved in this section generalize and extend the corresponding results proved in Banach spaces for strong proximinality in [1] and for simultaneous strong proximinality in [7].

3. Simultaneously strongly proximinality in quotient spaces

In this section, we discuss simultaneous strong proximinality in quotient spaces of metric linear spaces and see how simultaneous strong proximinaliy is transmitted to and from quotient spaces. The results proved in this section are motivated by the corresponding results proved for proximinality in [3], simultaneous proximinality in [8] and for simultaneous strong proximinality in [7].

The following results of [8] and [12] will be used in the sequel:

Lemma 1. *Let F and W be subspaces of a Banach space X such that F is simultaneously proximinal, W is finite dimensional and $F + W$ is closed then $F + W$ is simultaneously proximinal in X.*

Lemma 2. *Let F be a simultaneously proximinal subspace of a Banach space X and W a subspace of X such that $W + F$ is closed. If $W + F$ is simultaneously proximinal in X then $(W + F)/F$ is simultaneously proximinal in X/F.*

Lemma 3. *Let W be a proximinal subspace of a metric linear space (X, d) then for any bounded subset S of X, we have*

$$d(S, W) = \sup_{s\in S} \inf_{w\in W} d(s, w).$$

Theorem 4. *Let F be a simultaneously proximinal subspace of a metric linear space (X, d) and W a subspace of X. If $W + F$ is simultaneously strongly proximinal in X then $(W + F)/F$ is simultaneously strongly proximinal in X/F.*

Proof. Suppose $W + F$ is simultaneously strongly proximinal in X, then $W + F$ is simultaneously proximinal and so using Lemma 2, $(W + F)/F$ is simultaneously proximinal in X/F.

Let A be any bounded subset of X/F then $A = S/F$ for some bounded subset S of X (see [12]). Let $\{y_n + F\} \subseteq (W + F)/F$ be any minimizing sequence for S/F, i.e., $\lim_{n\to\infty} \sup_{s\in S} d(s+F, y_n+F) = d(S/F, (W+F)/F)$. Then $y_n + F \in L_{(W+F)/F}(S/F, \delta_n)$ for any $\delta_n > 0$ after some stage. Since $y_n + F \in L_{(W+F)/F}(S/F, \delta_n)$,

$$\sup_{s\in S} d(s + F, y_n + F) < \sup_{s\in S} d(s + F, g + F) + \delta_n$$

for all $g + F \in (W + F)/F$, and so

$$\sup_{s\in S} \inf_{f\in F} d(s - y_n, f) < \sup_{s\in S} \inf_{f\in F} d(s - g, f) + \delta_n \text{ for all } g \in (W + F).$$

Using Lemma 3 and the proximinality of F, we can find $f_n \in F$ such that

$$\sup_{s\in S} d(s - y_n, f_n) < \inf_{f\in F} \sup_{s\in S} d(s - g, f) + \delta_n \ \textit{for all}\ g \in (W + F).$$

Letting $\delta_n \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \sup_{s\in S} d(s - y_n, f_n) \le d(S, W + F).$$

Also $d(S, W+F) \le \lim_{n\to\infty} \sup_{s\in S} d(s-y_n, f_n)$ and so $\lim_{n\to\infty} \sup_{s\in S} d(s -y_n, f_n) = d(S, W + F)$, i.e., $\{y_n + f_n\}$ is a minimizing sequence for S in $W+F$. Since $W+F$ is simultaneously strongly proximinal for S, there exist a subsequence $\{y_{n_k}+f_{n_k}\}$ and $\{z_k\} \subseteq L_{W+F}(S)$ such that $d(y_{n_k}+f_{n_k}, z_k) \to 0$. Now $d(y_{n_k}+F, z_k+F) = \inf_{f\in F} d(y_{n_k}-z_k, f) \le d(y_{n_k}-z_k, f_{n_k}) \to 0$. Hence $(W + F)/F$ is simultaneously strongly proximinal for S/F. ■

Concerning strong proximinality in quotient spaces, we have

Theorem 5. *Let W and F be subspaces of a metric linear space (X, d) and $F \subseteq W$ is proximinal in X. If W is simultaneously strongly proximinal in X then W/F is simultaneously strongly proximinal in X/F.*

Proof. Since W is simultaneously proximinal, W/F is simultaneously proximinal (see [10]). Let S/F be any bounded subset of X/F and $\{y_n + F\} \subseteq W/F$ be any minimizing sequence for S/F, i.e.,

$$\lim_{n\to\infty} \sup_{s+F\in S/F} d(s + F, y_n + F) = d(S/F, W/F).$$

Thus $y_n + F \in L_{W/F}(S/F, \delta_n)$ for any $\delta_n > 0$ after some stage. Then $\sup_{s+F\in S/F} d(s+F, y_n+F) < \sup_{s+F\in S/F} d(s+F, w+F)+\delta_n$ for all $w+F \in W/F$, i.e., $\sup_{s\in S} \inf_{f\in F} d(s-y_n, f) < \sup_{s\in S} \inf_{f\in F} d(s-w, f) + \delta_n$ for all $w \in W$. Using Lemma 3.3 and proximinality of F, we can find $f_n \in F$ such that $\sup_{s\in S} d(s-y_n, f_n) < \inf_{f\in F} \sup_{s\in S} d(s-w, f)+\delta_n < \sup_{s\in S} d(s, w)+\delta_n$ for all $w \in W$. Therefore letting $n \to \infty$, we get $\lim_{n\to\infty} \sup_{s\in S} d(s, y_n + f_n) \leq d(S, W)$. We also have $d(S, W) \leq \lim_{n\to\infty} \sup_{s\in S} d(s, y_n+f_n)$. Therefore, $\lim_{n\to\infty} \sup_{s\in S} d(s, y_n + f_n) = d(S, W)$. This implies that $\{y_n + f_n\} \subseteq W$ is a minimizing sequence for S. Since W is simultaneously strongly proximinal for S there exist a subsequence $\{y_{n_k} + f_{n_k}\}$ and $\{z_k\} \subseteq L_W(S)$ such that $d(y_{n_k} + f_{n_k}, z_k) \to 0$. Now $d(y_{n_k} + F, z_k + F) = \inf_{f\in F} d(y_{n_k} - z_k, f) \leq d(y_{n_k} - z_k, f_{n_k}) \to 0$. Hence W/F is simultaneously strongly proximinal for S/F. ■

It was proved in [8] that for subspaces W and F of a Banach space X such that $F \subseteq W$ is simultaneously Chebyshev then W is simultaneously Chebyshev if and only if W/F is simultaneously Chebyshev. Therefore, we have

Corollary 2. *Let W and F be subspaces of a Banach space X and $F \subseteq W$ is simultaneously Chebyshev in X. If W is simultaneously strongly Chebyshev then W/F is simultaneously strongly Chebyshev.*

(*i*) *The converse of Theorem 3.5 is not true even if S is a singleton. It was proved in [11] that if F is an infinite dimensional proximinal Banach space then F can be embedded isometrically as a non-strongly proximinal hyperplane in another Banach space W. Thus, dim$W/F = 1$ and so it is strongly proximinal in all its super spaces* (*see [11]*). *Then W/F is proximinal in all its super spaces and so W is proximinal in all its super spaces* (*see [3]*). *Using the same technique, W can be embedded as a non-strongly proximinal hyperplane in another Banach space.*

(*ii*) *Theorems 3.4 and 3.5 extend the corresponding results of [7] proved for simultaneous strong proximinality in Banach spaces.*

(*iii*) *Taking S to be a singleton set, we obtain several results on strong proximinality proved in [1,11].*

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