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NEUTROSOPHIC CONNECTED TOPOLOGICAL SPACES

Abstract. We present the definition of neutrosophic connectedness and include some characterizations in this paper. We also introduce neutrosophic product space and demonstrate that this form of connectivity is not preserved neutrosophic product spaces. We also present and investigate the concepts of neutrosophic super-connected spaces and neutrosophic strongly connected spaces.

Key words: neutrosophic connectedness, neutrosophic super-connectedness, neutrosophic strongly connectedness, neutrosophic function, neutrosophic base, neutrosophic subbase, neutrosophic product topology.

AMS Mathematics Subject Classification: 54J05.

1. Introduction

The concept of connectedness is one of the principal topological properties used to distinguish topological spaces. Undoubtedly, the concept of connectedness has always been an indispensable characteristic in the world of topology. That concept has constituted the basis of many precious researches in general topology. These researches have played so significant roles in many areas of real life, and the findings of such researches have had reflections in many applications. However, as the technology has progressed, and the industry has been revolutionized, the needs of the individuals have also changed, and so the general topology has fallen behind in real life. Besides, the impacts of these findings have decreased on the real-life practises. Moreover, classical methods are insufficient in dealing with several practical problems in some other disciplines, such as economics, engineering, environment, social science, medical science, etc. Therefore, according to the new theories put forward, it has become inevitable for scientists to re-examine some of the fundamental issues of Mathematics and find new types of spaces [6, 7, 16, 19, 26, 34, 38]. Connected spaces have occupied an important area in these topological spaces. In 2005, Smarandache introduced the concept of a neutrosophic set [33] as a generalization of classical sets, and this idea has been the leading actor in many studies, as in [1, 2, 3, 4, 5, 8, 9, 11, 12, 13, 14, 15, 17, 18, 19, 20, 22, 23, 25, 27, 28, 29, 30, 32, 35, 36, 37]. Using this new concept in [33], Salma and Alblowi [31] have also put forth the neutrosophic topological theory of space. The concept of the univalent neutrosophic number (SVN-number) has been introduced in a different mode by examining its structural properties by Bera et al. [10]. They have developed an assignment problem model in the neutrosophic setting, along with the solution methodology. Yang et al. [37] have presented a Data Envelopment Analysis (DEA) model in the context of neutrosophic sets. This new model has been used in the healthcare system. Thus, they have achieved useful practical results. In [24], Edalatpanah has attempted to establish a new model of DEA, where the information on decision-making units is triangular neutrosophic numbers (TNNs). Duran et al. [22] have introduced an application of neutrosophic logic in the confirmatory data analysis of the life satisfaction scale. Dhar [20] has gived an application of the concept of an algorithm-based neutrosophic soft matrix to solve some of the problems in diagnosing a disease caused by the appearance of various symptoms in patients. Radha and Stanis Arul Mary [30] have applied the concept of quadripartitioned neutrosophic pythagorean sets to Lie Algebras. The idea of neutrosophic connectivity is introduced and its features are investigated in this study. We also investigate neutrosophic super-connectedness and neutrosophic strongly connectedness in neutrosophic topological spaces, as well as their characterizations. The idea of neutrosophic connectivity is introduced and its features are investigated in this study. We also investigate neutrosophic super-connectedness and neutrosophic strongly connectedness in neutrosophic topological spaces, as well as their characterizations. Throughout the paper, without any explanation, we use the symbols and definitions introduced in [21, 31, 33]. We use the notation N instead of neutrosophic due of its brevity. We hope that many researchers will benefit from the findings in this document to further their studies on neutrosophic topology to carry out a general framework for their applications in practical life.

2. Necessary definitions

Following are a few new descriptions that will be useful in the next section.

Definition 1 ([4]). *An N-point $x_{r,t,s}$ is said to be N-quasi-coincident*

(N-q-coincident, for short) with F*, denoted by* $x_{r,t,s}qF$ *iff* $x_{r,t,s} \not\subset F^c$*. If* $x_{r,t,s}$ *is not N-quasi-coincident with* F*, we denote by* $x_{r,t,s}\tilde{q}F$*.*

Definition 2 ([4])**.** *An N-set* F *is an N-topological space* (X, τ) *is said to be an N-q-neighbourhood of an N-point* $x_{r,t,s}$ *iff there exists an N-open set* G *providing that* $x_{r,t,s}qG \subset F$*.*

Definition 3 ([4])**.** *An N-set* G *is said to be N-quasi-coincident (N-q-coincident, for short) with* F*, denoted by* GqF *iff* $G \not\subset F^c$*. If* G *is not N-quasi-coincident with* F*, we denote by* $G\tilde{q}F$*.*

Definition 4. *An N-point* $x_{r,t,s}$ *is said to be N-interior point of an N-set* F *if and only if there exists an N-open q-neighbourhood* G *of* $x_{r,t,s}$ *providing that* $G \subset F$*. The union of all N-interior points of* F *is called the N-interior of* F *and denoted by* F°*.*

Definition 5 ([4])**.** *An N-point* $x_{r,t,s}$ *is said to be N-cluster point of an N-set* F *if and only if every N-open q-neighbourhood* G *of* $x_{r,t,s}$ *is q-coincident with* F*. The union of all N-cluster points of* F *is called the N-closure of* F *and denoted by* $\overline{F}$*.*

Definition 6. *An N-set* F *in an N-topological space* (X, τ) *is called an N-regular open (N-regular closed) set if and only if* $F = (\overline{F})^\circ (F = \overline{F^\circ})$*. The complement of an N-regular open set is called an N-regular closed set.*

Definition 7. *An N-set* F *in an N-topological space* (X, τ) *is called an N-semi open set if and only if there exists an N-open set* K *providing that* $K \subset F \subset \overline{K}$*. An N-set* F *is N-semi open if and only if* $F \subset \overline{F^\circ}$*. The complement of an N-semi open set is called an N-semi closed set.*
Equivalently, an N-set H *in an N-topological space* (X, τ) *is called an N-semi closed set if and only if there exists an N-closed set* G *providing that* $G^\circ \subset H \subset G$*. An N-set* H *is N-semi closed if and only if* $(\overline{H})^\circ \subset H$*.*

Definition 8 ([4])**.** *Consider that* f *is a function from* X *to* Y*. Let* B *be an N-set in* Y *with membership function* $T_B(y)$*, indeterminacy function* $I_B(y)$ *and non-membership function* $F_B(y)$*.Then, the inverse image of* B *under* f*, written as* $f^{-1}(B)$*, is an N-subset of* X *whose membership function, indeterminacy function and non-membership function are defined as* $T_{f^{-1}(B)}(x) = T_B(f(x))$*,* $I_{f^{-1}(B)}(x) = I_B(f(x))$ *and* $F_{f^{-1}(B)}(x) = F_B(f(x))$ *for all* x *in* X*, respectively.*
Conversely, let A *be an N-set in* X *with membership function* $T_A(x)$*, indeterminacy function* $I_A(x)$ *and non-membership function* $F_A(x)$*. The image of*

A under f, written as $f(A)$, is an N-subset of Y whose membership function, indeterminacy function and non-membership function are defined as

$$T_{f(A)}(y) = \begin{cases} \sup_{z\in f^{-1}(y)}\{T_A(z)\}, & \text{if } f^{-1}(y) \text{ is not empty}, \\ 0, & \text{if } f^{-1}(y) \text{ is empty}, \end{cases}$$

$$I_{f(A)}(y) = \begin{cases} \sup_{z\in f^{-1}(y)}\{I_A(z)\}, & \text{if } f^{-1}(y) \text{ is not empty}, \\ 0, & \text{if } f^{-1}(y) \text{ is empty}, \end{cases}$$

$$F_{f(A)}(y) = \begin{cases} \inf_{z\in f^{-1}(y)}\{F_A(z)\}, & \text{if } f^{-1}(y) \text{ is not empty}, \\ 1, & \text{if } f^{-1}(y) \text{ is empty}, \end{cases}$$

for all y in Y, where $f^{-1}(y) = \{x : f(x) = y\}$, respectively.
Let (X,τ) be an N-topological space and $A \subset X$. In this paper, we denote any N-set, whose supports are members of, by μ_A. It means that, if $x \in X - A$, then $T_{\mu_A}(x) = 0$, $I_{\mu_A}(x) = 0$ and $F_{\mu_A}(x) = 1$. Otherwise, $0 \le T_{\mu_A}(x) \le 1$, $0 \le I_{\mu_A}(x) \le 1$ and $0 \le F_{\mu_A}(x) \le 1$.

3. Neutrosophic connectedness

In this section, the concept of N-connectedness is introduced and its properties are investigated.

Definition 9. *An N-topological space (X,τ) said to be N-connected, if there don't exist proper N-open sets δ, μ in (X,τ) providing that $\delta\tilde{q}\mu$ and $\delta^c\tilde{q}\mu^c$. If (X,τ) is not N-connected, then it is said to be N-disconnected.*

Theorem 1. *Let (X,τ) be an N-topological space. If (X,τ) is an N-connected topological space then, it has no proper N-clopen set (N-closed and N-open). (An N-set μ is in (X,τ) is said to be proper, if it is neither a null N-set nor an absolute N-set).*

Proof. It follows directly from the above definition. ∎

Theorem 2. *An N-topological space (X,τ) is N-connected if and only if it doesn't have any N-open sets A and B providing that $T_A(x) = F_B(x)$, $F_A(x) = T_B(x)$ and $I_A(x) + I_B(x) = 1$.*

Proof. Suppose that there exist N-open sets A and B providing that $T_A(x) = F_B(x)$, $F_A(x) = T_B(x)$ and $I_A(x) + I_B(x) = 1$. Then, A and B are N-clopen sets (N-closed and N-open) in (X,τ). This implies that (X,τ) is not N-connected.
Conversely, assume that (X,τ) is not N-connected. This means that it has

a proper N-clopen set A. So, A^c is also N-open in (X, τ). Say $A^c = B$ Then, $T_A(x) = F_B(x)$, $F_A(x) = T_B(x)$ and $I_A(x) + I_B(x) = 1$. ■

Corollary 1. *An N-topological space (X, τ) is N-connected if and only if it does not have any N-open sets A and B providing that $T_A(x) = F_{\overline{B}}(x)$, $F_A(x) = T_{\overline{B}}(x)$ and $I_A(x) + I_{\overline{B}}(x) = 1$.*

Definition 10. *Let (X, τ) be an N-topological space. A subfamily β of τ is an N-base for τ if and only if each member of τ can be expressed as the union of some members of β.*

Definition 11. *Let (X, τ) be an N-topological space. A subfamily δ of τ is an N-subbase for τ if and only if the family of finite intersections of members of δ forms an N-base for τ.*

Definition 12. *Let $\{X_i\}_{i\in I}$ be a family of non-empty sets. Let $X = \Pi_{i\in I} X_i$ be be the usual product of X_i's and let P_i the projection from X to X_i. Suppose that (X_i, τ_i) be an N-topological space for each $i \in I$. N-topology generated by $\rho = \{P_i^{-1}(B_i) : foreachi \in I, B_i \in \tau_i\}$ as N-subbasis, is called the N-product topology in X.*
Clearly, if μ is a basic element in the product topology, then for $x = (x_i)_{i\in I} \in X$, there exist $i_1, i_2, i_3, ..., i_n \in I$ providing that

$$T_\mu(x) = \min\left\{T_{B_{i_k}} : \ k = 1, 2, 3, ... \ ..., \ n\right\},$$

$$I_\mu(x) = \min\left\{I_{B_{i_k}} : \ k = 1, 2, 3, ... \ ..., \ n\right\},$$

$$F_\mu(x) = \max\left\{F_{B_{i_k}} : \ k = 1, 2, 3, ... \ ..., \ n\right\}.$$

An N-product of N-topological spaces may not be N-connected as seen in the following example.

Example 1. Regard as sets $X_1 = \{a_1, b_1\}$, and $X_2 = \{a_2, b_2\}$. Then, (X_1, τ_1), (X_2, τ_2) are two N-topological spaces, where

$$\begin{aligned} \tau_1 &= \left\{0_{X_1}, 1_{X_1}, \{\langle a_1, 0.3, 0.3, 0.7\rangle, \langle b_1, 0.3, 0.3, 0.7\rangle\}\right\}, \\ \tau_2 &= \left\{0_{X_2}, 1_{X_2}, \{\langle a_2, 0.7, 0.7, 0.3\rangle, \langle b_2, 0.7, 0.7, 0.3\rangle\}\right\} \\ \textit{and} \quad \tau_3 &= \left\{0_{X_3}, 1_{X_3}\right\}. \end{aligned}$$

Then, (X_1, τ_1), (X_2, τ_2) and (X_3, τ_3) are N-connected topological spaces but $\Pi_{i=1}^3 X_i$ is not N-connected with the N-product topology.

Definition 13. *Let (X,τ) be an N-topological space and $Y \subseteq X$. Let H be an N-set over Y providing that*

$$T_H(x) = \begin{cases} 1, & x \in Y, \\ 0, & x \notin Y, \end{cases}$$

$$I_H(x) = \begin{cases} 1, & x \in Y, \\ 0, & x \notin Y, \end{cases}$$

$$F_H(x) = \begin{cases} 0, & x \in Y, \\ 1, & x \notin Y. \end{cases}$$

Let $\tau_Y = \{H \cap F : F \in \tau\}$, then (Y, τ_Y) is called N-subspace of (X,τ). If $H \in \tau$ (resp. $H^c \in \tau$), then (Y,τ_Y) is called N-open (resp. closed) subspace of (X,τ). And, the restriction of an N-set F to Y is denoted as F/Y.

Definition 14. *If $A \subset X$, (X,τ) is an N-topological space,then A is said to be an N-connected subset of X if A is an N-connected space as an N-subspace of X.*
Clearly, if $A \subset Y \subset X$, then A is an N-connected subset of the N-topological space X if and only if it is an N-connected subset of the N-subspace Y of X.

Theorem 3. *If (X,τ) is an N-topological space and Y is an N-connected subset of X. For any non-null N-open sets A and B in (X,τ) satisfying $T_A(x) = F_B(x)$,$F_A(x) = T_B(x)$ and $I_A(x)+I_B(x) = 1$ for all $x \in X$, either $T_{A/Y}(x) = 1$, $I_{A/Y}(x) = 1$, $F_{A/Y}(x) = 0$ or $T_{B/Y}(x) = 1$, $I_{B/Y}(x) = 1$, $F_{B/Y}(x) = 0$.*

Proof. Let (X,τ) is an N-topological space, Y be an N-connected subset of X and A and B be non-null N-open sets in (X,τ). Suppose that, for some points $a, b \in Y$, it isn't satisfied that $T_{A/Y}(a) = 1$, $I_{A/Y}(a) = 1$, $F_{A/Y}(a) = 0$ and $T_{B/Y}(b) = 1$, $I_{B/Y}(b) = 1$, $F_{B/Y}(b) = 0$. If $T_A(x) = F_B(x)$, $F_A(x) = T_B(x)$ and $I_A(x)+I_B(x) = 1$ for all $x \in X$, then $T_{A/Y}(y) = F_{B/Y}(y)$, $F_{A/Y}(y) = T_{B/Y}(y)$ and $I_{A/Y}(y)+I_{B/Y}(y) = 1$ for all $y \in Y$, where $T_{A/Y}(y) \neq 0$, $I_{A/Y}(y) \neq 0$, $F_{A/Y}(y) \neq 1$ and $T_{B/Y}(y) \neq 0$, $I_{B/Y}(y) \neq 0$, $F_{B/Y}(y) \neq 1$. From Theorem 11, Y is not N-connected. This discrepancy shows that $T_{A/Y}(x) = 1$, $I_{A/Y}(x) = 1$, $F_{A/Y}(x) = 0$ or $T_{B/Y}(x) = 1$, $I_{B/Y}(x) = 1$, $F_{B/Y}(x) = 0$ for all $x \in X$. ∎

Definition 15. *N-sets A and B in an N-topological space (X,τ) are said to be separated from each other if $\overline{A}\tilde{q}B$ and $A\tilde{q}\overline{B}$.*

Theorem 4. *Let (X,τ) be an N-topological space and $\{A_i\}_{i\in I}$ be a family of N-connected subsets in (X,τ) providing that for each $j,k \in I$, where $j \neq k$, μ_{A_j} and mu_{A_k} are not separated from each other. Then, $\bigcup_{i\in I} A_i$ is an N-connected subset of X.*

Proof. Assume that $Y = \bigcup_{i\in I} A_i$ is not N-connected. Then, there exist non-null N-open sets A and B in Y providing that $T_A(y) = F_B(y)$, $F_A(y) = T_B(y)$ and $I_A(y) + I_B(y) = 1$ for all $y \in Y$. Clearly, A and B are N-clopen in Y. Fix $i_0 \in I$. Then, A_{i_0} is N-connected in (X, τ). From Theorem 4, $T_{\mu_A/A_{i_0}}(x) = 1$, $I_{\mu_A/A_{i_0}}(x) = 1$, $F_{\mu_A/A_{i_0}}(x) = 0$ or $T_{\mu_B/A_{i_0}}(x) = 1$, $I_{\mu_B/A_{i_0}}(x) = 1$, $F_{\mu_B/A_{i_0}}(x) = 0$. This means that $\mu_{A_{i_0}} \subset A$ or $\mu_{A_{i_0}} \subset B$. Assume that $\mu_{A_{i_0}} \subset A$. So, $\overline{\mu_{A_{i_0}}} \subset \overline{A}$. Take $i_1 \in I$. Then, A_{i_1} is N-connected in (X, τ). Alike, $\mu_{A_{i_1}} \subset A$ or $\mu_{A_{i_1}} \subset B$. Assume that $\mu_{A_{i_1}} \subset B$. So, $\overline{\mu_{A_{i_1}}} \subset \overline{B}$. Since A and B are N-clopen in Y, $T_A(y) = F_{\overline{B}}(y)$, $F_A(y) = T_{\overline{B}}(y)$, $I_A(y) + I_{\overline{B}}(y) = 1$, and $T_{\overline{A}}(y) = F_B(y)$, $F_{\overline{A}}(y) = T_B(y)$ and $I_{\overline{A}}(y) + I_B(y) = 1$. This implies that $\overline{\mu_{A_{i_1}}} \tilde{q} \mu_{A_{i_0}}$ and $\mu_{A_{i_1}} \tilde{q} \overline{\mu_{A_{i_0}}}$. So, $\mu_{A_{i_0}}$ and $\mu_{A_{i_1}}$ are separated from each other. This discrepancy shows that Y is N-connected. ■

Corollary 2. *Let (X, τ) be an N-topological space and $\{A_i\}_{i\in I}$ be a family of N-connected subsets in (X, τ) providing that $\bigcap_{i\in I} A_i = \emptyset$. Then, $\bigcup_{i\in I} A_i$ is an N-connected subset of X.*

Corollary 3. *Let (X, τ) be an N-topological space and $\{A_i : i = 1, 2, 3, ...\}$ be a sequence of N-connected subsets in (X, τ) providing that μ_{A_j} and $\mu_{A_{j+1}}$ are not separated from each other, where $j = 1, 2, ...$. Then, $\bigcup_{i\in I}^{\infty} A_i$ is an N-connected subset of X.*

Theorem 5. *Let (X, τ) be an N-topological space , C be an N-connected subset of X, $V \subset X - C$ and $\mu_V/(X - C)$ be an N-clopen subset in $(X - C, \tau_{X-C})$. Then, $C \cup V$ is an N-connected subset of X.*

Proof. Assume that $Y = C \subset V$ is not N-connected. Then, there exist non-null N-open sets A and B in Y providing that $T_A(y) = F_B(y)$, $F_A(y) = T_B(y)$ and $I_A(y) + I_B(y) = 1$. From Theorem 4, $T_{A/C}(x) = 1$, $I_{A/C}(x) = 1$, $F_{A/C}(x) = 0$ or $T_{B/C}(x) = 1$, $I_{B/C}(x) = 1$, $F_{B/C}(x) = 0$. Since C is N-connected in (X, τ), $\mu_C \subset A$ or $\mu_C \subset B$. Assume that $\mu_C \subset A$. Then, $T_{B/C}(x) = 0$, $I_{B/C}(x) = 0$, $F_{B/C}(x) = 1$. As B is an N-open set in in (Y, τ_Y), B is an N-open set in (V, τ_V). Let us define an N-set B_1 in X as

$$T_{B_1}(x) = \begin{cases} T_B(x)\,, & \text{if } x \in V\,, \\ 0\,, & \text{if } x \notin V\,, \end{cases}$$

$$I_{B_1}(x) = \begin{cases} I_B(x)\,, & \text{if } x \in V\,, \\ 0\,, & \text{if } x \notin V\,, \end{cases}$$

$$F_{B_1}(x) = \begin{cases} F_B(x)\,, & \text{if } x \in V\,, \\ 1\,, & \text{if } x \notin V\,. \end{cases}$$

Now $B_1/V = B/V$ and B/V is N-closed in V. Therefore B_1/V is N-closed in V. Also μ_V is N-closed in $(X-C, \tau_{X-C})$. For this reason, $B_1/(X-C)$ is N-closed in $(X-C, \tau_{X-C})$. Now, $B_1/(X-C) = B/(X-C) \cap \mu_V/(X-C)$. For this reason, $B_1/(X-C)$ is N-open in $(X-C, \tau_{X-C})$. Thus, $B_1/(X-C)$ is N-clopen in $(X-C, \tau_{X-C})$. Further, $B_1/Y = B/Y$. As B/Y is N-clopen in Y, For this reason, B_1/Y is N-clopen in Y. So, B_1 is N-clopen in $(X-C) \cup Y = X$. As B_1 is a proper N-set in X. This implies that X is not connected. From this discrepancy, Y is N-connected. ■

Theorem 6. *Let A and B be subsets of X in an N-topological space (X, τ). If $\mu_A \subset \mu_B \subset \overline{\mu_A}$ and A is N-connected in an N-topological space (X, τ), then B is also N-connected in (X, τ).*

Proof. Use the method of prof by discrepancy. To do this, assume that B is not N-connected. Then, there exist N-open sets λ and δ in (X, τ), where λ/B and δ/B are not null N-sets and $T_{\lambda/B}(y) = F_{\delta/B}(y)$, $F_{\lambda/B}(y) = T_{\delta/B}(y)$ and $I_{\lambda/B}(y) + I_{\delta/B}(y) = 1$. We first show that λ/A is not a null N-set. If λ/A is a null N-set, then $\mu_A \subset \lambda^c$, which implies that $\overline{\mu_A} \subset \lambda^c$. So, $\mu_B \subset \lambda^c$. This implies that λ/B is a null N-set. From this discrepancy, λ/A is not a null N-set. Similarly we can show that δ/A is not a null N-set. Since $\mu_A \subset \mu_B$, $T_{\lambda/A}(y) = F_{\delta/A}(y)$, $F_{\lambda/A}(y) = T_{\delta/A}(y)$ and $I_{\lambda/A}(y) + I_{\delta/A}(y) = 1$. So, A is not N-connected. From this discrepancy, B is N-connected. ■

4. N-connected subsets

In this section, we present the concept of N-connected subset and investigate its properties.

Definition 16. *Let (X, τ) be an N-topological space. Then, (X, τ) is said to be an N-super-connected space, if there doesn't exist any proper N-regular open set in (X, τ). Since an N-clopen set is an N-regular open set, N-super-connectedness implies N-connectedness but the following example shows that the converse is not true.*

Example 2. Let $X = \{a, b\}$ and $\tau = \Big\{0_X, 1_X, \Big\{\langle a, \frac{1}{6}, \frac{1}{6}, \frac{5}{6}\rangle, \langle b, \frac{1}{6}, \frac{1}{6}, \frac{5}{6}\rangle\Big\}, \Big\{\langle a, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle, \langle b, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle\Big\}\Big\}$. Then, (X, τ) is N-connected but it is not N-super-connected, since it has a proper N-regular open set. We also define N-super-connected subsets of an N-topological space and study their properties.

Theorem 7. *Let (X, τ) be an N-topological space. Then, the following statements are equivalent:*

(a) *(X, τ) is N-super-connected.*

(b) *Closure of every non-null N-open set in X is 1_X.*

(*c*) *Interior of every non-absolute N-closed set in (X, τ) is a null N-set.*

(*d*) *There don't exist any non-null N-open sets λ and δ providing that $\lambda \subset \delta^c$.*

(*e*) *There don't exist any non-null N-sets λ and δ satisfying $\overline{\lambda} = \delta^c$ or $\overline{\delta} = \lambda^c$.*

(*f*) *There don't exist any non-null N-closed sets η and Ω satisfying $\eta^\circ = \Omega^c$ or $\Omega^\circ = \eta^c$.*

Proof.

$(a) \Longrightarrow (b)$ If there exists a non-null N-open set λ providing that $\overline{\lambda} \neq 1_X$, then $(\overline{\lambda})^\circ$ is a proper N-regular open set.

$(b) \Longrightarrow (c)$ Let λ be a non-absolute N-closed set in (X, τ). Now, $\lambda^\circ = (\overline{(\lambda^c)})^c = 0_X$, as λ^c is a non-null N-open set.

$(c) \Longrightarrow (d)$ If there exist non-null N-open sets λ and δ providing that $\lambda \subset \delta^c$, then $\overline{\lambda} \subset \delta^c$. For δ is non-null, $\overline{\lambda}$ is non-absolute. Since λ is non-null, $\overline{\lambda}^\circ$ is non-null. This contradicts with (c).

$(d) \Longrightarrow (a)$ If there exists a proper N-regular open set λ, then λ and $\delta = (\overline{\lambda})^c$ are non-null N-open sets satisfying $\lambda \subset \delta^c$. This is a discrepancy.

$(e) \Longrightarrow (a)$ Suppose that (X, τ) is not N-super-connected. Then, there exists a proper N-regular open set λ in (X, τ). If $\delta = (\overline{\lambda})^c$, then δ is non-null. In addition, $\overline{\delta} = \overline{(\overline{\lambda})^c} = \overline{(\lambda^c)^\circ} = \lambda^c$ as λ^c is an N-regular closed set. For this reason, $\overline{\delta} = \lambda^c$. This is a discrepancy.
Conversely, if there exist non-null N-open sets λ and δ providing that $\overline{\lambda} = \delta^c$ or $\overline{\delta} = \lambda^c$, then $(\overline{\lambda})^\circ = (\delta^c)^\circ = (\overline{\delta})^c = \lambda$. Since δ is non-null and $\overline{\delta} = \delta^c$, λ is non-absolute. In addition, λ is non-null. For this reason, λ is a proper N-regular open set. For this reason, (X, τ) is not N-super-connected. This is a discrepancy.
$(e) \Longrightarrow (f)$, $(e) \Longrightarrow (f)$ follows if we take $\eta = \lambda^c$ and $\Omega = \delta^c$. Reverse implication can be proved similarly. ∎

Theorem 8. *An N-topological space (X, τ) is N-super-connected if and only if there doesn't exist any proper N-open set which is also N-semi-closed or equivalently if and only if there doesn't exist any proper N-closed set which is also N-semi-open.*

Proof. It is clear from the definitions of N-regular open sets, N-semi-open sets and N-semi-closed sets. ∎

Theorem 9. *Let (X_1, τ_1), (X_2, τ_2) and f be an N-continuous function from (X_1, τ_1) to (X_2, τ_2). If (X_1, τ_1) is N-super-connected then (X_2, τ_2) is also N-super-connected.*

Proof. Assume that (X_2, τ_2) is not N-super-connected. Then there exists a non-null N-open set λ in (X_2, τ_2) providing that $\overline{\lambda} \neq 1_X$. Since f

is N-continuous $\overline{f^{-1}(\lambda)} \subset f^{-1}(\overline{\lambda})$. Since λ is non-null, there exist N-points $y_1 \in \lambda$ providing that $T_\lambda(y_1) \neq 0$ or $I_\lambda(y_1) \neq 0$ or $F_\lambda(y_1) \neq 1$. And, since $\overline{\lambda}$ is non-absolute, there exist N-points $(y_2) \in \overline{\lambda}$ providing that $T_{\overline{\lambda}}(y_2) \neq 1$ or $I_{\overline{\lambda}}(y_2) \neq 1$ or $F_{\overline{\lambda}}(y_2) \neq 0$. From f is onto, there exist N-points $x_1, x_2 \in X$ providing that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then, $T_{f^{-1}(\lambda)}(x_1) \neq 0$ and $I_{f^{-1}(\lambda)}(x_1) \neq 0$ and $F_{f^{-1}(\lambda)}(x_1) \neq 1$. In addition, $T_{f^{-1}(\overline{\lambda})}(x_1) \neq 0$ and $I_{f^{-1}(\overline{\lambda})}(x_1) \neq 0$ and $F_{f^{-1}(\overline{\lambda})}(x_1) \neq 1$. Then, $f^{-1}(\lambda)$ a non-null N-open set in (X_1, τ_1) providing that $\overline{f^{-1}(\lambda)} \neq 1_X$. As (X_1, τ_1) is N-super-connected, this is a discrepancy. ■

Theorem 10. *Finite product of N-super-connected spaces N-super-connected.*

Proof. Let (X, τ_X) and (Y, τ_Y) be N-super-connected topological spaces. Assume that (XxY, τ_{XxY}) is not N-super-connected. Then, there exist $\lambda, \mu \in \tau_X$ and $\delta, \eta \in \tau_Y$ providing that $\lambda x\delta \subset (\mu x\eta)^c$, where $\lambda x\delta$, $\mu x\eta \in \tau_{XxY}$, $\lambda x\delta = P_X^{-1}(\lambda) \cap P_Y^{-1}(\delta)$, P_X and P_Y are projection maps of XxY onto X and Y, respectively. So, $\min\Big\{T_\lambda(x), T_\delta(y)\Big\} \leq \max\Big\{F_\mu(x), F_\eta(y)\Big\}$ and $\min\Big\{I_\lambda(x), I_\delta(y)\Big\} + \min\Big\{I_{\mu(x)}, I_\eta(y)\Big\} \leq 1$ and $\max\Big\{F_\lambda(x), F_\delta(y)\Big\} \geq \min\Big\{T_\mu(x), T_\eta(y)\Big\}$ for all $(x, y) \in XxY$.
As (X, τ_X) and (Y, τ_Y) are N-super-connected topological spaces, if $\lambda \cap \mu \neq 0_X$ and $\delta \cap \eta \neq 0_Y$ then there exist $x_1 \in X$ and $y_1 \in Y$ providing that $T_{\lambda\in\mu}(x_1) > F_{\lambda\in\mu}(x_1)$, $I_{\lambda\in\mu}(x_1) > 0.5$ and $T_{\delta\in\eta}(y_1) > F_{\delta\in\eta}(y_1)$, $I_{\delta\in\eta}(y_1) > 0.5$. Therefore $\min\Big\{T_\lambda(x_1), T_\delta(y_1)\Big\} > \max\Big\{F_\mu(x_1), F_\eta(y_1)\Big\}$ and $\min\Big\{I_\lambda(x_1), I_\delta(y_1)\Big\} + \min\Big\{I_\mu(x_1), I_\eta(y_1)\Big\} > 1$ and $\max\Big\{F_\lambda(x_1), F_\delta(y_1)$ $< \min\Big\{T_\mu(x_1), T_\eta(y_1)\Big\}$.
If $\lambda \cap \mu = 0_X$, then for each $x \in X$, $\min\Big\{T_\lambda(x), T_\mu(x)\Big\} = 0$ and $\min\Big\{I_\lambda(x), I_\mu(x)\Big\} = 0$ and $\max\Big\{F_\lambda(x), F_\mu(x)\Big\} = 1$. If $T_\lambda(x) < F_\mu(x)$ and $I_\lambda(x) + I_\mu(x) \leq 1$, then $\lambda \subset \mu^c$. As $\lambda x\delta$ and $\mu x\eta$ are non-null, λ and μ are non-null. This implies that (X, τ_X) is not N-super-connected. Alike, it can be proved that (Y, τ_Y) is also not N-super-connected. This discrepancys shows that (XxY, τ_{XxY}) is N-super-connected. ■

Theorem 11. *Any N-product of N-super-connected spaces is tw$N$-super-connected.*

Proof. Let (X_i, τ_{X_i}) be a family of N-super-connected spaces and (X, τ_X) be an N-product-space, where $X = \Pi_{i\in I} X_i$ and τ_X is N-product topology.

Assume that there are two non-null N-open sets λ and μ in (X, τ_X) providing that $\lambda \subset \mu^c$.
Then, there exists $i_1, i_2, ..., i_m, j_1, j_2, ..., j_n \in I$ providing that $\lambda_{i_k} \in \tau_{X_{i_k}}$ for $k = 1, 2, ..., m$, $\mu_{j_p} \in \tau_{X_{j_p}}$ for $p = 1, 2, ..., n$ satisfying that $\min\Big\{T_{\lambda_{i_k}}(x_{i_k}) : k = 1, 2, ..., m\Big\} \leq \max\Big\{F_{\mu_{j_p}}(x_{j_p}) : p = 1, 2, ..., n\Big\}$ and $\min\Big\{I_{\lambda_{i_k}}(x_{i_k}) : k = 1, 2, ..., m\Big\} + \min\Big\{I_{\mu_{j_p}}(x_{j_p}) : p = 1, 2, ..., n\Big\} \leq 1$ for every $x_{i_k} \in X_{i_k}$, $k = 1, 2, ..., m$ and for every $x_{j_p} \in X_{j_p}, p = 1, 2, ..., n$.

Case 1. Let $\{i_1, i_2, ..., i_m\} \cap \{j_1, j_2, ..., j_n\} = \emptyset$. Since each X_i is N-super-connected, there exist $x'_{i_k} \in X_{i_k}$, $k = 1, 2, ..., m$ and $x'_{j_p} \in X_{j_p}$, $p = 1, 2, ..., n$ providing that $\min\Big\{T_{\lambda_{i_k}}(x'_{i_k}),\ k = 1, 2, ..., m\Big\} > \max\Big\{F_{\mu_{j_p}}(x'_{j_p}) : p = 1, 2, ..., n\Big\}$ and $\min\Big\{I_{\lambda_{i_k}}(x'_{i_k}),\ k = 1, 2, ..., m\Big\} + \min\Big\{I_{\mu_{j_p}}(x'_{j_p}) : p = 1, 2, ..., n\Big\} > 1$. This contradicts with our assumption.

Case 2. Let $\{i_1, i_2, ..., i_m\} \cap \{j_1, j_2, ..., j_n\} \neq \emptyset$. If $\min\Big\{T_{\lambda_{i_k}}(x_{i_k}) : k = 1, 2, ..., m\Big\}$, $\max\Big\{F_{\mu_{j_p}}(x_{j_p}) : p = 1, 2, ..., n\Big\}$, $\min\Big\{I_{\lambda_{i_k}}(x_{i_k}) : k = 1, 2, ..., m\Big\}$ and $\min\Big\{I_{\mu_{j_p}}(x_{j_p}) : p = 1, 2, ..., n\Big\}$ have different subscripts, we can complete our proof in the same sense as in Case 1. If they have the same subscript γ, then $T_{\lambda_\gamma}(x_\gamma) \leq F_{\mu_\gamma}(x_\gamma)$ and $I_{\lambda_\gamma}(x_\gamma) + I_{\mu_\gamma}(x_\gamma) \leq 1$ for all $x_\gamma \in X_\gamma$. But, $\lambda_\gamma \cap \mu_\gamma \in \tau_{X_\gamma}$. If $\min\Big\{T_{\lambda_\gamma}(x_\gamma), T_{\mu_\gamma}(x_\gamma)\Big\} > F_{\lambda_\gamma}(x_\gamma)$ for some $x_\gamma \in X_\gamma$, this is contradicts with our assumption. ■

Definition 17. *A subset A of X is said to be N-super-connected, if (A, τ_A) is an N-super-connected topological space as an N-subspace of (X, τ).*

Theorem 12. *If $A \subset Y \subset X$, then A is an N-super-connected subset of X if and only if it is an N-super-connected subset of the N-subspace Y of X.*

Proof. The proof is obvious. ■

Theorem 13. *Let A be an N-super-connected subset of X. If there exist N-closed sets λ and δ in (X, τ) providing that $\lambda^\circ = \delta^c$ and $\delta^\circ = \lambda^c$ then $\lambda \backslash A = 1_A$ or $\delta \backslash A = 1_A$.*

Proof. Suppose that $T_\lambda(x_0) \neq 1$, $I_\lambda(x_0) \neq 1$, $F_\lambda(x_0) \neq 0$ and $T_\delta(y_0) \neq 1$, $I_\delta(y_0) \neq 1$, $F_\delta(y_0) \neq 0$ for some x_0, $y_0 \in A$. Clearly, $T_{\lambda^\circ}(x_0) = F_\delta(y_0)$, $F_{\lambda^\circ}(x_0) = T_\delta(y_0)$, $T_{\delta^\circ}(y_0) = F_\lambda(x_0)$, $F^\circ_\delta(y_0) = T_\lambda(x_0)$, $I_{\lambda^\circ}(x_0) = 1 - I_\delta(y_0)$ and $I^\circ_\delta(y_0) = 1 - I_\lambda(x_0)$.
So, $T_{\lambda^\circ}(x_0) \leq F_{\delta^\circ}(y_0)$, $F_{\lambda^\circ}(x_0) \geq T_{\delta^\circ}(y_0)$,$T_{\delta^\circ}(y_0) \leq F_{\lambda^\circ}(x_0)$, $F_{\delta^\circ}(y_0) \geq$

$T_{\lambda^\circ}(x_0)$, $I_{\lambda^\circ}(x_0) \leq 1 - I_{\delta^\circ}(y_0)$ and $I_{\delta^\circ}(y_0) \leq 1 - I_{\lambda^\circ}(x_0)$. Thus $\lambda^\circ \backslash A$ and $\delta^\circ \backslash A$ are non-null N-open sets in A providing that $\lambda^\circ \backslash A \subset (\delta^\circ \backslash A)^c$. This is a discrepancy. So, $T_\lambda(x_0) = 1$, $I_\lambda(x_0) = 1$, $F_\lambda(x_0) = 0$ and $T_\delta(y_0) = 1$, $I_\delta(y_0) = 1$, $F_\delta(y_0) = 0$. ■

Theorem 14. *Let (X, τ) be an N-topological space and $A \subset X$ be N-super-connected subset providing that A is an N-open set in (X, τ). If λ is an N-regular open set in (X, τ), then either $\mu_A \subset \lambda$ or $\mu_A \subset \lambda^c$.*

Proof. If $\lambda = 1_X$ or $\lambda = 0_X$, then it is clear. Suppose that $\lambda \neq 1_X$ and $\lambda \neq 0_X$. Then, $(\overline{\lambda})^\circ = (\lambda^c)^c$ and $(\lambda^c)^\circ = (\overline{\lambda})^c$. According to the prior theorem, $\overline{\lambda} \backslash A = 1_A$ or $\lambda^c \backslash A = 1_A$. Then $\mu_A \subset \overline{\lambda}$ or $\mu_A \subset \lambda^c$. Since μ_A is n-open, $\mu_A \subset (\overline{\lambda})^\circ$ or $\mu_A \subset (\lambda^c)^\circ$. For this reason, $\mu_A \subset \lambda$ or $\mu_A \subset \lambda^c$. ■

Theorem 15. *Let $\{A_i\}_{i \in I}$ be a family of subsets of X providing that each μ_{A_i} is n-open. If $\bigcap_{i \in I} A_i$ and each A_i is an N-super-connected subset of X, then $\bigcup_{i \in I} A_i$ is also an N-super-connected subset of X.*

Proof. Let $Y = \bigcup_{i \in I} A_i$ and assume that Y is not an N-super-connected subset of X. Then, there exists a proper N-regular open set λ_A in the N-subspace Y of X. Each μ_{A_i} is N-open in (X, τ). So, each $\mu_{A_i} \backslash Y$ is N-open in (Y, τ_Y). In addition, each A_i is an N-super-connected subset in (Y, τ_Y). From the previous theorem, for each $i \in I$, $\mu_{A_i} \backslash Y \subset \lambda_Y$ or $\mu_{A_i} \backslash Y \subset (\lambda_Y)^c$. Suppose $x_0 \in \bigcap_{i \in I} A_i$. Then, either $T_{\lambda_Y}(x_0) = 1$, $I_{\lambda_Y}(x_0) = 1$, $F_{\lambda_Y}(x_0) = 0$ or $T_{\lambda_Y}(x_0) = 0$, $I_{\lambda_Y}(x_0) = 0$, $F_{\lambda_Y}(x_0) = 1$. If $T_{\lambda_Y}(x_0) = 1$, $I_{\lambda_Y}(x_0) = 1$, $F_{\lambda_Y}(x_0) = 0$, then $\mu_{A_i} \backslash Y \subset \lambda_Y$ for every $i \in I$. Hence, $\mu_Y \backslash Y = \bigcup_{i \in I} \mu_{A_i} \backslash Y \subset \lambda_Y$. But, $\lambda_Y \subset \mu_Y \backslash Y$. So, λ_Y is an absolute N-set. Since λ_Y is a proper N-set, this is a discrepancy. Alike, if $T_{\lambda_Y}(x_0) = 0$, $I_{\lambda_Y}(x_0) = 0$, $F_{\lambda_Y}(x_0) = 1$, it will be seen that λ_Y is a null N-set, which is also a discrepancy. ■

Theorem 16. *Let A and B be N-super-connected subsets of X. If $(\mu_B)^\circ \backslash A \neq 0_X$ or $(\mu_A)^\circ \backslash A \neq 0_X$ then, $A \cup B$ is an N-super-connected subset of X.*

Proof. Assume that $Y = A \cup B$ is not an N-super-connected subset of X. Then, there exist N-open sets λ and δ providing that $\lambda \backslash Y \neq 0_X$, $\delta \backslash Y \neq 0_X$ and $\lambda \backslash Y \subset (\delta \backslash Y)^c$. Since A is an N-super-connected subset of X either $\lambda \backslash A \neq 0_X$ or $\delta \backslash A = 0_X$. Suppose that $\delta \backslash A = 0_X$. Since B is also N-super-connected, we $\lambda \backslash A \neq 0_X$, $\delta \backslash B \neq 0_X$, $\delta \backslash A = 0_X$ and $\lambda \backslash B = 0_X$. From $\lambda \backslash B = 0_X$, $\lambda \backslash A \subset ((\mu_B)^\circ \backslash A)^c$. Let $(\mu_B)^\circ \neq 0_X$. Since $\lambda \backslash A \neq 0_X$ and $\lambda \backslash A \subset ((\mu_B)^\circ \backslash A)^c$,$A$ is not an N-super-connected subset of X. Alike, let $(\mu_A)^\circ \neq 0_X$. Since $\delta \backslash B \neq 0_X$ and $\delta \backslash B \subset ((\mu_A)^\circ \backslash B)^c$, B is not an N-super-connected subset of X. This is a discrepancy. ■

Theorem 17. *Let $\{A_i\}_{i\in I}$ be a family of N-super-connected subsets of X. If $(\bigcap_{i\in I}\mu_{A_i})^\circ \neq 0_X$, then $\bigcup_{i\in I}\mu_{A_i}$ is an N-super-connected subset of X.*

Proof. Assume that $Y = \bigcup A_{i i\in I}$ is not an N-super-connected subset of X. Then, there exist N-open sets λ and δ in (X,τ) providing that $\lambda\backslash Y \neq 0_X$, $\delta\backslash Y \neq 0_X$ and $\lambda\backslash Y \subset (\delta\backslash Y^c$. Since $\lambda\backslash Y \neq 0_X$ and $\delta\backslash Y \neq 0_X$, for some i_1 and $i_2 \in I$, $\lambda\backslash A_{i_1} \neq 0_X$, $\delta\backslash A_{i_2} \neq 0_X$.

Case 1. Let $i_1 = i_2$. Then, A_{i_1} isn't an N-super-connected subset of X. This is a discrepancy.

Case 2. Let $i_1 \neq i_2$. Then, $(\bigcap_{i\in I}\mu_{A_i})^\circ \subset \bigcap_{i\in I}(A_i)^\circ$ and $(\bigcap_{i\in I}\mu_{A_i})^\circ \neq 0_X$. Then, $(\mu_{A_{i_1}})^\circ \cap (\mu_{A_{i_2}})^\circ \neq 0_X$. This implies that $(\mu_{A_{i_1}})^\circ\backslash A_{i_2} \neq 0_X$. From the previous theorem, $A_{i_1} \cup A_{i_2}$ is an N-super-connected subset of X . Whereas, it is obvious that $\lambda\backslash A_{i_1} \cup A_{i_2} \neq 0_X$, $\delta\backslash A_{i_1} \cup A_{i_2} \neq 0_X$ and $\lambda\backslash A_{i_1} \cup A_{i_2} \subset (\delta\backslash A_{i_1} \cup A_{i_2})^c$. This means that $A_{i_1} \cup A_{i_2}$ is not N-super-connected. This is a discrepancy. ∎

Theorem 18. *Let (X,τ) be an N-super-connected topological space and $C \subset X$ be N-super-connected subset. If there exists a subset $V \subset X$ providing that $V \cap C = \emptyset$ and $\mu_V\backslash(X-C) \in \tau_{(X-C)}$, then $C \cup V$ is an N-super-connected subset of X.*

Proof. Suppose that $Y = C \cup V$ is not an N-super-connected subset of X. Then, there exist N-open sets λ and δ providing that $\lambda\backslash Y \neq 0_X$, $\delta\backslash Y \neq 0_X$ and $\lambda\backslash Y \subset (\delta\backslash Y)^c$. As C is an N-super-connected subset of X, either $\lambda\backslash C = 0_X$ or $\delta\backslash C = 0_X$. Assume that $\lambda\backslash C = 0_X$. For this reason, $\lambda\backslash V \neq 0_X$. Let $\lambda_V = \lambda \cap \mu_V$. Then, λ_V is N-open in (X,τ). So, λ_V is N-regular closed in (X,τ). Since $\lambda\backslash Y \subset (\delta\backslash Y)^c$, $\lambda_V \subset \delta^c$. This implies that $\overline{\lambda_V} \subset \delta^c$. From $\delta \neq 0_X$, $\overline{\lambda_V} \neq 1_X$. In addition, $\overline{\lambda_V} \neq 0_X$. Because, $\overline{\lambda_V} \neq 0_X$ implies $\lambda_V = 0_X$. Then, $\lambda\backslash V = 0_X$. This discrepancy implies that $C \cup V$ is N-super-connected. ∎

Theorem 19. *Let (X,τ) be an N-topological space. If A and B are subsets of X and $\mu_A \subset \mu_B \subset \overline{\mu_A}$ and A is an N-super-connected subset of X, then B is also N-super-connected.*

Proof. Similar to that of Theorem 6. ∎

5. N-eutrosophic strong connectedness

This part contains the following information that we present the concept of N-strong connectedness and investigate its properties.

Definition 18. *Let (X,τ) be an N-topological space. (X,τ) is said to be N-strongly connected if it has no non-null N-closed sets f and k providing that $k \subset f^c$. If (X,τ) is not N-strongly connected then it will be called N-weakly disconnected.*

Theorem 20. *Let (X,τ) be an N-topological space. This implies that, it is weakly connected if and only if there exist non-null N-closed sets f and k providing that $f \subset k^c$. So, it is weakly connected if and only if there exist non-null N-open sets f^c and k^c providing that $(k^c)^c \subset f^c$. Clearly, N-strong connectedness implies N-connectedness. However, the converse statement is not always true. In addition, the following example shows that N-strong connectedness and N-super-connectedness are unrelated.*

Example 3. Let $X = \{a,b\}$ and $\tau_1 = \Big\{0_X, 1_X, \Big\{\langle a, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle, \langle b, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle\Big\}$ and $\tau_2 = \Big\{0_X, 1_X, \Big\{\langle a, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\rangle, \langle b, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\rangle\Big\}\Big\}$. Then, (X,τ_1) is N-connected, N-super-connected, but not N-strongly connected and (X,τ_2) is N-strongly connected but not N-super-connected.

Theorem 21. *Let (X,τ) be an N-topological space and $A \subset X$. Then, A is an N-strongly connected subset of X if and only if, for all N-open sets λ and δ providing that $T_{\mu_A}(x) \leq T_\lambda(x) + T_\delta(x)$, $I_{\mu_A}(x) \leq I_\lambda(x) + I_\delta(x)$, $F_{\mu_A}(x) \geq F_\lambda(x) + F_\delta(x)$ for all $x \in X$, either $\mu_A \subset \lambda$ or $\mu_A \subset \delta$.*

Proof. Regard as an N-weakly connected subset A of X. Then, there exist N-closed sets f and k in (X,τ) providing that $f\backslash A \neq 0_A$, $k\backslash A \neq 0_A$ and $f\backslash A \subset (k\backslash A)^c$. Let $\lambda = f^c$ and $\delta = k^c$. Then, $\lambda\backslash A = (f\backslash A)^c$ and $\delta\backslash A = (k\backslash A)^c$. Clearly, $T_{\mu_A}(x) \leq T_\lambda(x) + T_\delta(x)$, $I_{\mu_A}(x) \leq I_\lambda(x) + I_\delta(x)$, $F_{\mu_A}(x) \neq F_\lambda(x) + F_\delta(x)$, for all $x \in X$. But, $\mu_A \not\subset \lambda$ and $\mu_A \not\subset \delta$. Conversely, assume that there exist N-open sets λ and δ providing that $T_{\mu_A}(x) \leq T_\lambda(x) + T_\delta(x)$, $I_{\mu_A}(x) \leq I_\lambda(x) + I_\delta(x)$, $F_{\mu_A}(x) \neq F_\lambda(x) + F_\delta(x)$ for all $x \in X$, but neither $\mu_A \subset \lambda$ or $\mu_A \subset \delta$. Then $\lambda\backslash A \neq 1_A$, $\delta\backslash A \neq 1_A$ and $(\lambda\backslash A)^c \subset \delta\backslash A$. So, A is N-weakly connected. ■

Theorem 22. *Let (X,τ) be an N-topological space and F be a subset of X providing that μ_F is N-closed in (X,τ). If (X,τ) is N-strongly connected then F is an N-strongly connected subset of X.*

Proof. Let F be a subset of X providing that μ_F is N-closed in an N-strongly connected topological space (X,τ). Suppose that F is not N-strongly connected. Then, there exist N-closed sets f and k in (X,τ) providing that $f\backslash F \neq 0_F$, $k\backslash F \neq 0_F$ and $f\backslash F \subset (k\backslash F)^c$. So, $f \cap \mu_F \neq 0_X$, $k \cap \mu_F \neq 0_X$ and $f \cap \mu_F \subset (f \cap \mu_F)^c$. This means that (X,τ) is not N-strongly connected. This is a discrepancy. ■

Theorem 23. *Let* (X_1, τ_1) *and* (X_2, τ_2) *N-topological spaces and* $f : (X_1, \tau_1) \longrightarrow (X_2, \tau_2)$ *be an N-continuous function. Then, if* (X_1, τ_1) *is N-strongly connected then* (X_2, τ_2) *is also N-strongly connected.*

Proof. Omitted. ■

Theorem 24. *A finite product of N-strongly connected spaces is N-strongly connected.*

Proof. Let (X, τ_X) and (Y, τ_Y) be N-strongly connected topological spaces. Suppose that $(X \times Y, \tau_{X\times Y})$ is not N-strongly connected. The members of $\tau_{X\times Y}$ are of the $\lambda \times \delta$, where $\lambda \in \tau_X$ and $\delta \in \tau_Y$. Then, there exist non-absolute N-sets $\lambda\times\delta$, $\mu\times\eta$ providing that $(\mu\times\eta)^c \subset \lambda\times\delta$, $\lambda, \mu \in \tau_X$ and $\delta, \eta \in \tau_Y$. Suppose that λ is an absolute N-set. Then, δ is a non-absolute N-set. Then, $\max\{F_\mu(x), F_\eta(y)\} \leq T_\delta(y)$ and $1 - \min\{I_\mu(x), I_\eta(y)\} \leq I_\delta(y)$ and $\min\{T_\mu(x), T_\eta(y)\} \geq F_\delta(y)$, for any $x \in X$ and $y \in Y$. Neither μ nor η is null. Because, if μ or η was null, then it would be impossible that $\max\{F_\mu(x), F_\eta(y)\} \leq T_\delta(y)$ and $1 - \min\{I_\mu(x), I_\eta(y)\} \leq I_\delta(y)$ and $\min\{T_\mu(x), T_\eta(y)\} \geq F_\delta(y)$.
Since $\mu \times \eta$ is non-absolute, μ or η is non-absolute,

Case 1. μ is non-absolute. For any $x \in X$ and providing that $F_\mu(x) > T_\delta(y)$ or $1 - I_\mu(x) > I_\delta(y)$ or $T_\mu(x) < F_\delta(y)$, $(\mu \times \eta)^c \not\subset \lambda \times \delta$. This is a discrepancy.

Case 2. η is non-absolute. Since δ is a non-absolute N-set and (Y, τ_Y) is N-strongly connected, there exists $y_1 \in Y$ providing that $F_\eta(y_1) > T_\delta(y_1)$ or $1 - I_\eta(y_1) > I_\delta(y_1)$ or $T_\eta(y_1) < F_\delta(y_1)$. Then, for any $x \in X$, $\max\{F_\mu(x), F_\eta(y_1)\} > T_\delta(y_1)$ or $1 - \min\{I_\mu(x), I_\eta(y_1)\} > T_\delta(y_1)$ or $\min\{T_\mu(x), T_\eta(y_1)\} < F_\delta(y_1)$. Since $(\mu \times \eta)^c \subset \lambda \times \delta$, this is a discrepancy.
So, λ is a non-absolute N-set. Alike, it can be proved that δ is a non-absolute N-set. So, neither $(\mu)^c \subset \lambda$ nor $(\eta)^c \subset \delta$. Then, $(\mu \times \eta)^c \not\subset \lambda \times \delta$. From this discrepancy, $(X \times Y, \tau_{X\times Y})$ is N-strongly connected.
An infinite N-product of N-strongly connected spaces may not be N-strongly connected as seen in the following example. ■

Example 4. Let $X_n = \{a, b\}$ and $\tau_n = \Big\{0_X, 1_X, \lambda_n\Big\}$, $\lambda_n = \Big\{\langle a, \frac{n}{2(n+1)}, \frac{n}{2(n+1)}, 1 - \frac{n}{2(n+1)}\rangle, \langle b, \frac{n}{2(n+1)}, \frac{n}{2(n+1)}, 1 - \frac{n}{2(n+1)}\rangle\Big\}$ for any $n \in N$. Then, (X_n, τ_n) is N-strongly connected. But, (X, τ_X) is not N-strongly connected, where $X = \Pi_{i\in N} X_i$ and τ_X is the N-product topology.Because, $T_{\bigcup_{i\in N} P^{-1}(\lambda_n)}(x) = \frac{1}{2}$, $I_{\bigcup_{i\in N} P^{-1}(\lambda_n)}(x) = \frac{1}{2}$ and $F_{\bigcup_{i\in N} P^{-1}(\lambda_n)}(x) = \frac{1}{2}$, for all $x \in X$.

6. Conclusion

We describe the definition of neutrosophic connectedness and include some characterizations in this paper. We also introduce neutrosophic product space and demonstrate that this form of connectivity is not preserved neutrosophic product spaces. We also present and investigate the concepts of neutrosophic super-connected spaces and neutrosophic strongly connected spaces. As a result, we have given the world of topology a new perspective connectedness in N-topological spaces. Furthermore, we have provided a new description for the N-function, and we assume that the N-function will be useful in other mathematical research, particularly in topology. It is expected that the results in this document will encourage and lead scientists to develop their further work on neutrosophic topology to run it in a general framework for its applications in practical life. We also hope the new terms and concepts we propose to help develop new research disciplines and life-changing innovations.

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