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GENERALIZED ULAM-HYERS STABILITY OF GROUP AND STABILITY OF GROUP AND RING HOMOMORPHISMS VIA A FIXED POINT METHOD

Abstract. By using the fixed point method, we investigate the generalized Hyers-Ulam stability of group homomorphisms and ring homomorphisms. Our work brings some complements and some continuations to the results obtained previously by R. Badora (*On approximate ring homomorphisms* published in J. Math. Anal. Appl, 276, (2002), 589-597) and those of D. Zhang and H-X. Cao (*Stability of group and ring homomorphisms*, Mathematical Inequalities & Applications, 9(3)(2006), 521-528).

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1. Introduction

In 1940, S. M. Ulam (see [32]) raised the following question:

Given a group G_1, a metric group (G_2, d) and a positive number ϵ, does there exist a $\delta > 0$ such that if a function $f : G_1 \longrightarrow G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) \leq \delta$ for all $x, y \in G_1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) \leq \epsilon$ for all $x \in G_1$?

If the answer to this question is affirmative, then we say that the homomorphisms from G_1 to G_2 are stable or that the functional equation defining homomorphisms is stable in the sense of Ulam.

One of the first results concerning the question of Ulam was given in 1941 by D. H. Hyers [16], where he gave an affirmative answer to Ulam's question for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. Hyers proved that each solution of the inequality

$$\|f(x+y) - f(x) - f(x)\| \leq \epsilon, \quad \forall x, y \in G_1$$

can be approximated by an exact solution. That is by an additive mapping.

In 1949, D. G. Bourgin [5], investigated the approximately isometric and multiplicative transformations on continuous function rings.

In 1950, T. Aoki [2] studied Ulam stability for additive mappings. In fact the work of T. Aoki was submitted in 1947, where a generalization of [16] was done in the following way. Let E and E' be two real Banach spaces and let f be a map from E into E'. T. Aoki defined f to be "approximately linear", when there exists $K \geq 0$ and a number $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq K(\|x\|^p + \|y\|^p),$$

for any x and y in E.

Let f and φ be transformations from E into E'. T. Aoki called these mappings "near", when there exists $K \geq 0$ and $p \in [0, 1)$ such that

$$||f(x) - \varphi(x)|| \leq K\|x\|^p, \quad \forall x \in E.$$

Using the concepts above, T. Aoki established the following result.

Theorem 1 (T. Aoki [2]). *Let E and E' be two real Banach spaces and let f be a map from E into E'.*

If f is an approximately linear transformation from E into E', then there is a linear transformation φ near f. And such φ is unique.

For more informations and discussions concerning the work of T. Aoki, the reader is invited to read the following paper [20] of L. Maligranda.

According to [20], T. Aoki was the first author dealing with unbounded Cauchy differences.

In 1978, Th. M. Rassias [24] extended the results of T. Aoki and established an important generalization of the result of Hyers by considering the stability problem for unbounded Cauchy differences. This phenomenon of stability studied by Th. M. Rassias in [24] was called by Th. M. Rassias (see [27]) the generalized Hyers–Ulam stability. Many authors used also the name of Hyers–Ulam–Rassias stability.

Theorem 2 (Th. M. Rassias [24]). *Let $f : E_1 \longrightarrow E_2$ be a mapping from a real normed vector space E_1 into a Banach space E_2 satisfying the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1}$$

for all $x, y \in E_1$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E_1 \longrightarrow E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p, \quad \forall x \in E_1. \tag{2}$$

If $p < 0$ then inequality (1) *holds for all $x, y \neq 0$, and* (2) *for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E_2 is continuous for each fixed $x \in E$, then T is linear.*

In [14], Gajda considered also the stability problem with unbounded Cauchy differences. From the papers of Hyers, Rassias and Gajda, we have the following Theorem which completes the results of Theorem 2.

Theorem 3 (Hyers-Rassias-Gajda [14], [17], [24])**.** *Suppose that E_1 is a real normed space, E_2 is a real Banach space, $f : E_1 \longrightarrow E_2$ is a given function, and the following condition holds*

$$\|f(x+y) - f(x) - f(y)\|_{E_2} \leq \theta(\|x\|_{E_1}^p + \|y\|_{E_1}^p), \quad \forall x, y \in E_1, \qquad (C_p)$$

for some $p \in [0, +\infty) \setminus \{1\}$. Then there exists a unique additive function $T : E_1 \longrightarrow E_2$ such that

$$\|f(x) - T(x)\|_{E_2} \leq \frac{2\theta}{2 - 2^p}\|x\|_{E_1}^p, \quad \forall x \in E_1. \qquad (Est_p)$$

The methods used in the previous theorems are called direct methods.

L. Székelyhidi (see [29], [30] and [31]) has developped other methods to treat the stability of functional equations.

In 1994, P. Găvruta (see [15]) established a generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings by using a new method.

In 1991, J. A. Baker (see [4]) studied the stability of certain functional equations by using the Banach contraction principle.

In 2003, V. Radu [23] obtained some stability results via the alternative fixed point theorem. In their paper [7], L. Cădariu and V. Radu have used the same fixed point method to establish the stability of functional equations of Jensen type.

In 2011, M. Akkouchi (see [1]) extended the results of [4] by using a Ćirić fixed point theorem.

Fixed point methods are now successfully used to investigate the stability of various algebraic–differential–integral–functional equations. See for example [9], [6], [19], [22], [23] and others.

During the last decades, the results of Hyers, Rassias, Székelyhidi, Gajda, Găvruta and other leading mathematicians have been generalized in various directions and contexts by using different methods. The reader is invited to consult the list of references given at the end of this paper and the references therein.

R. Badora in [3] proved the following result concerning the stability of a ring homomorphism.

Theorem 4 (R. Badora)**.** *Let R be a ring and $\mathcal{B}$ be a Banach algebra and let $\epsilon, \delta > 0$. Assume that $f : R \longrightarrow \mathcal{B}$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

and

$$\|f(xy) - f(x)f(y)\| \leq \delta,$$

for all $x, y \in R$. Then there exists a unique ring homomorphism $T : R \longrightarrow \mathcal{B}$ such that

$$\|f(x) - T(x)\| \leq \epsilon, \quad \forall x \in R.$$

D. Zhang and H-X. Cao [33], by using direct methods have established the following generalization of Badora's result.

Theorem 5 (D. Zhang and H-X. Cao)**.** *Let R be a ring and $\mathcal{B}$ be a Banach algebra and $r \in \mathbb{N}$, $r \geq 2$ and $\epsilon, \delta > 0$. Assume that $f : R \longrightarrow \mathcal{B}$ satisfies the following conditions:*

$$\left\| f(\sum_{k=1}^{r} x_k) - \sum_{k=1}^{r} f(x_k) \right\| \leq \epsilon, \quad \forall\, x_1, x_2, \ldots, x_r \in R \tag{3}$$

and

$$\|f(x_1 x_2 \ldots x_r) - f(x_1)f(x_2)\ldots f(x_r)\| \leq \delta, \quad \forall\, x_1, x_2, \ldots, x_r \in R. \tag{4}$$

Then there exists a unique ring homomorphism $T : R \longrightarrow \mathcal{B}$ such that

$$T(x_1 x_2 \ldots x_r) = T(x_1)T(x_2)\ldots T(x_r), \quad \forall\, x_1, x_2, \ldots, x_r \in R \tag{5}$$

and

$$\|f(x) - T(x)\| \leq \frac{1}{r-1}\epsilon, \quad \forall x \in R. \tag{6}$$

Before proving this theorem, D. Zhang and H-X. Cao [33] have investigated the Ulam-Hyers stability of the following functional equation of Cauchy type:

$$f(\sum_{k=1}^{r} x_k) = \sum_{k=1}^{r} f(x_k), \quad \forall x_1, \ldots, x_r \in G \tag{7}$$

where $r \geq 2$ is an integer and f is a function defined on an Abelian group G and taking values in a Banach space F.

If G is a ring, then beside the equation (7), one should consider the following (multiplicative) Cauchy type functional equation:

$$f(x_1 x_2 \ldots x_r) = f(x_1)f(x_2)\ldots f(x_r), \quad \forall x_1, \ldots, x_r \in G. \tag{8}$$

The first aim of this paper is to investigate the generalized Ulam-Hyers stability of equation (7) by a fixed point method using the Banach contraction principle. The main result obtained in this direction is Theorem 7 which is established in Section 2.

In the third section, we apply Theorem 7 to establish the generalized Ulam-Hyers stability of the system of functional equations (7) and (8) on a ring. The result obtained is stated in Theorem 8 which is a general version of Badora's result and Theorem 1.4 of Zhang and Cao.

At the end, we apply Theorem 8 to deduce in Theorem 3.3 the generalized Ulam-Hyers stability of ring homomorphisms yielding an extension of a result of [33]. So, in particular, the results of our study complete and extend those obtained by Badora [3] and D. Zhang and H-X. Cao in [33].

2. Generalized Ulam-Hyers stability of group homomorphisms

Our result will make use of the Banach contraction principle. For sake of completeness, we recall this theorem.

Theorem 6 (Banach's contraction principle). *Let (X, d) be a complete metric space, and consider a mapping $\Lambda : X \to X$, which is strictly contractive, that is*

$$(B_1)\qquad d(\Lambda x, \Lambda y) \le L\, d(x, y), \quad \forall x, y \in X,$$

for some (Lipschitz constant) $0 \le L < 1$.

Then

(i) The mapping Λ has one, and only one, fixed point $x^* = \Lambda x^*$;

(ii) The fixed point x^* is globally attractive, that is

$$(B_2)\qquad \lim_{n\to\infty} \Lambda^n x = x^*;$$

for any starting point x in X;

(iii) One has the following estimation inequalities:

$$(B_3)\qquad d(\Lambda^n x, x^*) \le L^n\, d(x, x^*), \quad \forall n \ge 0, \forall x \in X;$$

$$(B_4)\qquad d(\Lambda^n x, x^*) \le \frac{1}{1-L}\, d(\Lambda^n x, \Lambda^{n+1} x), \quad \forall n \ge 0, \forall x \in X;$$

$$(B_4)\qquad d(x, x^*) \le \frac{1}{1-L} d(x, \Lambda x), \quad \forall x \in X.$$

Throughout this paper, $\mathbb{N}$ will designate the set of all nonnegative integers. We fix a positive integer r such that $r \geq 2$. Let G be an Abelian group and let F be (a real or complex) Banach space. We study the generalized Ulam-Hyers-Rassias stability of equation (5). To be precise, we introduce the following definition.

Definition 1. *Let $\varphi : G^r := G \times G \times \ldots \times G \longrightarrow [0, +\infty)$ be a mapping. We say that the equation (5) is generalized Ulam-Hyers-Rassias stable with respect to φ, if there exists a positive constant c such that for all function $f : G^r \longrightarrow F$ satisfying*

$$\left\| f(\sum_{k=1}^{r} x_k) - \sum_{k=1}^{r} f(x_k) \right\| \leq \varphi(x_1, \ldots, x_r), \quad \forall x_1, \ldots, x_r \in G, \tag{9}$$

there exists a mapping $g : G \longrightarrow F$ which is additive (that is $g(x+y) = g(x) + g(y)$ for all $x, y \in X$) and satisfying the following estimation inequality:

$$\|f(x) - g(x)\| \leq c\,\varphi(x, \ldots, x), \quad \forall x \in G. \tag{10}$$

The first main result of this papers reads as follows.

Theorem 7. *Let $(G, +)$ be an Abelian group. Let $r \geq 2$ be an integer. Let $(F, \|\cdot\|)$ be a complete (real or complex) normed vector space endowed with a norm $\|\cdot\|$. Let $f : G \longrightarrow F$ be a mapping for which there exists a function $\varphi : G^r \to [0, \infty)$ such that*

$$\|f(\sum_{k=1}^{r} x_k) - \sum_{k=1}^{r} f(x_k)\| \leq \varphi(x_1, \ldots, x_r), \quad \forall x_1, \ldots, x_r \in G, \tag{11}$$

We suppose also that there exists a constant L, $0 < L < 1$ such that

$$\varphi(rx, \ldots, rx) \leq rL\varphi(x, \ldots, x), \quad \forall x \in G \tag{12}$$

and

$$\lim_{n\to\infty} \frac{\varphi(r^n x_1, \ldots, r^n x_r)}{r^n} = 0, \quad \forall x_1, \ldots, x_r \in G. \tag{13}$$

Then there exists a unique additive mapping $f^* : G \to F$ such that

$$\|f(x) - f^*(x)\| \leq \frac{1}{r}\frac{1}{1-L}\varphi(x, \ldots, x), \quad \forall x \in G. \tag{14}$$

Proof. By setting $x_1 = x_2 = \ldots = x_r = x$ in the inequality (11), we obtain

$$\|f(rx) - rf(x)\| \leq \varphi(x, \ldots, x), \quad \forall x \in G,$$

which implies that

$$\|\frac{1}{r}f(rx) - f(x)\| \leq \frac{1}{r}\varphi(x,\ldots,x), \quad \forall x \in G. \tag{15}$$

We consider the set

$$\mathcal{X} = \{h : G \to F\}.$$

For all $g \in \mathcal{X}$, we set

$$(\Lambda g)(x) := \frac{1}{r}g(rx), \quad \forall x \in G. \tag{16}$$

For each pair g, h of elements of $\mathcal{X}$, we consider the set given by

$$I_\varphi(g,h) := \{c \in [0,+\infty) : \|g(x) - h(x)\| \leq c\,\varphi(x,\ldots,x), \forall x \in G\}. \tag{17}$$

We observe that $I_\varphi(g,h) = I_\varphi(h,g)$.

We introduce the set $\mathcal{X}_\varphi$ defined by the following.

$$\mathcal{X}_\varphi := \{h \in \mathcal{X} : I_\varphi(h,f) \neq \emptyset\} \tag{18}$$

Obviousely, we have $f \in \mathcal{X}_\varphi$. Also, from (15), we deduce that $\Lambda f \in \mathcal{X}_\varphi$. Hence the set $\mathcal{X}_\varphi$ is not empty.

By using the triangle inequality, it is easy to see that $I_\varphi(g,h)$ is not empty for all $g, h \in \mathcal{X}_\varphi$.

For all $g, h \in \mathcal{X}_\varphi$ we set

$$d_\varphi(g,h) := \inf\{c \in [0,\infty) : \|g(x) - h(x)\| \leq c\varphi(x,\ldots,x), \text{for all}\, x \in G\},$$

Then it is easy to see that d_φ is a distance on the set $\mathcal{X}_\varphi$.

We observe that for all $g, h \in \mathcal{X}_\varphi$, the number d_φ satisfies the following property:

$$\|g(x) - h(x)\| \leq d_\varphi(g,h).\varphi(x,\ldots,x), \quad \text{for all } x \in G. \tag{19}$$

By classical arguments, one can prove that the metric space $(\mathcal{X}_\varphi, d_\varphi)$ is complete.

Now, we prove that $(\mathcal{X}_\varphi, d_\varphi)$ is invariant under the map Λ. To this end, let $g \in \mathcal{X}_\varphi$ be given. By the triagle inequality, for all $x \in G$, we have

$$\|f(x) - (\Lambda g)(x)\| \leq \|f(x) - (\Lambda f)(x)\| + \|(\Lambda f)(x) - (\Lambda g)(x)\|.$$

By the assumption (15), we have

$$\|f(x) - (\Lambda f)(x)\| \leq \frac{1}{r}\varphi(x,\ldots,x), \quad \forall x \in G. \tag{20}$$

By using the assumption (12), we have the following inequalities:

$$\begin{aligned} \|(\Lambda g)(x) - (\Lambda h)(x)\| &= \left\| \frac{g(rx) - h(rx)}{r} \right\| \\ &\le d_\varphi(g,h) \frac{\varphi(rx,\dots,rx)}{r} \\ &\le d_\varphi(g,h) L\varphi(x,\dots,x), \quad \forall x \in G. \end{aligned} \tag{21}$$

From (20) and (21), we obtain that

$$\|f(x) - (\Lambda g)(x)\| \le \left[L\, d_\varphi(g,h) + \frac{1}{r} \right] \varphi(x,\dots,x),\ \forall x \in G,$$

which implies that $\Lambda g \in \mathcal{X}_\varphi$.

The inequality (20) implies that

$$d_\varphi(f, \Lambda f) \le \frac{1}{r}$$

The inequality (21) implies that

$$d_\varphi(\Lambda g, \Lambda h) \le L\, d_\varphi(g,h), \quad \forall g, h \in \mathcal{X}_\varphi.$$

That is Λ is a strict contractive self-mapping of the complete metric space $(\mathcal{X}_\varphi, d_\varphi)$. By applying the Banach contraction principle (see 6). It follows that there exists a unique function f^* in the set $\mathcal{X}_\varphi$ which is fixed by Λ, i.e, $\Lambda(f^*) = f^*$ such that $\lim_{n\to\infty} d_\varphi(\Lambda^n g, f^*) = 0$ for each $g \in \mathcal{X}_\varphi$. In particular, we have $\lim_{n\to\infty} d_\varphi(\Lambda^n f, f^*) = 0$, i.e,

$$\lim_{n\to\infty} \frac{1}{r^n} f(r^n x) = f^*(x), \quad \forall x \in G. \tag{22}$$

From (B_4) of (*iii*) of Theorem 6, we obtain

$$d_\varphi(f, f^*) \le \frac{1}{1-L} d_\varphi(\Lambda f, f) \le \frac{1}{r} \frac{1}{1-L}, \tag{23}$$

which implies that the inequality (14) is true for all $x \in G$.

Now, we prove that f^* is additive. To this respect, we start by replacing each x_k by $r^n x_k$ for $k = 1, 2, \dots, r$ in (11). We obtain

$$\left\| f(\sum_{k=1}^{r} (r^n x_k)) - \sum_{k=1}^{r} f(r^n x_k) \right\| \le \varphi(r^n x_1, r^n x_2, \dots, r^n x_r),$$

which gives after dividing by r^n the following inequality

$$\left\| (\Lambda^n f)(\sum_{k=1}^{r} x_k) - \sum_{k=1}^{r} (\Lambda^n f)(x_k) \right\| \le \frac{\varphi(r^n x_1, r^n x_2, \dots, r^n x_r)}{r^n}, \tag{24}$$

for all $x_1, x_2, \ldots, x_r \in G$.

According to (13) and (22), by letting n tend to infinity in (24), we get

$$f^*(\sum_{k=1}^{r} x_k) = \sum_{k=1}^{r} f^*(x_k), \quad \forall x_1, x_2, \ldots, x_r \in G.$$

Thus f^* satisfies the functional equation (5). This implies that $f^*(0) = 0$ and that f^* is additive.

Finally, we prove that f^* is uniquely determined. Assume that inequality (14) is also satisfied with another additive function $f^\sharp : G \to F$ besides f^*. As $f^\sharp$ is an additive function, $f^\sharp$ satisfies that

$$(\Lambda f^\sharp)(x) = \frac{1}{r} f^\sharp(rx) = f^\sharp(x), \quad \forall x \in G.$$

That is, $f^\sharp$ is a fixed point of Λ. Since $f^\sharp$ satisfies (14), it follows that $f^\sharp$ is in the space $\mathcal{X}_\varphi$. By the Banach contraction principle (see Theorem 6), it follows that $f^\sharp = f^*$. This ends the proof. ■

3. Generalized Ulam-Hyers stability of ring homomorphisms

In this section, we intend to study the generalized Ulam-Hyers stability of ring homomorphisms.

We start by proving a general stability result in the sense of Ulam-Hyers for a system of two functional equations of Cauchy type on rings. This result generalizes both R. Badora's theorem and Theorem 3.1 of [33].

Theorem 8. *Let R be a ring. Let $r \geq 2$ be an integer. Let $(\mathcal{B}, \|.\|)$ be a (real or complex) Banach algebra endowed with a norm $\|.\|$. Let $\varphi, \psi : R^r \to [0, \infty)$ be functions satisfying the following conditions:*

$(1)_\varphi$: *There exists a constant L, $0 < L < 1$ such that*

$$\varphi(rx, \ldots, rx) \leq rL\varphi(x, \ldots, x), \quad \forall x \in R. \tag{25}$$

$(2)_\varphi$:

$$\lim_{n\to\infty} \frac{\varphi(r^n x_1, \ldots, r^n x_r)}{r^n} = 0, \quad \forall x_1, \ldots, x_r \in R. \tag{26}$$

$(3)_\psi$: *There exists an index $j \in \{1, 2, \ldots, r\}$ such that*

$$\lim_{n\to\infty} \frac{\psi(x_1, \ldots, x_{j-1}, r^n x_j, x_{j+1}, \ldots, x_r)}{r^n} = 0, \quad \forall x_1, \ldots, x_r \in R. \tag{27}$$

Let $f : R \longrightarrow \mathcal{B}$ be a mapping satisfying the following inequalities:

$$\|f(\sum_{k=1}^{r} x_k) - \sum_{k=1}^{r} f(x_k)\| \leq \varphi(x_1, \ldots, x_r), \quad \forall x_1, \ldots, x_r \in R, \tag{28}$$

and

$$\|f(x_1 x_2 \ldots x_r) - f(x_1)f(x_2)\ldots f(x_r)\| \leq \psi(x_1, \ldots, x_r), \quad \forall x_1, \ldots, x_r \in R. \tag{29}$$

Then there exists a unique additive mapping $f^* : R \to \mathcal{B}$ such that

$$f^*(x_1 x_2 \ldots x_r) = f^*(x_1)f^*(x_2)\ldots f^*(x_r), \quad \forall x_1, \ldots, x_r \in R, \tag{30}$$

and

$$\|f(x) - f^*(x)\| \leq \frac{1}{r}\frac{1}{1-L}\varphi(x, \ldots, x) \tag{31}$$

for all $x \in R$.

Proof. From Theorem 7, it follows that there exists a unique additive mapping $f^* : R \longrightarrow \mathcal{B}$ which satisfies the inequality (30). From the proof of Theorem 7, we know that the additive mapping f^* is given by

$$f^*(x) = \lim_{n\to\infty} \frac{1}{r^n} f(r^n x), \quad \forall x \in R. \tag{32}$$

For all $x_1, \ldots, x_r \in R$, we put

$$g(x_1, \ldots, x_r) := f(x_1 x_2 \ldots x_r) - f(x_1)f(x_2)\ldots f(x_r).$$

Let $j \in \{1, 2, \ldots, r\}$ be such that

$$\lim_{n\to\infty} \frac{\psi(x_1, \ldots, x_{j-1}, r^n x_j, x_{j+1}, \ldots, x_r)}{r^n} = 0, \quad \forall x_1, \ldots, x_r \in R. \tag{33}$$

From (28) and (32), it follows that

$$\lim_{n\to\infty} \frac{g(x_1, \ldots, x_{j-1}, r^n x_j, x_{j+1}, \ldots, x_r)}{r^n} = 0, \quad \forall x_1, \ldots, x_r \in R. \tag{34}$$

Therefore, for all $x_1, \ldots, x_r \in R$, we have

$$\begin{aligned}
f^*(x_1 x_2 \ldots x_r) &= \lim_{n\to\infty} \frac{f(x_1 \ldots x_{j-1} r^n x_j x_{j+1} \ldots x_r)}{r^n} \\
&= \lim_{n\to\infty} \Bigg[\frac{g(x_1, \ldots, x_{j-1}, r^n x_j, x_{j+1}, \ldots, x_r)}{r^n} \\
&\qquad + \frac{f(x_1)\ldots f(x_{j-1})f(r^n x_j)f(x_{j+1})\ldots f(x_r)}{r^n}\Bigg] \\
&= f(x_1)\ldots f(x_{j-1})f^*(x_j)f(x_{j+1})\ldots f(x_r).
\end{aligned}$$

Hence, we have proved the following identity:

$$f^*(x_1x_2\dots x_r) = f(x_1)\dots f(x_{j-1})f^*(x_j)f(x_{j+1})\dots f(x_r), \tag{35}$$

for all $x_1,\dots,x_r \in R$.

We recall that $\mathbb{N}$ is the set of nonnegative integers. Let $m_1,\dots m_r \in \mathbb{N}$ with $m_j = 0$. By using (35) and the additivity of f^*, we have the following inequalities:

$$\begin{aligned} r^{m_1+\dots+m_r} & f^*(x_1x_2\dots x_r) \\ &= f^*((r^{m_1}x_1)\dots(r^{m_{j-1}}x_{j-1})x_j(r^{m_{j+1}}x_{j+1})\dots(r^{m_r}x_r)) \\ &= f(r^{m_1}x_1)\dots f(r^{m_{j-1}}x_{j-1})f^*(x_j)f(r^{m_{j+1}}x_{j+1})\dots f(r^{m_r}x_r), \\ &\quad \forall\, x_1,\dots,x_r \in R, \end{aligned}$$

from which, we deduce the following identity

$$\begin{aligned} f^*(x_1x_2\dots x_r) \;=\; & \frac{f(r^{m_1}x_1)}{r^{m_1}}\dots\frac{f(r^{m_{j-1}}x_{j-1})}{r^{m_{j-1}}}f^*(x_j)\frac{f(r^{m_{j+1}}x_{j+1})}{r^{m_{j+1}}} \\ & \dots\frac{f(r^{m_r}x_r)}{r^{m_r}}, \end{aligned} \tag{36}$$

for all $x_1,\dots,x_r \in R$.

By sending the integers $m_1,\dots,m_{j-1},m_{j+1},\dots,m_r$ to infinity, we obtain the following identity

$$f^*(x_1x_2\dots x_r) = f^*(x_1)f^*(x_2)\dots f^*(x_r), \quad \forall\, x_1,\dots,x_r \in R,$$

which is the desired identity (29). This ends the proof. ■

A first consequence of Theorem 8 is the next corollary.

Corollary 1. *Let R be a ring. Let $r \geq 2$ be an integer. Let $(\mathcal{B},\|\cdot\|)$ be a (real or complex) Banach algebra endowed with a norm $\|\cdot\|$. Let $\varphi,\psi : R^r \to [0,\infty)$ be functions satisfying the following conditions:*

$(4)_\varphi$: *There exists a constant L, $0 < L < 1$ such that*

$$\varphi(rx_1,rx_2,\dots,rx_r) \leq rL\,\varphi(x_1,x_2,\dots,x_r), \quad \forall x_1,\dots,x_r \in R. \tag{37}$$

$(3)_\psi$: *There exists an index $j \in \{1,2,\dots,r\}$ such that*

$$\lim_{n\to\infty}\frac{\psi(x_1,\dots,x_{j-1},r^nx_j,x_{j+1},\dots,x_r)}{r^n} = 0, \quad \forall x_1,\dots,x_r \in R.$$

Let $f : R \longrightarrow \mathcal{B}$ be a mapping satisfying the following inequalities:

$$\|f(\sum_{k=1}^{r}x_k) - \sum_{k=1}^{r}f(x_k)\| \leq \varphi(x_1,\dots,x_r), \quad \forall x_1,\dots,x_r \in R, \tag{38}$$

and

$$\|f(x_1x_2\dots x_r) - f(x_1)f(x_2)\dots f(x_r)\| \le \psi(x_1,\dots,x_r), \quad \forall x_1,\dots,x_r \in R. \tag{39}$$

Then there exists a unique additive mapping $f^* : R \to \mathcal{B}$ such that

$$f^*(x_1x_2\dots x_r) = f^*(x_1)f^*(x_2)\dots f^*(x_r), \quad \forall x_1,\dots,x_r \in R, \tag{40}$$

and

$$\|f(x) - f^*(x)\| \le \frac{1}{r}\frac{1}{1-L}\varphi(x,\dots,x) \tag{41}$$

for all $x \in R$.

Proof. Clearly the condition $(4)_\varphi$ implies the condition $(1)_\varphi$ of Theorem 8. By induction, the condition $(4)_\varphi$ yields to the following:

$$\frac{\varphi(r^n x_1,\dots,r^n x_r)}{r^n} \le L^n\,\varphi(x_1,\dots,x_r) \longrightarrow 0, \text{ as } n \longrightarrow \infty, \forall x_1,\dots,x_r \in R.$$

Thus, the condition $(2)_\varphi$ is also satisfied. Hence, all the conditions of Theorem 8 are satisfied. Therefore, By applying Theorem 8, we obtain the desired conclusions. ■

Remark 1. Thorem 1.4 of D. Zhang and H-X. Cao [33] is a consequence of Corollary 1. Indeed, Let R be a ring and let $\epsilon > 0$ and $\delta > 0$ be given. We set $\varphi(x_1,x_2,\dots,x_r) = \epsilon$ and $\psi(x_1,x_2,\dots,x_r) = \delta$, for all $x_1,\dots,x_r \in R$. Then it is easy to see that φ satisfies the condition $(4)_\varphi$ with $L = \frac{1}{r}$ and that ψ satisfies the condition $(3)_\psi$ for all index $j \in \{1,2,\dots,r\}$.

Concerning the generalized Ulam-Hyers-Rassias stability of ring homomorphisms, we have the following theorem.

Theorem 9. *Let $r \ge 2$ be an integer. Let R be a ring with a unit 1 and $(\mathcal{B},\|\cdot\|)$ be a (real or complex) Banach algebra endowed with a norm $\|\cdot\|$ and a unit e. Let $\varphi,\psi : R^r \to [0,\infty)$ be functions satisfying the conditions $(1)_\varphi$, $(2)_\varphi$ and $(3)_\psi$ of Theorem 8.*

If a mapping $f : R \longrightarrow \mathcal{B}$ satisfies the conditions (28) and (29) and $f(1) = e$, then there exists a unique homomorphism $f^* : R \to \mathcal{B}$ such that

$$\|f(x) - f^*(x)\| \le \frac{1}{r}\frac{1}{1-L}\varphi(x,\dots,x), \quad \forall x \in R. \tag{42}$$

Proof. From Theorem 8, there exists a unique additive mapping $f^* : R \to \mathcal{B}$ satisfying the identity (30) and the inequality (42).

Let $j \in \{1, 2, \dots, r\}$ be an index for which $(3)_\psi$ is true. We set $k := j - 1$ if $j > 1$ and $k := j + 1$ if $j < r$. By using the identity (36), it is easy to derive the following identity

(43)

$$f^*(x_1 x_2 \dots x_r) = f^{s_1}(x_1) f^{s_2}(x_2) \dots f^{s_{j-1}}(x_{j-1}) f^*(x_j) f^{s_{j+1}}(x_{j+1}) \dots f^{s_r}(x_r),$$

for all $x_1, \dots, x_r \in R$, where the powers $s_1, \dots, s_{j-1}, s_{j+1}, \dots s_r$ are in the set of symbols $\{1, *\}$.

In (43), we put $x_l = 1$ for all index $l \notin \{k, k+1\}$ and $x_k = x$ and $x_{k+1} = y$, where x, y are arbitrary elements in R. Since $f(1) = e$, it follows the following identity:

$$f^*(xy) = f^*(x) f^*(y). \tag{44}$$

Since (44) holds true for all $x, y \in R$, it follows that the additive mappings $f^* : R \to \mathcal{B}$ is a ring homomorphism. This compltes the proof. ■

Remark 2. The assumptions of Therem 9 do not imply that $f^*(1) = e$, where f^* is the function involved in (42) and defined by (22). For example, let $R = \mathcal{B} = \mathbb{R}$ be the real field. We put $f(1) = 1$ and $f(x) = 0$ for all $x \in \mathbb{R} \setminus \{1\}$. Let $r \in \mathbb{N}$ with $r \geq 2$. We set

$$\varphi(x_1, x_2, \dots, x_r) = r \quad \text{and} \quad \psi(x_1, x_2, \dots, x_r) = 1, \quad \forall x_1, \dots, x_r \in \mathbb{R}.$$

From Remark 1, we know that φ satisfies the condition $(4)_\varphi$ with $L = \frac{1}{r}$ and that ψ satisfies the condition $(3)_\psi$ for all index $j \in \{1, 2, \dots, r\}$.

It is easy to prove the following inequalities:

$$\|f(x_1 + x_2 + \dots + x_r) - f(x_1) - f(x_2) - \dots - f(x_r)\| \leq \varphi(x_1, \dots, x_r), \quad \forall x_1, \dots, x_r \in \mathbb{R}$$

and

$$\|f(x_1 x_2 \dots x_r) - f(x_1) f(x_2) \dots f(x_r)\| \leq \psi(x_1, \dots, x_r), \ \forall x_1, \dots, x_r \in \mathbb{R}.$$

So, all the conditions of Theorem 9 are satisfied. It is easy to show that the function f^* (involved in (42) and) defined by (22) is identically zero. Thus, in this example, we have $f^*(1) = 0$ but $f(1) = 1$.

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