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Bruno de Malafosse

SOLVABILITY OF NEW (SSIE) INVOLVING THE CONTINUOUS AND RESIDUAL SPECTRA OF THE GENERALIZED DIFFERENCE OPERATOR $B(r,s)$ ON c_0

Abstract. Let U^+ be the set of all positive sequences. Then, given any sequence $z = (z_n)_{n\geq 1} \in U^+$ and any set E of complex sequences, we write E_z for the set of all sequences $y = (y_n)_{n\geq 1}$ such that $y/z = (y_n/z_n)_{n\geq 1} \in E$. We use the notation $\mathbf{s}_z = (\ell_\infty)_z$. In this paper, for given r, $s \neq 0$ and for every $\lambda \in \mathbb{C}$, we determine the set of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy the (SSIE) with an operator $(c_0)_{B(r,s)-\lambda I} \subset \mathcal{E} + \mathbf{s}_x$, where $\mathcal{E} \subset \mathbf{s}_\theta$ for some $\theta \in U^+$ is a linear space of sequences, in each of the cases, (1) $|\lambda - r| > |s|$, or $\lambda = r$, (2) $|\lambda - r| = |s|$ and (3) $|\lambda - r| < |s|$ and $\lambda \neq r$. These cases are associated with the continuous and residual spectra $\sigma_c(B(r,s), c_0)$ and $\sigma_r(B(r,s), c_0)$, of $B(r,s)$ on c_0, determined by Altay and Başar in [2]. We apply these results to the solvability of the (SSIE) $(c_0)_{B(r,s)-\lambda I} \subset \mathbf{s}_R^{(c)} + \mathbf{s}_x$ for all $\lambda \in \mathbb{C}$ and $R > 0$. Then we deal with the (SSIE) $(c_0)_{\Delta-\lambda I} \subset bv_p + \mathbf{s}_x$ and $(c_0)_{B(r,s)-\lambda I} \subset E_{R_a} + \mathbf{s}_x$, for $E = c_0$, c, or ℓ_∞, where R_a, $a \in U^+$, is the Rhaly matrix. These results extend those stated in [21].

Key words: sequence space, BK space, (SSIE), Banach algebra, operator $B(r,s)$, bounded linear operator, fine spectrum.

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1. Preliminary results

Let $A = (\mathbf{a}_{nk})_{n,k\geq 1}$ be an infinite matrix and consider the sequence $x = (x_n)_{n\geq 1}$. We define the sequence $Ax = (A_n(x))_{n\geq 1}$ with $A_n(x) = \sum_{k=1}^{\infty} \mathbf{a}_{nk} x_k$ whenever the series are convergent for all $n \geq 1$. Let ω denote the set of all complex sequences. We write c_0, c and ℓ_∞ for the sets of all null, convergent and bounded sequences respectively. By U^+ we define the set of all positive sequences. For any given subsets E, F of ω, we say that the operator represented by the infinite matrix $A = (\mathbf{a}_{nk})_{n,k\geq 1}$ maps E into F, that is, $A \in (E, F)$, see [15], if the series defined by $A_n(x) = \sum_{k=1}^{\infty} \mathbf{a}_{nk} x_k$ are convergent for all $n \geq 1$ and for all $x \in E$, and $Ax \in F$ for all $x \in E$. If F is a subset of ω, then we denote the so-called matrix domain of A in E by $F_A = \{x \in \omega : y = Ax \in F\}$. Let $E \subset \omega$ be a *Banach space*, with norm $\|\|_E$. By $\mathcal{B}(E)$ we denote the *set of all bounded linear operators, mapping E into itself.* We say that $L \in \mathcal{B}(E)$ if and only if $L : E \mapsto E$ is a *linear operator* and $\|L\|^*_{\mathcal{B}(E)} = \sup_{x\neq 0} (\|Lx\|_E / \|x\|_E) < \infty$. It is well known that $\mathcal{B}(E)$ is a Banach algebra with the norm $\|L\|^*_{\mathcal{B}(E)}$, see [1]. A Banach space $E \subset \omega$ is a BK space if the projection $P_n : x \mapsto x_n$ *from E into $\mathbb{C}$ is continuous for all n.* A BK space $E \supset \varphi$ *is said to have AK if for every $x \in E$, then* $x = \lim_{p\to\infty} \sum_{k=1}^{p} x_k e^{(k)}$,*where* $e^{(k)} = (0, ..., 1, ...)$, 1 *being in the* $k-th$ *position.* It is well known that if E has AK then $\mathcal{B}(E) = (E, E)$. If E is a BK space with the norm $\|\|_E$, then $(E, E) \subset \mathcal{B}(E)$. Indeed, by ([26], Theorem 4.2.8 p. 57), the matrix map $A \in (E, E)$ is continuous and there is $M > 0$ such that $\|Ax\|_E \leq M \|x\|_E$ for all $x \in E$.

We call *sequence spaces inclusion equations* (SSIE) an inclusion, for which each term is a *sum* or *a sum of products of sets of the form* $(E_a)_T$ *and* $\left(E_{f(x)}\right)_T$, where f maps U^+ to itself, E is any linear space of sequences and T is a triangle, (cf. [7], [19], [20], [21]).

In this manuscript, we establish a *connection between the fine spectrum of the operator* $B(r, s)$ *and the solvability of some* (*SSIE*) *involving this operator.* For $s \neq 0$, we write $B(r, s)$ for *the generalized difference operator*, which is a double band matrix, entries of which are determined by $[B(r, s)]_{nn} = r$ for all n, and $[B(r, s)]_{n,n-1} = s$ for $n \geq 2$. If $r = -s = 1$, we obtain the *operator* $\Delta = B(1, -1)$ *of the first difference.* Then, for every $\lambda \in \mathbb{C}$, using some results obtained in the *fine spectrum theory*, we determine the set of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy the (SSIE) with operator of the form $(c_0)_{B(r,s)-\lambda I} \subset \mathcal{E} + \mathbf{s}_x$, where $\mathcal{E}$ is a linear space of sequences in each of the cases,

(*a*) $\lambda \in \rho(B(r, s), c_0) \cup \{r\}$,

(*b*) $\lambda \in \sigma_c(B(r, s), c_0)$,

(*c*) $\lambda \in \sigma_r(B(r, s), c_0) \setminus \{r\}$.

We apply these results to the solvability of the (SSIE) $(c_0)_{B(r,s)-\lambda I} \subset \mathbf{s}_R^{(c)} + \mathbf{s}_x$ for all $\lambda \in \mathbb{C}$ and $R > 0$. Then we study the solvability of each of the (SSIE)

$(c_0)_{\Delta-\lambda I} \subset bv_p + \mathbf{s}_x$, and

$(c_0)_{B(r,s)-\lambda I} \subset E_{R_a} + \mathbf{s}_x$,

for $E = c_0$, c, or ℓ_∞, where R_a, $a \in U^+$, is the Rhaly matrix.

This paper is organized as follows. In Section 2, we define the Banach algebra S_a and the multipliers of some sets. In Section 3, we recall some definitions and results on the spectrum and the fine spectrum of operators represented by triangles. In Section 4, we give some definitions and results related to sequence spaces inclusions. Finally, in Section 5, we deal with the solvability of the (SSIE) of the form $(c_0)_{B(r,s)-\lambda I} \subset \mathcal{E} + \mathbf{s}_x$.

2. The Banach algebra S_a and the multipliers of classical sets

For $a \in U^+$ we write $\mathbf{s}_a = (\ell_\infty)_a$, $\mathbf{s}_a^0 = (c_0)_a$, and $\mathbf{s}_a^{(c)} = c_a$. Each of the sets $\mathbf{s}_a$, $\mathbf{s}_a^0$, and $\mathbf{s}_a^{(c)}$ is a BK space with the norm $\|x\|_{\mathbf{s}_a} = \sup_n (|x_n| / a_n)$.

Then, the set S_a of all infinite matrices $A = (\mathbf{a}_{nk})_{n,k\geq 1}$, that satisfy $\|A\|_{S_a} = \sup_{n\geq 1} \left(a_n^{-1} \sum_{k=1}^{\infty} |\mathbf{a}_{nk}| a_k\right) < \infty$, is a Banach algebra with identity normed by $\|A\|_{S_a}$, (cf. [21], p. 161). Recall that, if $A \in (\mathbf{s}_a, \mathbf{s}_a)$, then we have $\|Ax\|_{\mathbf{s}_a} \leq \|A\|_{S_a} \|x\|_{\mathbf{s}_a}$ for all $x \in \mathbf{s}_a$. We have $\mathcal{B}(\mathbf{s}_a) \bigcap (\mathbf{s}_a, \mathbf{s}_a) = S_a = (\mathbf{s}_a, \mathbf{s}_a)$, for any given $a \in U^+$. Each of the sets $\mathcal{B}(E)$ for $E \in \left\{\mathbf{s}_\alpha, \mathbf{s}_a^0, \mathbf{s}_a^{(c)}\right\}$, is a Banach algebra, and $\mathcal{B}\left(\mathbf{s}_a^0\right) = \left(\mathbf{s}_a^0, \mathbf{s}_a^0\right)$, since $\mathbf{s}_a^0$ has AK. When $a = (r^n)_{n\geq 1}$, $r > 0$, S_a, $\mathbf{s}_a$, $\mathbf{s}_a^0$ *and* $\mathbf{s}_a^{(c)}$ are denoted by S_r and $\mathbf{s}_r$, $\mathbf{s}_r^0$ *and* $\mathbf{s}_r^{(c)}$. When $r = 1$, $\mathbf{s}_1 = \ell_\infty$, $\mathbf{s}_1^0 = c_0$ and $\mathbf{s}_1^{(c)} = c$. It is well-known that for each $E \in \{c_0, c, \ell_\infty\}$, we have $(E, \ell_\infty) = S_1$, for $E \in \{c_0, c, \ell_\infty\}$. We will use the next elementary result, where D_u, $u \in \omega$ denotes the diagonal matrix, determined by $[D_u]_{nn} = u_n$ for all n. If $\xi \neq 0$, then we write D_ξ for the diagonal matrix $D_{(\xi^n)_{n\geq 1}}$.

Lemma 1. *Let a, $b \in U^+$, and E, $F \subset \omega$. Then $A \in (E_a, F_b)$ if and only if $D_{1/b} A D_a \in (E, F)$.*

Now, let y and z be sequences and let E and F be two subsets of ω, we then write $yz = (y_n z_n)_{n\geq 1}$. Then we denote by

$$M(E, F) = \{y \in \omega : \ yz \in F \text{ for all } z \in E\},$$

the *multiplier space of E and F*. In this way, we recall the following well-known results.

Lemma 2. *Let E, $\widetilde{E}$, F and $\widetilde{F}$ be arbitrary subsets of ω. Then*

(i) $M(E, F) \subset M\left(\widetilde{E}, F\right)$ *for all* $\widetilde{E} \subset E$,

(ii) $M(E, F) \subset M\left(E, \widetilde{F}\right)$ *for all* $F \subset \widetilde{F}$.

By ([22], Lemma 3.1, p. 648) and ([23], Example 1.28, p. 157), we obtain the next lemma.

Lemma 3. *We have: (i)* $M(c, c_0) = M(\ell_\infty, c) = M(\ell_\infty, c_0) = c_0$ *and* $M(c, c) = c$. *(ii)* $M(E, \ell_\infty) = M(c_0, F) = \ell_\infty$ *for* E, $F = c_0$, c, *or* ℓ_∞.

3. On the spectrum and the fine spectrum of operators represented by triangles

In this section, we give a short survey on the fine spectrum and we recall some notions on this topic. We also deal with the fine spectrum of $B(r,s)$.

3.1. The point spectrum, continuous spectrum and residual spectrum

Let E be a BK space and let T be an operator mapping E to itself, (note that T is continuous since E is a BK space). We denote by $\sigma(T,E)$ the set of all complex numbers λ such that $T-\lambda I$ considered as an operator from E to itself is not invertible. Then we write $\rho(T,E)=[\sigma(T,E)]^c$ *for the resolvent set, which is the set of all complex numbers* λ *such that* $T-\lambda I$ *considered as an operator from* E *to itself is invertible.* Recall that the resolvent set of a linear operator on E is an open subset of the complex plane $\mathbb{C}$. We use the notation $D(\lambda_0,\mathrm{r})=\{\lambda\in\mathbb{C}:|\lambda-\lambda_0|<\mathrm{r}\}$, with $\lambda_0\in\mathbb{C}$ and $\mathrm{r}>0$, for the disk centered at λ_0 and of radius r.

Recall that the spectrum and the fine spectrum of the linear operators defined by infinite matrices over certain sequence spaces have been studied by many authors. We only give a short survey on this study. Recently, the fine spectra of the operator of the first difference over the sequence spaces ℓ_p and bv_p, were studied in [3], where bv_p is the space of $p-$bounded variation sequences, with $1\leq p<\infty$. In [7], there is a study on the *fine spectrum of the generalized difference operator* $B(r,s)$ on the each of the sets ℓ_p and bv_p. In [25], Srivastava and Kumar dealt with the fine spectrum of the generalized difference operator Δ_v over ℓ_1, where Δ_v is the triangle the nonzero entries of which are defined by $(\Delta_v)_{nn}=v_n$ and $(\Delta_v)_{n+1,n}=-v_n$. Then, Akhmedov and El-Sabrawy [4] determined the *spectrum of the generalized difference operator* $\Delta_{a,b}$ *defined as a double band matrix mapping in* c. In [20], using the generalized operator of the first difference $B(\widetilde{r},\widetilde{s})$, where $\widetilde{r}=(r_n)_{n\geq1},\widetilde{s}=(s_n)_{n\geq1}$ are two convergent sequences, we determined its spectrum over each of the spaces E_a, where $E=w_0(\Lambda)$, or $w_\infty(\Lambda)$ and Λ *is a nondecreasing exponentially bounded sequence.* In [17], we determined the spectrum of the operator represented by $B(\widetilde{r},\widetilde{s})$ over each of the spaces $\mathbf{s}_a$, $\mathbf{s}_a^0$, $\mathbf{s}_a^{(c)}$, ℓ_a^p, W_a^0 and W_a, with $1\leq p<\infty$. In [27] Yeşilkayagil and Başar gave a survey on the spectrum of triangles. Some other results on this topic, were stated in [12], [10], [8], [6], [4], [2], [13].

Now we briefly recall some definitions and results on the partition of the spectrum $\sigma(T,E)$ of the operator T over the normed space E. We refer the reader to ([14], pp. 370-371), for the basic concepts in spectral theory, concerning the operators T, $T_\lambda=T-\lambda I$ and T_λ^{-1} and recalled as follows. The spectrum $\sigma(T,E)$ is partitioned into the next three sets, (see

[14], Definition 7.2.1, p. 371, for more precisions), defined as follows. We begin with the discrete *point spectrum* $\sigma_p(T,E)$, which is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} does not exist. Then, the *continuous spectrum* $\sigma_c(T,E)$ is the set of all $\lambda \in \mathbb{C}$ such that $(T-\lambda I)^{-1}$ exists and is unbounded, and the domain of T_λ^{-1} is dense in E. Finally, the *residual spectrum* $\sigma_r(T,E)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists and is unbounded, but the domain of T_λ^{-1} is not dense in E.

To avoid confusion, notice that, in this theory, the expression, "T_λ^{-1} exists" means T_λ is injective, that is, $KerT_\lambda \cap E = \{0\}$, where $KerT_\lambda \cap E$, is the set of all sequences $x \in E$ such that $T_\lambda x = 0$.

In ([2], were stated the following results on the fine spectrum of $B(r,s)$.

Lemma 4. *Let r, $s \in \mathbb{C}$ and $s \neq 0$. Then we have,*
(i) $\sigma_p(B(r,s), c_0) = \varnothing$,
(ii) $\sigma_c(B(r,s), c_0) = \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\}$,
(iii) $\sigma_r(B(r,s), c_0) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$.

Then we have $\sigma(B(r,s), c_0) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$ and $\sigma(B(r,s), c_0) = \sigma_c(B(r,s), c_0) \cup \sigma_r(B(r,s), c_0)$. Then we write $a^\bullet = (a_n^\bullet)_{n\geq 1}$, where $a_n^\bullet = a_{n-1}/a_n$, for $n \geq 1$ with the convention $a_1^\bullet = 1$. In the next lemma, we use the set ℓ^p, $(p \geq 1)$, of $p-$absolutely convergent series, determined by $\ell^p = \left\{x = (x_k)_{k\geq 1} : \sum_{k=1}^{\infty} |x_k|^p < \infty\right\}$, and the spaces w_∞ and w_0, of sequences that are strongly bounded or strongly summable to zero by the Cesàro method of order 1, (cf. [16]).

Lemma 5 ([17], Theorem 5.1.1, p.14)**.** *Let r, $s \neq 0$, let $a \in U^+$, and let E be any of the sets c_0, c, ℓ_∞, ℓ^p, w_0, or w_∞, with $1 \leq p < \infty$ and assume $a^\bullet \in c$. Then we have*

$$\sigma(B(r,s), E_a) = \left\{\lambda \in \mathbb{C} : |\lambda - r| \leq |s| \lim_{n\to\infty} a_n^\bullet\right\}.$$

We obtain the next result, where we recall that $E_R = E_{(R^n)_n}$, $R > 0$. As a direct consequence of ([17], Corollary 5.1.4, p.15), we obtain the following result.

Lemma 6. *Let r, s, R be reals with $r \neq 0$ and $R > 0$ assume $E \in \{c_0, c, \ell_\infty, \ell^p, w_0, w_\infty\}$ with $1 \leq p < \infty$. Then we have: $\sigma(B(r,s), E_R) = \sigma(B(r,s/R), E) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|/R\}$.*

4. Some definitions and results related to the sequence spaces inclusions equations

Let $a \in U^+$ and let $\mathcal{E}$, F and F' be linear spaces of sequences. Our aim is to determine the set of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy the *sequence spaces inclusion equation*, (SSIE) $F \subset \mathcal{E} + F'_x$, (cf. [21]). Then we use the notation $\mathcal{I}(\mathcal{E}, F, F') = \{x \in U^+ : F \subset \mathcal{E} + F'_x\}$, (cf. [21], p. 236). Of course, if $F \subset \mathcal{E}$, then $\mathcal{I}(\mathcal{E}, \mathcal{F}, F') = U^+$, then, for any set χ of sequences we let $\overline{\chi} = \{x \in U^+ : 1/x \in \chi\}$. We begin to state the next elementary properties of the set $\mathcal{I}(\mathcal{E}, F, F')$.

Lemma 7 ([18], Lemma 10, p. 4)**.** *Let $\mathcal{E}$, $\mathcal{E}_1$, F, F', $\mathcal{F}$ and F'' be linear spaces of sequences. Then we have:*
(*i*) *If $\mathcal{E}_1 \subset \mathcal{E}$, then $\mathcal{I}(\mathcal{E}_1, F, F') \subset \mathcal{I}(\mathcal{E}, F, F')$,*
(*ii*) *If $\mathcal{F} \subset F$, then $\mathcal{I}(\mathcal{E}, F, F') \subset \mathcal{I}(\mathcal{E}, \mathcal{F}, F')$,*
(*iii*) *If $F' \subset F''$, then $\mathcal{I}(\mathcal{E}, F, F') \subset \mathcal{I}(\mathcal{E}, F, F'')$,*

To state the following results, we need the next lemma.

Lemma 8 ([18], Lemma 10, p. 5)**.** *Let $\mathcal{E}$, $\mathcal{E}_0$, F, G, $\mathcal{F}$ and F' be linear spaces of sequences. Then we have:*
(*i*) *$\overline{M(F, F')} \subset \mathcal{I}(\mathcal{E}, \mathcal{F}, F')$,*
(*ii*) *If $\mathcal{I}(\mathcal{E}_0, F, F') \subset \overline{M(F, F')}$, for any linear space of sequences $\mathcal{E}_0$ such that $\mathcal{E} \subset \mathcal{E}_0$, then we have $\mathcal{I}(\mathcal{E}, F, F') = \overline{M(F, F')}$,*
(*iii*) *If $\mathcal{I}(\mathcal{E}, \mathcal{F}, F') \subset \overline{M(F, F')}$, for some linear space of sequences $\mathcal{F} \subset F$, then we have $\mathcal{I}(\mathcal{E}, F, F') = \overline{M(F, F')}$.*

5. The main results. Solvability of the (SSIE) of the form $(c_0)_{B(r,s)-\lambda I} \subset \mathcal{E} + s_x$

In this section, we solve some (SSIE) with the operator, involving the fine spectrum of $B(r, s)$.

5.1. Some properties of the operator $B(r, s)$

In all that follows, we use the notation $\alpha = -s/r$, where r, $s \neq 0$, belong to $\mathbb{C}$. Then we write $\mathbf{s}_{1/|\alpha|} = \mathbf{s}_{(|r/s|^n)_{n \geq 1}} = \mathbf{s}_{|r/s|}$.

Lemma 9. *Let r, $s \neq 0$ and $\alpha = -s/r$, and let $E \in \{c_0, c, s_1\}$. Then we have:*
(*i*) *The operator $B(r, s) \in (E, E)$, is bijective if and only if $|s| < |r|$,*
(*ii*) *If $|s| > |r|$, then we have $E_{B(r,s)} \subset s_{|\alpha|}$, and $B^{-1}(r, s) \in (E, s_{|\alpha|})$.*

Proof. (*i*) follows from ([17], Corollary 5.3.1, p. 17), see also, ([13], Theorem 3.1, p. 1303). (*ii*) It is well known that the nonzero entries of the triangle $B^{-1}(r, s)$ are determined by $[B^{-1}(r, s)]_{nk} = \alpha^{n-k}/r$, for $k \leq n$, for all n. Then, the entries of the triangle $D_{1/\alpha}B^{-1}(r, s)$ are given by $[D_{1/\alpha}B^{-1}(r, s)]_{nk} = r^{-1}\alpha^{-k}$, for $k \leq n$, for all n. Then we have $|\alpha| > 1$ and

$\left(|\alpha|^{-k}\right)_{k\geq 1} \in \ell_1$. This implies $D_{1/\alpha}B^{-1}(r,s) \in (E, s_1)$, and $E_{B(r,s)} \subset s_{|\alpha|}$, for $E \in \{c_0, c, s_1\}$. ■

5.2. Determination of the multiplier $M\left((c_0)_{B(r,s)-\lambda I}, \mathbf{s}_1\right)$

In this Part, we determine the multiplier $\mathcal{M}_{0,1} = M\left((c_0)_{B(r,s)-\lambda I}, \mathbf{s}_1\right)$ for all $\lambda \in \mathbb{C}$. For this study, the set $\mathbb{C}$ is partitioned into the next three sets, $\sigma_1 = \rho(B(r,s), c_0) \cup \{r\}$, $\sigma_2 = \sigma_c(B(r,s), c_0)$ and $\sigma_3 = \sigma_r(B(r,s), c_0) \setminus \{r\}$. We can state the next result, where we let $\alpha_\lambda = s/(r-\lambda)$ and we write $\mathbf{s}_{1/|\alpha_\lambda|} = \mathbf{s}_{(|(r-\lambda)/s|^n)_{n\geq 1}} = \mathbf{s}_{|(r-\lambda)/s|}$.

Lemma 10. *Let r, $s \neq 0$ and $\lambda \in \mathbb{C}$. Then we have*

$$\mathcal{M}_{0,1} = \begin{cases} \mathbf{s}_1 & \text{if } \lambda \in \sigma_1, \\ \mathbf{s}_{(1/n)_{n\geq 1}} & \text{if } \lambda \in \sigma_2, \\ \mathbf{s}_{1/|\alpha_\lambda|} & \text{if } \lambda \in \sigma_3. \end{cases}$$

Proof. We show that the condition $\lambda \in \sigma_1$ implies $\mathcal{M}_{0,1} = \mathbf{s}_1$. Indeed, if $\lambda = r$, then we have $(c_0)_{B(r,s)-\lambda I} = (c_0)_{B(0,s)}$ and $(c_0)_{B(0,s)} = c_0$ and we conclude $M\left((c_0)_{B(r,s)-rI}, \mathbf{s}_1\right) = \mathbf{s}_1$. Then, if $\lambda \in \rho(B(r,s), c_0)$, by Lemma 9, we have $(c_0)_{B(r-\lambda,s)} = c_0$ and again we have $\mathcal{M}_{0,1} = M(c_0, \mathbf{s}_1) = \mathbf{s}_1$.

Then we let $\lambda \in \sigma_c(B(r,s), c_0)$. Then we have $|\alpha_\lambda| = 1$, and the condition $\gamma \in \mathcal{M}_{0,1}$ is equivalent to $D_\gamma B^{-1}(r-\lambda, s) \in (\mathbf{s}_1, \mathbf{s}_1)$, and to

$$\chi_n = |\gamma_n| \sum_{i=0}^{n-1} |\alpha_\lambda|^i \leq K \text{ for some } K > 0 \text{ and for all } n. \tag{1}$$

The statement in (1), means, $n|\gamma_n| \leq K$ for all n and $\mathcal{M}_{0,1} = \mathbf{s}_{(1/n)_{n\geq 1}}$.

Now we show that the condition $\lambda \in \sigma_3$ implies $\mathcal{M}_{0,1} = \mathbf{s}_{1/|\alpha_\lambda|}$. We have $|\alpha_\lambda| = |s/(r-\lambda)| > 1$, that is, $|r-\lambda| < |s|$, and $\chi_n \sim K'|\gamma_n||\alpha_\lambda|^n$ $(n \to \infty)$ for some $K' > 0$. So the condition in (1), implies there is K'' such that $|\gamma_n| \leq K''|\alpha_\lambda|^{-n}$ for all n, and we have shown that the condition $\lambda \in \sigma_3$ implies $\mathcal{M}_{0,1} = \mathbf{s}_{1/|\alpha_\lambda|}$. This completes the proof. ■

5.3. Application to the solvability of the (SSIE) with an operator $(c_0)_{B(r,s)-\lambda I} \subset \mathcal{E} + \mathbf{s}_x$, for $\lambda \in \mathbb{C}$ and $\mathcal{E} \subset \mathbf{s}_\theta$

In this part, we deal with the set $\mathcal{I}\left(\mathcal{E}, (c_0)_{B(r,s)-\lambda I}, \mathbf{s}_1\right)$ of all positive sequences x that satisfy the (SSIE)

$$(c_0)_{B(r,s)-\lambda I} \subset \mathcal{E} + \mathbf{s}_x, \text{ for } \lambda \in \mathbb{C}. \tag{2}$$

For instance, the condition $x \in \mathcal{I}\left(c, (c_0)_{B(r,s)-\lambda I}, \mathbf{s}_1\right)$ means that for every $y \in \omega$, such that $\lim_{n\to\infty}[(r-\lambda)y_n + sy_{n-1}] = 0$, there are u, $v \in \omega$ with $y = u + v$, such that $\lim_{n\to\infty} u_n = L$ and $|v_n|/x_n \leq K$ for some scalars L and K, with $K > 0$ and for all n.

We need the next lemma.

Lemma 11. *Let a, $b \in U^+$ and let $\widetilde{i_\infty}(a,b)$ be the set of all $b \in U^+$ such that $x_n + a_n \geq Kb_n$ for some $K > 0$ and for all n. If $a/b \in c_0$, then $\widetilde{i_\infty}(a,b) \subset \overline{\mathbf{s}_{1/b}}$.*

Proof. Let $x \in \widetilde{i_\infty}(a,b)$. Then we have $x_n + a_n \geq Kb_n$, and

$$x_n \geq Kb_n - a_n \geq b_n(K - a_n/b_n) \text{ for all } n.$$

Since $a/b \in c_0$, there is an integer N such that for every $n \geq N$, we have $a_n/b_n \leq K/2$ and

$$K - a_n/b_n \geq K/2.$$

If we let $\chi = \min\{x_1/b_1, x_2/b_2, ..., x_{N-1}/b_{N-1}\} > 0$, and $K' = \min\{K/2, \chi\} > 0$, then we have $1/x_n \leq K_1/b_n$ for all n, with $K_1 = 1/K'$, and $x \in \overline{\mathbf{s}_{1/b}}$. So we have shown the inclusion $\widetilde{i_\infty}(a,b) \subset \overline{\mathbf{s}_{1/b}}$. This concludes the proof. ■

To state the following theorem on the solvability of the (SSIE) in (2), with $\mathcal{E} \subset \mathbf{s}_\theta$, and $\theta \in U^+$, we use the set

$$\widetilde{\sigma}_\theta = \left\{\lambda \in \mathbb{C} : \alpha_\lambda \in \overline{\mathbf{s}^0_{1/\theta}}\right\} = \left\{\lambda \in \mathbb{C} : \lim_{n\to\infty} \theta_n \left|\frac{\lambda - r}{s}\right|^n = 0\right\}.$$

Theorem 1. *Let $a \in U^+$, $\lambda \in \mathbb{C}$, r, $s \neq 0$, $\alpha_\lambda = s/(r-\lambda)$, and let $\mathcal{E} \subset \mathbf{s}_\theta$ be a linear space of sequences. Then, the set $\mathcal{I}_0^1(\mathcal{E},\lambda) = \mathcal{I}\big(\mathcal{E}, (c_0)_{B(r,s)-\lambda I}, \mathbf{s}_1\big)$ of all sequences $x \in U^+$, that satisfy the (SSE) in (2), is determined in the following way,*

$$\mathcal{I}_0^1(\mathcal{E},\lambda) = \begin{cases} \overline{\mathbf{s}_1} & \text{if } \lambda \in \sigma_1 \text{ and } \theta \in c_0, \\ \overline{\mathbf{s}_{(1/n)_{n\geq 1}}} & \text{if } \lambda \in \sigma_2 \text{ and } \theta \in \mathbf{s}^0_{(n)_{n\geq 1}}, \\ \overline{\mathbf{s}_{1/|\alpha_\lambda|}} & \text{if } \lambda \in \sigma_3 \cap \widetilde{\sigma}_\theta. \end{cases}$$

Proof. By Part (ii) of Lemma 8, it is enough to show the inclusion $\mathcal{I}_0^1(\mathbf{s}_\theta,\lambda) \subset \overline{\mathcal{M}_{0,1}}$ in each of the cases (a) $\lambda \in \sigma_1$ and $\theta \in c_0$, (b) $\lambda \in \sigma_2$ and $\theta \in \mathbf{s}^0_{(n)_{n\geq 1}}$ and (c) $\lambda \in \sigma_3 \cap \widetilde{\sigma}_\theta$ with $\lambda \neq r$

Case (a). Let $\theta \in c_0$ and $\lambda = r$. We have $(c_0)_{B(r,s)-\lambda I} = (c_0)_{B(0,s)} = c_0$, since $y \in (c_0)_{B(0,s)}$ means that $sy_{n-1} \to 0$ $(n \to \infty)$ and $y \in c_0$. Then we have $c_0 \subset \mathbf{s}_{\theta+x}$ and $1/(\theta+x) \in \mathbf{s}_1$. This implies there is $K>0$ such that $\theta_n + x_n \geq K$, and since $\theta \in c_0$, by Lemma 11, with $a=\theta$ and $b=e$, we obtain $x \in \overline{\mathbf{s}_1}$ and $\mathcal{I}_0^1(\mathbf{s}_\theta, \lambda) \subset \overline{\mathbf{s}_1}$. By Part (ii) of Lemma 8 and Lemma 10, where $\mathcal{M}_{0,1} = \mathbf{s}_1$, we conclude $\mathcal{I}_0^1(\mathbf{s}_\theta, \lambda) = \overline{\mathbf{s}_1}$.

Now, let $\lambda \in \rho(B(r,s), c_0)$ and $\theta \in c_0$. This implies $|r-\lambda| > |s|$, and by Part (i) of Lemma 9, we have $(c_0)_{B(r,s)-\lambda I} = c_0$. Then $c_0 \subset \mathbf{s}_{\theta+x}$ and $1/(\theta+x) \in \mathbf{s}_1$. As we have just seen, the condition $\theta \in c_0$ implies there is $K>0$ such that $x_n \geq K > 0$ for some $K>0$ and for all n, and $\mathcal{I}_0^1(\mathbf{s}_\theta, \lambda) \subset \overline{\mathbf{s}_1}$. As above, by Part (ii) of Lemma 8 and Lemma 10, where $\mathcal{M}_{0,1} = \mathbf{s}_1$, we conclude $\mathcal{I}_0^1(\mathbf{s}_\theta, \lambda) = \overline{\mathbf{s}_1}$.

Now we consider the cases (b) and (c). For this study, notice that $\mathbf{s}_\theta + \mathbf{s}_x = \mathbf{s}_{\theta+x}$, (cf. [21], Remark 4.1, p. 162), and the inclusion $(c_0)_{B(r,s)-\lambda I} \subset \mathbf{s}_\theta + \mathbf{s}_x$ is equivalent to $(c_0)_{B(r,s)-\lambda I} \subset \mathbf{s}_{\theta+x}$ and to $1/(\theta+x) \in \mathcal{M}_{0,1}$.

Case (b). We have $1/(\theta+x) \in \mathcal{M}_{0,1} = \mathbf{s}_{(1/n)_{n\geq1}}$, and $n/(\theta_n+x_n) \leq K$ and $\theta_n + x_n \geq K^{-1}n$ for some $K>0$ and for all n. Since $\theta \in \mathbf{s}^0_{(n)_{n\geq1}}$, we can apply Lemma 11, with $a=\theta$ and $b=(n)_{n\geq1}$ and we conclude $1/x \in \mathbf{s}_{(1/n)_{n\geq1}}$. So we have shown that, under the conditions $\theta \in \mathbf{s}^0_{(n)_{n\geq1}}$ and $\lambda \in \sigma_2$, the inclusion $\mathcal{I}_0^1(\mathbf{s}_\theta, \lambda) \subset \overline{\mathbf{s}_{(1/n)_{n\geq1}}}$ holds. Then, by Lemma 10, the condition $\lambda \in \sigma_2$ implies $\mathcal{M}_{0,1} = \mathbf{s}_{(1/n)_{n\geq1}}$ and again, by Part (ii) of Lemma 8, we conclude $\mathcal{I}_0^1(\mathbf{s}_\theta, \lambda) = \overline{\mathbf{s}_{(1/n)_{n\geq1}}}$.

Case (c). Here we have $1/(\theta+x) \in \mathcal{M}_{0,1} = \mathbf{s}_{1/|\alpha_\lambda|}$, and $\theta_n + x_n \geq K|\alpha_\lambda|^n$ for some $K>0$ and for all n. Then, using the condition $\lambda \in \widetilde{\sigma}_\theta$, and applying Lemma 11, with $a=\theta$ and $b=(|\alpha_\lambda|^n)_{n\geq1}$, and conclude $1/x \in \mathbf{s}_{1/|\alpha_\lambda|}$, and we have shown the inclusion $\mathcal{I}_0^1(\mathbf{s}_\theta, \lambda) \subset \overline{\mathbf{s}_{1/|\alpha_\lambda|}}$. By Lemma 10, we have $\mathcal{M}_{0,1} = \mathbf{s}_{1/|\alpha_\lambda|}$ for $\lambda \in \sigma_3$, and again, by Part (ii) of Lemma 8, we conclude $\mathcal{I}_0^1(\mathbf{s}_\theta, \lambda) = \overline{\mathbf{s}_{1/|\alpha_\lambda|}}$. This completes the proof. ■

As a direct consequence of Theorem 1, we obtain the next corollary.

Corollary 1. *Let $\theta \in U^+$, $\lambda \in \mathbb{C}$, r, $s \neq 0$, and let $\mathcal{E} \subset \mathbf{s}_\theta$ be a linear space of sequences. Then,the identity*

$$\mathcal{I}_0^1(\mathcal{E}, \lambda) = \overline{\mathbf{s}_{1/|\alpha_\lambda|}} \tag{3}$$

holds in each of the cases: (i) $1/\theta \in \ell_\infty$, and $\lambda \in \widetilde{\sigma}_\theta \setminus \{r\}$, and (ii) $\theta \in \ell_\infty$, and $\lambda \in \sigma_r(B(r,s), c_0) \setminus \{r\}$.

Proof. (i) Assume $1/\theta \in \ell_\infty$. Then we have $\theta_n |(\lambda-r)/s|^n \geq K\big|(\lambda-r)/s\big|^n$, for some $K>0$ and for all n, and since $\lambda \in \widetilde{\sigma}_\theta$, we conclude $|(\lambda-r)/s| < 1$ and $\lambda \in \sigma_r(B(r,s), c_0)$. So we have

$$\sigma_r\left(B\left(r,s\right),c_0\right)\cap\widetilde{\sigma}_\theta=\widetilde{\sigma}_\theta,$$

and by Theorem 1, the condition $\lambda\in\sigma_r\left(B\left(r,s\right),c_0\right)\setminus\{r\}$ implies the identity in (3).

(ii) If $\theta\in\ell_\infty$, then the condition $\left|\left(\lambda-r\right)/s\right|<1$ implies $\lim_{n\to\infty}\theta_n\left|\left(\lambda-r\right)/s\right|^n=0$ and

$$\sigma_r\left(B\left(r,s\right),c_0\right)\cap\widetilde{\sigma}_\theta=\sigma_r\left(B\left(r,s\right),c_0\right),$$

and again by Theorem 1, the condition $\lambda\in\sigma_r\left(B\left(r,s\right),c_0\right)$ implies (3). This concludes the proof. ∎

5.4. Application to the solvability of the (SSIE) with operator $(c_0)_{B(r,s)-\lambda I}\subset\mathbf{s}_R^{(c)}+\mathbf{s}_x$

In this part, we solve the (SSIE) $(c_0)_{B(r,s)-\lambda I}\subset\mathbf{s}_R^{(c)}+\mathbf{s}_x$ for all $R>0$ and all $\lambda\in\mathbb{C}$. We have $D_{1/R}B\left(r,s\right)D_R=B\left(r,s/R\right)$, and the condition $B\left(r,s\right)\in\left(\mathbf{s}_R^0,\mathbf{s}_R^0\right)$ holds if and only if $B\left(r,s/R\right)\in\left(c_0,c_0\right)$. So we have

$$\varkappa\left(B\left(r,s\right),\mathbf{s}_R^0\right)=\varkappa\left(B\left(r,s/R\right),c_0\right),$$

where $\varkappa$ is any of the symbols σ_r, σ_c, and ρ. In the following result, we solve the (SSIE) $(c_0)_{B(r,s)-\lambda I}\subset\mathbf{s}_R^{(c)}+\mathbf{s}_x$, for all $\lambda\in\mathbb{C}$ and $R>0$, except, for $\lambda\in\sigma_c\left(B\left(r,s\right),\mathbf{s}_R^0\right)$ with $R>1$. More precisely, for $R\leq1$, we consider the cases, (1) $\left|r-\lambda\right|>\left|s\right|$, or $\lambda=r$, (2) $\left|r-\lambda\right|=\left|s\right|$ and (3) $\left|r-\lambda\right|<\left|s\right|$ and $\lambda\neq r$. Then, for $R>1$ and inside this condition, we deal with the cases, (1) $\left|r-\lambda\right|>\left|s\right|/R$, or $\lambda=r$ and (2) $\left|r-\lambda\right|<\left|s\right|/R$ and $\lambda\neq r$.

We can state the next result.

Corollary 2. *Let $R>0$, r, $s\neq0$. The set $\mathcal{I}_0^1\left(\mathbf{s}_R^{(c)},\lambda\right)$ of all the solutions of the (SSIE) $(c_0)_{B(r,s)-\lambda I}\subset\mathbf{s}_R^{(c)}+\mathbf{s}_x$, where $\lambda\in\mathbb{C}$,is determined in the following way.*

(i) Let $R<1$. Then

$$\mathcal{I}_0^1\left(\mathbf{s}_R^{(c)},\lambda\right)=\begin{cases}\overline{\mathbf{s}_1} & \text{if} \quad \lambda\in\sigma_1,\\ \overline{\mathbf{s}_{(1/n)_{n\geq1}}} & \text{if} \quad \lambda\in\sigma_2,\\ \overline{\mathbf{s}_{\left|\frac{r-\lambda}{s}\right|}} & \text{if} \quad \lambda\in\sigma_3.\end{cases}$$

(ii) Let $R=1$. Then

$$\mathcal{I}_0^1\left(\mathbf{s}_R^{(c)},\lambda\right)=\begin{cases}U^+ & \text{if} \quad \lambda\in\sigma_1,\\ \overline{\mathbf{s}_{(1/n)_{n\geq1}}} & \text{if} \quad \lambda\in\sigma_2,\\ \overline{\mathbf{s}_{\left|\frac{r-\lambda}{s}\right|}} & \text{if} \quad \lambda\in\sigma_3.\end{cases}$$

(iii) Let $R > 1$. Then

$$\mathcal{I}_0^1\left(\mathbf{s}_R^{(c)}, \lambda\right) = \begin{cases} U^+ & \text{if } \lambda \in \rho\left(B(r,s), \mathbf{s}_R^0\right) \cup \{r\}, \\ \overline{\mathbf{s}_{\left|\frac{r-\lambda}{s}\right|}} & \text{if } \lambda \in \sigma_r\left(B(r,s), \mathbf{s}_R^0\right) \setminus \{r\}. \end{cases}$$

Proof. We apply Theorem 1, with $\theta = (R^n)_{n\geq 1}$, and we write $\widetilde{\sigma}_\theta = \widetilde{\sigma}_R$.

Case $R < 1$. By Theorem 1, since $(R^n)_{n\geq 1} \in c_0$, the condition $\lambda \in \sigma_1$ implies $\mathcal{I}_0^1\left(\mathbf{s}_R^{(c)}, \lambda\right) = \overline{\mathbf{s}_1}$. In the same way, by Theorem 1, since $(R^n)_{n\geq 1} \in \mathbf{s}^0_{(n)_{n\geq 1}}$, the condition $\lambda \in \sigma_2$, implies $\mathcal{I}_0^1\left(\mathbf{s}_R^{(c)}, \lambda\right) = \overline{\mathbf{s}_{(1/n)_{n\geq 1}}}$. Then, by Part (ii) of Corollary 1, where $(R^n)_{n\geq 1} \in \ell_\infty$, the condition $\lambda \in \sigma_3$ implies $\mathcal{I}_0^1\left(\mathbf{s}_R^{(c)}, \lambda\right) = \overline{\mathbf{s}_{1/|\alpha_\lambda|}}$. So we have shown Part (i).

Case $R = 1$. If $\lambda \in \rho(B(r,s), c_0)$, it can easily be seen that since $|\alpha_\lambda| < 1$ we have $B^{-1}(r-\lambda, s) \in (c_0, c)$ and $(c_0)_{B(r,s)-\lambda I} \subset c$. This implies $\mathcal{I}_0^1(c, \lambda) = U^+$ for all $\lambda \in \rho(B(r,s), c_0)$. The remainder of the proof follows from Theorem 1.

Case $R > 1$. First, for $\lambda = r$ we have seen that $(c_0)_{B(r,s)-\lambda I} = c_0 \subset \mathbf{s}_R^{(c)}$ and $\mathcal{I}_0^1\left(\mathbf{s}_R^{(c)}, r\right) = U^+$. Then, the inclusion $(c_0)_{B(r,s)-\lambda I} \subset \mathbf{s}_R^{(c)}$ holds if and only if $D_{1/R}B^{-1}(r-\lambda, s) \in (c_0, c)$. By the characterization of (c_0, c), (cf. [21], Theorem 1.23, p. 23), we obtain

$$\frac{1}{R^n}\sum_{i=0}^{n-1} |\alpha_\lambda|^i \leq K, \text{ for some } K > 0 \text{ and for all } n, \tag{4}$$

and

$$\lim_{n\to\infty}\left(\frac{\alpha_\lambda}{R}\right)^n = L \text{ for some scalar } L. \tag{5}$$

We can see that the conditions in (4), and (5) hold for $|\alpha_\lambda| < R$. Indeed, if $1 < |\alpha_\lambda| < R$, then we have $\tau_n = \sum_{i=0}^{n-1} |\alpha_\lambda|^i \sim |\alpha_\lambda|^n / (|\alpha_\lambda| - 1)$ $(n \to \infty)$ which implies the condition in (4), and we have $L = 0$ in (5). Then, the conditions in (4), and (5) hold, if $|\alpha_\lambda| \leq 1$, since the inequality $|\alpha_\lambda| < 1$, implies $\tau_n = O(1)$ $(n \to \infty)$, and the identity $|\alpha_\lambda| = 1$, implies $\tau_n = nO(1)$ $(n \to \infty)$. We conclude the inclusion $(c_0)_{B(r,s)-\lambda I} \subset \mathbf{s}_R^{(c)}$ holds and $\mathcal{I}_0^1\left(\mathbf{s}_R^{(c)}, \lambda\right) = U^+$, if $|r-\lambda| > |s|/R$.

Then, assume $|r-\lambda| < |s|/R$ and $\lambda \neq r$. Then we have $\widetilde{\sigma}_R = D(r, |s|/R)$, and we conclude by Part (i) of Corollary 1, with $\theta = (R^n)_{n\geq 1}$, that $\mathcal{I}_0^1\left(\mathbf{s}_R^{(c)}, \lambda\right) = \overline{\mathbf{s}_{|(r-\lambda)/s|}}$. This completes the proof. ∎

Remark 1. In the case $R > 1$, with r, $s \neq 0$, using the same arguments as in Corollary 2, the set of the solutions of the (SSIE) $(c_0)_{B(r,s)-\lambda I} \subset \mathbf{s}_R + \mathbf{s}_x$, is determined by,

$$\mathcal{I}_0^1\left(\mathbf{s}_R^{(c)}, \lambda\right) = \begin{cases} U^+ & \text{if} \quad |r-\lambda| \geq |s|/R, \text{ or } \lambda = r, \\ \overline{\mathbf{s}_{\left|\frac{r-\lambda}{s}\right|}} & \text{if} \quad |r-\lambda| < |s|/R, \text{ and } \lambda \neq r. \end{cases}$$

5.5. On the (SSIE) $(c_0)_{\Delta-\lambda I} \subset bv_p + \mathbf{s}_x$, and $(c_0)_{B(r,s)-\lambda I} \subset E_{R_a} + \mathbf{s}_x$, for $E = c_0$, c, or ℓ_∞

In this part, we solve each of the (SSIE) $(c_0)_{\Delta-\lambda I} \subset bv_p + \mathbf{s}_x$, with $\lambda \in \sigma(\Delta, c_0) \setminus \{1\}$, and $(c_0)_{B(r,s)-\lambda I} \subset E_{R_a} + \mathbf{s}_x$, for $E = c_0$, c, or ℓ_∞, where R_a, $a \in U^+$, is the Rhaly matrix.

5.5.1. Solvability of the (SSIE) $(c_0)_{\Delta-\lambda I} \subset bv_p + \mathbf{s}_x$

In the following, we use the set ℓ^p, for $p \geq 1$ of all sequences $x = (x_k)_{k\geq 1}$ such that $\sum_{k=1}^\infty |x_k|^p < \infty$, and the set $bv_p = \ell^p_\Delta$, $(1 \leq p \leq \infty)$ of sequences of p-bounded variation introduced by Başar and Altay [5], if $p = 1$ we write $bv_1 = bv$. We can state the next result.

Proposition 1. *Let $p \geq 1$. The set of all the positive sequences $x = (x_k)_{k\geq 1}$, that satisfy the (SSIE) $(c_0)_{\Delta-\lambda I} \subset bv_p + \mathbf{s}_x$, where $\lambda \in \sigma(\Delta, c_0) \setminus \{1\}$, is determined by*

$$\mathcal{I}_0^1(bv_p, \lambda) = \begin{cases} \overline{\mathbf{s}_{(1/n)_{n\geq 1}}} & \text{if} \quad \lambda \in \sigma_c(\Delta, c_0), \\ \overline{\mathbf{s}_{|\lambda-1|}} & \text{if} \quad \lambda \in \sigma_r(\Delta, c_0) \setminus \{1\}. \end{cases}$$

Proof. *Case* $p > 1$. The condition $bv_p \subset \mathbf{s}_\theta$, is equivalent to

$$D_{1/\theta}\Sigma \in (\ell^p, \mathbf{s}_1),$$

and by the characterization of $(\ell^p, \mathbf{s}_1)$, (cf. [21], Theorem 1.23, p. 23), this condition is equivalent to $n\theta_n^{-q} \leq K$, for some $K > 0$ and for all n, where $q = p/(p-1)$. So we can apply Theorem 1, with $\mathcal{E} = bv_p$, and since $\theta = \left(n^{1/q}\right)_{n\geq 1} \in \mathbf{s}^0_{(n)_{n\geq 1}}$, the condition $\lambda \in \sigma_c(\Delta, c_0)$, implies $\mathcal{I}_0^1(bv_p, \lambda) = \overline{\mathbf{s}_{(1/n)_{n\geq 1}}}$. In a similar way, the condition $\lim_{n\to\infty} n^{1/q}|1-\lambda|^n = 0$ holds if and only if $|1-\lambda| < 1$. This implies the identity $\sigma_r(\Delta, c_0) = \widetilde{\sigma}_\theta$, and by Theorem 1, with $r = -s = 1$, the condition $\lambda \in \sigma_r(\Delta, c_0) \setminus \{1\}$ implies $\mathcal{I}_0^1(bv_p, \lambda) = \overline{\mathbf{s}_{|1-\lambda|}}$. So we have shown the proposition for $p > 1$.

Case $p = 1$. We have $\ell_{\Delta}^{1} \subset c$, and for $\lambda \in \sigma(\Delta, c_0) \setminus \{1\}$, we can apply Part (*ii*) of Lemma 8, which implies $\mathcal{I}_0^1(bv, \lambda) = \mathcal{I}_0^1(c, \lambda)$, where $\mathcal{I}_0^1(c, \lambda)$ is defined in Part (*ii*) of Corollary 2, with $r = -s = 1$. This concludes the proof. ■

5.5.2. Solvability of the (SSIE) $(c_0)_{B(r,s)-\lambda I} \subset E_{R_a} + \mathbf{s}_x$

Let $a \in U^+$, and for $\lambda \in \mathbb{C}$, we write $\mathcal{I}(E_{R_a}, \lambda)$ for the set of all positive sequences x, that satisfy the (SSIE) $(c_0)_{B(r,s)-\lambda I} \subset E_{R_a} + \mathbf{s}_x$, for $E = c_0$, c, or ℓ_∞, where R_a, $a \in U^+$, is the Rhaly matrix, (cf. [24], [28]). The nonzero entries of R_a are defined by $[R_a]_{nk} = a_n$, for $k \leq n$ and for all n. We have $R_a = D_a \Sigma$, where Σ is the sum operator defined by $\Sigma_n y = \sum_{k=1}^{n} y_k$. Then, the condition $x \in \mathcal{I}(c_{R_a}, \lambda)$ means that for every $y \in \omega$, such that $\lim_{n\to\infty} [(r - \lambda) y_n + s y_{n-1}] = 0$, there are u, $v \in \omega$ with $y = u + v$, such that $\lim_{n\to\infty} a_n \left(\sum_{k=1}^{n} u_k\right) = L$, and $|v_n| / x_n \leq K$, for some scalars L and K, with $K > 0$ and for all n. By similar arguments as above, we can state the following result, whose the proof follows from Theorem 1 and Corollary 1.

Corollary 3. *Let $a \in U^+$ and let $E \in \{c_0, c, \ell_\infty\}$. Then we have,*

(*i*) *If $1/a \in c_0$, then the condition $\lambda \notin \sigma(B(r,s), c_0)$ implies $\mathcal{I}(E_{R_a}, \lambda) = \overline{\mathbf{s}_1}$.*

(*ii*) *If $\lim_{n\to\infty} (1/na_n) = 0$, then the condition $\lambda \in \sigma_c(B(r,s), c_0)$ implies $\mathcal{I}(E_{R_a}, \lambda) = \overline{\mathbf{s}_{(1/n)_{n\geq 1}}}$.*

(*iii*) *If $a \in \mathbf{s}_{(1/n)_{n\geq 1}}$ and $\lambda = r$, then we have $\mathcal{I}(E_{R_a}, \lambda) = U^+$.*

(*iv*) (*a*) *Let $1/a \in \ell_\infty$. Then, the condition $\lambda \in \sigma_r(B(r,s), c_0) \setminus \{r\}$ implies $\mathcal{I}(E_{R_a}, \lambda) = \overline{\mathbf{s}_{|(r-\lambda)/s|}}$,*

(*b*) *Let $a \in \ell_\infty$. Then, the condition $\lambda \in \widetilde{\sigma}_{1/a} \setminus \{r\}$ implies $\mathcal{I}(E_{R_a}, \lambda) = \overline{\mathbf{s}_{|(r-\lambda)/s|}}$.*

If $a = e$ in Corollary 3, then we obtain the next application.

Example 1. For every $\lambda \in \sigma(B(r,s), c_0)$, the set $\mathcal{I}(E_\Sigma, \lambda)$ of all positive sequences x, that satisfy the (SSIE) $(c_0)_{B(r,s)-\lambda I} \subset E_\Sigma + \mathbf{s}_x$, for $E = c_0$, c, or ℓ_∞, is determined by
$$\mathcal{I}(E_\Sigma, \lambda) = \begin{cases} \overline{\mathbf{s}_{(1/n)_{n\geq 1}}} & \text{if } |r - \lambda| = |s|, \\ \overline{\mathbf{s}_{|(r-\lambda)/s|}} & \text{if } |r - \lambda| < |s| \text{ and } \lambda \neq r. \end{cases}$$

These results lead to the next conclusion.

Conclusion. In all this article, we only have considered the solvability of the (SSIE) $(c_0)_{B(r,s)-\lambda I} \subset \mathcal{E} + \mathbf{s}_x$, for positive sequences. If $\mathcal{E} = \{\theta\}$, where θ is the zero sequence, then we may solve the (SSIE) $D_x * E_{B(r,s)-\lambda I} \subset F$, where E and F are linear spaces of sequences, for all $x \in \omega$, since this resolution consists in determining the set $M\left(E_{B(r,s)-\lambda I}, F\right)$. In future, we may extend the results stated above to the study of the (SSIE) of the form $E_{B(r,s)-\lambda I} \subset \mathcal{E} + F_x$, where E and F are any of the sets c_0, c, or ℓ_∞. For instance, the solvability of the (SSIE) $c_{B(r,s)-\lambda I} \subset \mathcal{E} + \mathbf{s}_x^{(c)}$, is associated with the multiplier $M\left(c_{B(r,s)-\lambda I}, c\right)$, and the solutions of this (SSIE) may be stated using the continuous and residual spectra of the generalized difference operator $B(r,s)$ on c. Then we may also extend these results, to the solvability of the (SSIE) $E_{B(r,s,t)-\lambda I} \subset \mathcal{E}+F_x$, where r, s, $t \neq 0$, and E and F are any of the sets c_0, c, or ℓ_∞, (cf. [9]). The solutions of these (SSIE) can be stated, using the continuous and residual spectra of the infinite tridiagonal matrix $B(r,s,t)$ on E, (cf. [11]).

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BRUNO DE MALAFOSSE
UNIVERSITÉ DU HAVRE 76600
e-mail: bdemalaf@wanadoo.fr

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