F A S C I C U L I   M A T H E M A T I C I

Nr 66 2023

DOI: 10.21008/j.0044-4413.2023.0008

Gagandeep Singh and Gurcharanjit Singh

ON MULTIVALENT CLOSE-TO-STAR FUNCTIONS

Abstract. The present paper deals with certain generalized subclasses of multivalent close-to-star functions defined with subordination. Various properties of these classes such as the coefficient estimates, growth theorems, argument theorems and inclusion relations are studied. Some earlier known results will follow as particular cases.

Key words: subordination, Univalent functions, analytic functions, multivalent functions, close-to-star functions, close-to-convex functions.

AMS Mathematics Subject Classification: 30C45, 30C50.

1. Introduction

Let the unit disc is defined as $E = \{z : z \in \mathbb{C}, |z| < 1\}$ where $\mathbb{C}$ denotes the complex plane. $\mathcal{A}_p (p \geq 1)$ denotes the class of analytic functions f in the unit disc E and of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k. \tag{1}$$

In particular, $\mathcal{A}_p \equiv \mathcal{A}_1$, the class of analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and which are normalized by the conditions $f(0) = f'(0) - 1 = 0$. By $\mathcal{S}$ we denote the class of functions in $\mathcal{A}_1$ which are univalent in E.

Let $\mathcal{U}$ denotes the class of analytic functions in E which can be expressed as

$$w(z) = \sum_{k=1}^{\infty} c_k z^k,$$

and with the conditions $w(0) = 0$, $|w(z)| < 1$. The functions in the class $\mathcal{U}$ are known as Schwarz functions. It was proved in [15] that for $w \in \mathcal{U}$, $|c_1| \leq 1$ and $|c_2| \leq 1 - |c_1|^2$.

For two analytic functions f and g in E, f is said to be subordinate to g if there exists a Schwarz function $w \in \mathcal{U}$ such that $f(z) = g(w(z))$ and symbolically it is written as $f \prec g$. Further, if the function g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

By $\mathcal{S}_p^*(\alpha)$ and $\mathcal{K}_p(\alpha)$ $(0 \leq \alpha < p)$, we denote the subclasses of $\mathcal{A}_p$ which are respectively the classes of p-valently starlike functions and p-valently convex functions of order α and defined as

$$\mathcal{S}_p^*(\alpha) = \left\{ f : f \in \mathcal{A}_p, Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in E \right\}$$

and

$$\mathcal{K}_p(\alpha) = \left\{ f : f \in \mathcal{A}_p, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > \alpha, z \in E \right\}.$$

The classes $\mathcal{S}_p^*(\alpha)$ and $\mathcal{K}_p(\alpha)$ were investigated by Goluzin [5]. It can be easily seen that $f \in \mathcal{K}_p(\alpha)$ if and only if $\dfrac{zf'}{p} \in \mathcal{S}_p^*(\alpha)$. For $0 \leq \alpha < 1$, $\mathcal{S}_1^*(\alpha) \equiv \mathcal{S}^*(\alpha)$ and $\mathcal{K}_1(\alpha) \equiv \mathcal{K}(\alpha)$, the classes of starlike functions of order α and convex functions of order α respectively, introduced by Robertson [19]. Also $\mathcal{S}_p^*(0) \equiv \mathcal{S}_p^*$ and $\mathcal{K}_p(0) \equiv \mathcal{K}_p$, the classes of p-valent starlike functions and p-valent convex functions respectively. Further $\mathcal{S}_1^*(0) \equiv \mathcal{S}^*$ and $\mathcal{K}_1(0) \equiv \mathcal{K}$, the well known classes of starlike functions and convex functions respectively

Umezawa [21] established the class $\mathcal{C}_p(\alpha)$ of p-valent close-to-convex functions defined as

$$\mathcal{C}_p(\alpha) = \left\{ f : f \in \mathcal{A}_p, Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha, g \in \mathcal{S}_p^*, z \in E \right\}.$$

For $p = 1$, $\alpha = 0$, the class $\mathcal{C}_p(\alpha)$ reduces to $\mathcal{C}$, the class of close-to-convex functions introduced by Kaplan [8].

Reade [18] introduced the class $\mathcal{CS}^*$ of close-to-star functions defined as

$$\mathcal{CS}^* = \left\{ f : f \in \mathcal{A}_1, Re\left(\frac{f(z)}{g(z)}\right) > 0, g \in \mathcal{S}^*, z \in E \right\}.$$

The class of close-to-star functions has the same relation with the class of close-to-convex functions as the class of starlike functions bear to the class of convex functions.

The corresponding class of p-valent close-to-star functions is denoted by $\mathcal{CS}^*(p)$ and defined as

$$\mathcal{CS}^*(p) = \left\{ f : f \in \mathcal{A}_p, Re\left(\frac{pf(z)}{g(z)}\right) > 0, g \in \mathcal{S}_p^*, z \in E \right\}.$$

For $p = 1$, $\mathcal{CS}^*(p)$ agrees with the class $\mathcal{CS}^*$. Various subclasses of close-to-star functions were studied in [3, 10, 12, 13, 14, 16].

For $-1 \leq B < A \leq 1$ and $0 \leq \alpha < p$, Aouf [1] established the class $\mathcal{P}(A, B; p; \alpha)$ which consists of the functions of the form $p(z) = p + \sum_{k=1}^{\infty} p_k z^k$ such that $p(z) \prec \frac{p + [pB + (A-B)(p-\alpha)]z}{1 + Bz}$. In particular,

(i) $\mathcal{P}(A, B; 1; \alpha) \equiv \mathcal{P}(A, B; \alpha)$, the class introduced by Polatoglu et al. [17].

(ii) $\mathcal{P}(A, B; 1; 0) \equiv \mathcal{P}(A, B)$, a subclass of $\mathcal{A}_1$ introduced by Janowski [7].

Further, for $-1 \leq B < A \leq 1$ and $0 \leq \alpha < p$, Aouf [1, 2], introduced the following useful classes:

$$\mathcal{S}^*(A, B; p; \alpha) = \left\{ f : f \in \mathcal{A}_p, \frac{zf'(z)}{f(z)} \prec \frac{p + [pB + (A-B)(p-\alpha)]z}{1 + Bz}, z \in E \right\}$$

and

$$\mathcal{K}(A, B; p; \alpha) = \left\{ f : f \in \mathcal{A}_p, \frac{(zf'(z))'}{f'(z)} \prec \frac{p + [pB + (A-B)(p-\alpha)]z}{1 + Bz}, z \in E \right\}.$$

The following observations are obvious:

(i) $\mathcal{S}^*(1, -1; p; \alpha) \equiv \mathcal{S}_p^*(\alpha)$ and $\mathcal{K}(1, -1; p; \alpha) \equiv \mathcal{K}_p(\alpha)$.

(ii) $\mathcal{S}^*(A, B; p; 0) \equiv \mathcal{S}_p^*(A, B)$ and $\mathcal{K}(A, B; p; 0) \equiv \mathcal{K}_p(A, B)$, the classes studied by Hayami and Owa [6].

(iii) $\mathcal{S}^*(A, B; 1; \alpha) \equiv \mathcal{S}^*(A, B; \alpha)$, the class studied by Polatoglu et al. [17].

(iv) $\mathcal{S}^*(A, B; 1; 0) \equiv \mathcal{S}^*(A, B)$ and $\mathcal{K}(A, B; 1; 0) \equiv \mathcal{K}(A, B)$, the subclasses of starlike and convex functions respectively, introduced by Janowski [7] and studied further by Goel and Mehrok [4].

(v) $\mathcal{S}^*(1, -1; 1; \alpha) \equiv \mathcal{S}^*(\alpha)$ and $\mathcal{K}(1, -1; 1; \alpha) \equiv \mathcal{K}(\alpha)$.

(vi) $\mathcal{S}^*(1, -1; 1; 0) \equiv \mathcal{S}^*$ and $\mathcal{K}(1, -1; 1; 0) \equiv \mathcal{K}$.

Throughout this paper, we assume that $-1 \leq D < C \leq 1$, $-1 \leq B < A \leq 1$, $0 \leq \alpha < p$, $0 \leq \beta < p$ and $z \in E$.

Getting motivated by the above work, we take into account the following definitions:

Definition 1. *Let $\mathcal{CS}^*(A, B; C, D; p; \beta; \alpha)$ denote the class of functions $f \in \mathcal{A}_p$ and satisfying the condition*

$$\frac{pf(z)}{g(z)} \prec \frac{p + [pD + (C-D)(p-\beta)]z}{1 + Dz},$$

where

$$g(z) = z^p + \sum_{k=p+1}^{\infty} d_k z^k \in \mathcal{S}^*(A, B; p; \alpha).$$

Definition 2. *$\mathcal{CS}_1^*(A, B; C, D; p; \beta; \alpha)$ is the class of functions $f \in \mathcal{A}_p$ which satisfy the condition*

$$\frac{pf(z)}{h(z)} \prec \frac{p + [pD + (C-D)(p-\beta)]z}{1 + Dz},$$

where

$$h(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \in \mathcal{K}(A, B; p; \alpha).$$

The following points are to be noted:

(i) $\mathcal{CS}^*(A, B; C, D; p; 0; 0) \equiv \mathcal{CS}^*(A, B; C, D; p)$.

(ii) $\mathcal{CS}^*(A, B; C, D; 1; 0; 0) \equiv \mathcal{CS}^*(A, B; C, D)$, *the subclass of close-to-star functions investigated by Mehrok and Singh [12].*

(iii) $\mathcal{CS}^*(1, -1; C, D; 1; 0; 0) \equiv \mathcal{CS}^*(C, D)$, *the subclass of close-to-star functions studied by Mehrok et al. [13].*

(iv) $\mathcal{CS}^*(1, -1; 1, -1; 1; 0; 0) \equiv \mathcal{CS}^*$.

(v) $\mathcal{CS}_1^*(A, B; C, D; p; 0; 0) \equiv \mathcal{CS}_1^*(A, B; C, D; p)$.

(vi) $\mathcal{CS}_1^*(1, -1; C, D; 1; 0; 0) \equiv \mathcal{CS}_1^*(C, D)$, *the subclass of close-to-star functions studied by Mehrok et al. [14].*

In this paper, we investigate various properties such as the coefficient estimates, growth theorems, argument theorems and inclusion relations for the classes $\mathcal{CS}^*(A, B; C, D; p; \beta; \alpha)$ and $\mathcal{CS}_1^*(A, B; C, D; p; \beta; \alpha)$. The results already proved by various authors, will follow as special cases.

2. Preliminary results

Lemma 1 ([1]). *If* $P(z) = \dfrac{p + [pD + (C - D)(p - \beta)]w(z)}{1 + Dw(z)} = p + \sum_{k=1}^{\infty} p_k z^k \in \mathcal{P}(C, D; p; \beta)$, *then*

$$|p_n| \leq (C - D)(p - \beta), \quad n \geq p.$$

The bounds are sharp for $w(z) = z^n$ *and for the function*

$$P(z) = p + (C - D)(p - \beta)z^n + \dots$$

Lemma 2 ([11]). *Let* $-1 \leq D_2 \leq D_1 < C_1 \leq C_2 \leq 1$, *then*

$$\frac{1 + C_1 z}{1 + D_1 z} \prec \frac{1 + C_2 z}{1 + D_2 z}.$$

Lemma 3 ([20]). *If* $\psi(z)$ *is regular in* E, $\phi(z)$ *and* $h(z)$ *are convex univalent in* E *such that* $\psi(z) \prec \phi(z)$, *then* $\psi(z) * h(z) \prec \phi(z) * h(z)$, $z \in E$.

Lemma 4 ([1]). *For* $g(z) = z^p + \sum_{k=p+1}^{\infty} d_k z^k \in \mathcal{S}^*(A, B; p; \alpha)$,

$$|d_n| \leq \Pi_{j=0}^{n-(p+1)} \frac{|(B - A)(p - \alpha) + Bj|}{j + 1}, \quad n \geq 1.$$

Lemma 5 ([1]). *Let $g \in \mathcal{S}^*(A,B;p;\alpha)$, then for $|z| = r$, $0 < r < 1$, we have*

$$r^p(1-Br)^{\frac{(A-B)(p-\alpha)}{B}} \le |g(z)| \le r^p(1+Br)^{\frac{(A-B)(p-\alpha)}{B}}, \ \ B \neq 0,$$
$$r^p e^{-A(p-\alpha)r} \le |g(z)| \le r^p e^{A(p-\alpha)r}, \ \ B = 0.$$

Lemma 6 ([1]). *For $g \in \mathcal{S}^*(A,B;p;\alpha)$,*

$$\left| arg \frac{g(z)}{z^p} \right| \le \begin{cases} \frac{(A-B)(p-\alpha)}{B} sin^{-1}(Br) & \text{if } B \neq 0, \\ A(p-\alpha)r & \text{if } B = 0. \end{cases}$$

Lemma 7 ([2]). *For $h(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \in \mathcal{K}(A,B;p;\alpha)$,*

$$|b_n| \le \frac{p}{n[(n-p)!]} \Pi_{j=0}^{n-(p+1)} |(B-A)(p-\alpha) + Bj|, \ \ n \ge p+1.$$

Lemma 8 ([2]). *Let $h \in \mathcal{K}(A,B;p;\alpha)$, then for $|z| = r$, $0 < r < 1$, we have*

$$p\int_0^r t^{p-1}(1-Bt)^{\frac{(A-B)(p-\alpha)}{B}} dt \le |h(z)| \le p\int_0^r t^{p-1}(1+Bt)^{\frac{(A-B)(p-\alpha)}{B}} dt, \ B \neq 0,$$
$$p\int_0^r t^{p-1} e^{-A(p-\alpha)t} dt \le |h(z)| \le p\int_0^r t^{p-1} e^{A(p-\alpha)t} dt, \ B = 0,$$

Lemma 9. *If $g(z) = z^p + \sum_{k=p+1}^{\infty} d_k z^k \in \mathcal{S}^*(A,B;p;\alpha)$, then*

$$|d_{p+1}| \le \frac{(A-B)(p-\alpha)}{p+1}$$

and

$$|d_{p+2}| \le \frac{(A-B)(p-\alpha)}{2} \max\left\{1, \frac{|(A-B)(p-\alpha) - B|}{p+1}\right\}.$$

Proof. As $g \in \mathcal{S}^*(A,B;p;\alpha)$, therefore we have

$$\frac{zg'(z)}{g(z)} - p = \frac{(A-B)(p-\alpha)w(z)}{(1+Bw(z))}, \ \ w(z) = \sum_{k=1}^{\infty} \gamma_k z^k \in \mathcal{U}.$$

On expanding the above expression and equating the coefficients of z^{p+1} and z^{p+2}, it yields

$$d_{p+1} = \frac{(A-B)(p-\alpha)\gamma_1}{p+1}$$

and

$$d_{p+2} = \frac{(A-B)(p-\alpha)\gamma_2}{2} + \frac{[(A-B)(p-\alpha) - B](A-B)(p-\alpha)}{2(p+1)}\gamma_1^2.$$

On applying the triangle inequality in the above equations, we obtain

$$|d_{p+1}| \leq \frac{(A-B)(p-\alpha)}{p+1}|\gamma_1|$$

and

$$|d_{p+2}| \leq \frac{(A-B)(p-\alpha)}{2}\left|\gamma_2 + \frac{[(A-B)(p-\alpha)-B]}{p+1}\gamma_1^2\right|.$$

It is well known [9] that for $w(z) = \sum_{k=1}^{\infty}\gamma_k z^k \in \mathcal{U}$, $|\gamma_1| \leq 1$ and $\left|\gamma_2 - s\gamma_1^2\right| \leq \max\{1, |s|\}$.

Using these results in the above inequalities, the proof of the Lemma is obvious. ■

3. Results for the class $\mathcal{CS}^*(A, B; C, D; p; \beta; \alpha)$

Theorem 1. *If $f \in \mathcal{CS}^*(A, B; C, D; p; \beta; \alpha)$, then for $n \geq 1$,*

$$|a_n| \;\leq\; \Pi_{j=0}^{n-(p+1)}\frac{|(B-A)(p-\alpha)+Bj|}{j+1} + \frac{(C-D)(p-\beta)}{p} \times\left[1+\sum_{m=p+1}^{n-1}\Pi_{j=0}^{m-(p+1)}\frac{|(B-A)(p-\alpha)+Bj|}{j+1}\right]. \tag{2}$$

The result is sharp.

Proof. As $f \in \mathcal{CS}^*(A, B; C, D; p; \beta; \alpha)$, therefore by Definition 1, we have

$$pf(z) = g(z)P(z), \tag{3}$$

where

$$g(z) = z^p + \sum_{k=p+1}^{\infty} d_k z^k \in \mathcal{S}^*(A, B; p; \alpha)$$

and

$$P(z) = p + \sum_{k=1}^{\infty} p_k z^k \in \mathcal{P}(C, D; p; \beta).$$

Expansion of (3) yields,

$$\begin{aligned} p[1 + a_{p+1}z + a_{p+2}z^2 + ... + a_n z^{n-p} + ... \\ = [1 + d_{p+1}z + d_{p+2}z^2 + ... + d_n z^{n-p} + ...] \\ \times [p + p_1 z + p_2 z^2 + ... + p_n z^n + ...]. \end{aligned} \tag{4}$$

On equating the coefficients of z^{n-p} on both sides of (4), we have

$$pa_n = pd_n + p_1 d_{n-1} + p_2 d_{n-2} ... + p_{n-p-1} d_{p+1} + p_{n-p}. \tag{5}$$

Application of triangle inequality and using Lemma 1 in (5), it gives

$$p|a_n| \le p|d_n| + (C-D)(p-\beta)\left[1 + |d_{p+1}| + |d_{p+2}| + ... + |d_{n-1}|\right]. \tag{6}$$

Using Lemma 4 in (6), we can easily obtain the result (2). Equality is attained in (2), for the functions f_p defined by

$$f_p(z) = \frac{z^p}{(1 - B\delta_1 z)^{\frac{(B-A)(p-\alpha)}{B}}}\left[\frac{p + \{pD + (C-D)(p-\beta)\}\delta_2 z}{1 + D\delta_2 z}\right], \tag{7}$$

$|\delta_1| = |\delta_2| = 1$, $B \ne 0$. ■

Remark 1. (*i*) On putting $\alpha = 0$, $\beta = 0$ in Theorem 1, we can easily obtain the result for the class $\mathcal{CS}^*(A, B; C, D; p)$.

(*ii*) For $p = 1$, $\alpha = 0$, $\beta = 0$, Theorem 1 agrees with the result due to Mehrok and Singh [12].

(*iii*) For $A = 1$, $B = -1$, $\alpha = 0$, $\beta = 0$, $p = 1$, Theorem 1 leads to the result established by Mehrok et al. [13].

(*iv*) On putting $A = 1$, $B = -1$, $C = 1$, $D = -1$, $\alpha = 0$, $\beta = 0$, $p = 1$ in Theorem 1, we can easily obtain the result derived by Reade [18].

Theorem 2. *For $f \in \mathcal{CS}^*(A, B; C, D; p; \beta; \alpha)$ and for $|z| = r$, $0 < r < 1$, we have for $B \ne 0$,*

$$\begin{aligned} \frac{1}{p} r^p (1 - Br)^{\frac{(A-B)(p-\alpha)}{B}}\left[\frac{p - \{pD + (C-D)(p-\beta)\}r}{1 - Dr}\right] &\le |f(z)| \\ &\le \frac{1}{p} r^p (1 + Br)^{\frac{(A-B)(p-\alpha)}{B}}\left[\frac{p + \{pD + (C-D)(p-\beta)\}r}{1 + Dr}\right]; \end{aligned} \tag{8}$$

for $B = 0$,

$$\begin{aligned} \frac{1}{p} r^p e^{-A(p-\alpha)r}\left[\frac{p - \{pD + (C-D)(p-\beta)\}r}{1 - Dr}\right] &\le |f(z)| \\ &\le \frac{1}{p} r^p e^{A(p-\alpha)r}\left[\frac{p + \{pD + (C-D)(p-\beta)\}r}{1 + Dr}\right]. \end{aligned} \tag{9}$$

Estimates are sharp.

Proof. By taking modulus, (3) yields

$$|pf(z)| = |g(z)||P(z)|. \tag{10}$$

It was proved in [2] that,

$$\begin{aligned}(11)\qquad \frac{p-[pD+(C-D)(p-\beta)]r}{1-Dr} &\leq |P(z)| \\ &\leq \frac{p+[pD+(C-D)(p-\beta)]r}{1+Dr}.\end{aligned}$$

Using the result (11) and Lemma 5 in (10), the results (8) and (9) can be easily obtained. Sharpness follows for the functions f_p defined as

$$(12)\quad f_p(z)=\begin{cases}\frac{1}{p}(1+B\delta_3 z)^{\frac{(A-B)(p-\alpha)}{B}}\left[\frac{p+\{pD+(C-D)(p-\beta)\}\delta_4 z}{1+D\delta_4 z}\right] & \text{if } B\neq 0,\\ \frac{1}{p}e^{A(p-\alpha)\delta_5 z}\left[\frac{p+\{pD+(C-D)(p-\beta)\}\delta_4 z}{1+D\delta_4 z}\right] & \text{if } B=0,\end{cases}$$

where $|\delta_3|=|\delta_4|=|\delta_5|=1$. ■

Remark 2. (i) On putting $\alpha=0$, $\beta=0$ in Theorem 2, we can easily obtain the result for the class $\mathcal{CS}^*(A,B;C,D;p)$.

(ii) For $p=1$, $\alpha=0$, $\beta=0$, Theorem 2 agrees with the result due to Mehrok and Singh [12].

(iii) For $A=1$, $B=-1$, $\alpha=0$, $\beta=0$, $p=1$, Theorem 2 leads to the result established by Mehrok et al. [13].

(iv) On putting $A=1$, $B=-1$, $C=1$, $D=-1$, $\alpha=0$, $\beta=0$, $p=1$ in Theorem 2, we can easily obtain the result derived by Reade [18].

Theorem 3. *If $f\in\mathcal{CS}^*(A,B;C,D;p;\beta;\alpha)$, then*

$$(13)\quad \left|arg\frac{f(z)}{z^p}\right|\leq\begin{cases}\frac{1}{p}\left[\frac{(A-B)(p-\alpha)}{B}\sin^{-1}(Br)\right.\\ \qquad\left.+\ \sin^{-1}\left(\frac{(C-D)(p-\beta)r}{p-[pD+(C-D)(p-\beta)]Dr^2}\right)\right] & \text{if } B\neq 0,\\ \frac{1}{p}\left[A(p-\alpha)r\right.\\ \qquad\left.+\ \sin^{-1}\left(\frac{(C-D)(p-\beta)r}{p-[pD+(C-D)(p-\beta)]Dr^2}\right)\right] & \text{if } B=0.\end{cases}$$

The results are sharp.

Proof. Again from (3), we have

$$pf(z)=g(z)P(z),$$

which implies

$$(14)\qquad \left|arg\frac{f(z)}{z^p}\right|\leq|argP(z)|+\left|arg\frac{g(z)}{z^p}\right|.$$

It was proved by Aouf [1] that

$$(15)\qquad |argP(z)|\leq\sin^{-1}\left(\frac{(C-D)(p-\beta)r}{p-[pD+(C-D)(p-\beta)]Dr^2}\right).$$

On using (15) and Lemma 6, the results (13) can be easily obtained. Results are sharp for the function defined in (12). ■

Remark 3. (*i*) On putting $\alpha = 0$, $\beta = 0$ in Theorem 3, we can easily obtain the result for the class $\mathcal{CS}^*(A, B; C, D; p)$.

(*ii*) For $p = 1$, $\alpha = 0$, $\beta = 0$, Theorem 3 agrees with the result due to Mehrok and Singh [12].

(*iii*) For $A = 1$, $B = -1$, $\alpha = 0$, $\beta = 0$, $p = 1$, Theorem 3 leads to the result established by Mehrok et al. [13].

(*iv*) On putting $A = 1$, $B = -1$, $C = 1$, $D = -1$, $\alpha = 0$, $\beta = 0$, $p = 1$ in Theorem 3, we can easily obtain the result derived by Reade [18].

Theorem 4. *Let $f \in \mathcal{CS}^*(A, B; C, D; p; \beta; \alpha)$, then*

$$|a_{p+1}| \leq \frac{(p-\beta)(C-D)}{p} + \frac{(p-\alpha)(A-B)}{p+1} \tag{16}$$

and

$$|a_{p+2}| \leq \begin{cases} \frac{(A-B)(p-\alpha)}{2} + \frac{(C-D)(p-\beta)}{p}\left[1 + \frac{(A-B)(p-\alpha)}{p+1}\right] \\ \qquad \text{if } |(A-B)(p-\alpha) - B| \leq p+1, \\ \frac{(A-B)(p-\alpha)}{2}\left[\frac{|(A-B)(p-\alpha)-B|}{p+1}\right] \\ \qquad + \frac{(C-D)(p-\beta)}{p}\left[\frac{(A-B)(p-\alpha)}{p+1} + 1\right] \\ \qquad \text{if } |(A-B)(p-\alpha) - B| > p+1. \end{cases} \tag{17}$$

The bounds are sharp.

Proof. From Definition 1, using the principle of subordination, it gives

$$\frac{pf(z)}{g(z)} = \frac{p + [pD + (C-D)(p-\beta)]w(z)}{1 + Dw(z)}, \quad w(z) \in \mathcal{U}.$$

On expanding and comparing the coefficients, it leads to

$$a_{p+1} = d_{p+1} + \frac{(C-D)(p-\beta)}{p} c_1 \tag{18}$$

and

$$a_{p+2} = d_{p+2} + \frac{1}{p}[(C-D)(p-\beta)]d_{p+1}c_1 + \frac{(C-D)(p-\beta)}{p}[c_2 - Dc_1^2]. \tag{19}$$

From Lemma 9, for $g(z) = z^p + \sum_{k=p+1}^{\infty} d_k z^k \in \mathcal{S}^*(A, B; p; \alpha)$, we have

$$|d_{p+1}| \leq \frac{(A-B)(p-\alpha)}{p+1} \tag{20}$$

and

$$|d_{p+2}| \leq \frac{(A-B)(p-\alpha)}{2} \max\left\{1, \frac{|(A-B)(p-\alpha)-B|}{p+1}\right\}. \tag{21}$$

Keogh and Merkes [9] proved that, for any complex number γ,

$$|c_2 - \gamma c_1^2| \leq \max\{1, |\gamma|\}. \tag{22}$$

Application of triangle inequality and using (20), (21) and (22) in (18) and (19), along with the inequality $|c_1| \leq 1$, the results (16) and (17) are obvious. The extremal function for (16) and first inequality of (17) is given by

$$\begin{aligned} f_1(z) &= z^p + \left[\frac{(p-\beta)(C-D)}{p} + \frac{(p-\alpha)(A-B)}{p+1}\right] z^{p+1} \\ &+ \left[\frac{(A-B)(p-\alpha)}{2} + \frac{(C-D)(p-\beta)}{p}\left[1 + \frac{(A-B)(p-\alpha)}{p+1}\right]\right] z^{p+2} \\ &+ \ldots \end{aligned}$$

The extremal function for (16) and second inequality of (17) is given by

$$\begin{aligned} f_2(z) &= z^p + \left[\frac{(p-\beta)(C-D)}{p} + \frac{(p-\alpha)(A-B)}{p+1}\right] z^{p+1} \\ &+ \left[\frac{(A-B)(p-\alpha)}{2}\left[\frac{|(A-B)(p-\alpha)-B|}{p+1}\right]\right. \\ &+ \left.\frac{(C-D)(p-\beta)}{p}\left[\frac{(A-B)(p-\alpha)}{p+1} + 1\right]\right] z^{p+2} + \ldots \end{aligned}$$

■

Theorem 5. *Let $-1 \leq D_2 = D_1 < C_1 \leq C_2 \leq 1$ and $0 \leq \beta_2 \leq \beta_1 < p$, then*

$$\mathcal{CS}^*(A, B; C_1, D_1; p; \beta_1; \alpha) \subset \mathcal{CS}^*(A, B; C_2, D_2; p; \beta_2; \alpha).$$

Proof. Since $f \in \mathcal{CS}^*(A, B; C_1, D_1; p; \beta_1; \alpha)$, so

$$\frac{pf(z)}{g(z)} \prec \frac{p + [pD_1 + (C_1 - D_1)(p-\beta_1)]z}{1 + D_1 z}.$$

As $-1 \leq D_2 = D_1 < C_1 \leq C_2 \leq 1$ and $0 \leq \beta_2 \leq \beta_1 < p$, we have

$$-1 \leq D_1 + \frac{(p-\beta_1)(C_1-D_1)}{p} \leq D_2 + \frac{(p-\beta_2)(C_2-D_2)}{p} \leq 1.$$

Thus by Lemma 2, we obtain

$$\frac{pf(z)}{g(z)} \prec \frac{p + [pD_2 + (C_2 - D_2)(p-\beta_2)]z}{1 + D_2 z},$$

which implies $f \in \mathcal{CS}^*(A, B; C_2, D_2; p; \beta_2; \alpha)$. ■

Theorem 6. *If $f \in \mathcal{CS}^*(A,B;C,D;p;\beta;\alpha)$, then there exists $P(z) \in \mathcal{P}(C,D;p;\alpha)$ such that for all s and t with $|s| \le 1$, $|t| \le 1$ $(s \neq t)$,*

$$\frac{f(sz)P(tz)(tz)^p}{f(tz)P(sz)(sz)^p} = \begin{cases} \left(\frac{1+Bsz}{1+Btz}\right)^{\left(\frac{A-B}{B}\right)(p-\alpha)}, & \text{if } B \neq 0, \\ e^{A(p-\alpha)(s-t)z}, & \text{if } B = 0. \end{cases}$$

Proof. Firstly we discuss the case when $B \neq 0$. On differentiating (3) logarithmically, we get

$$\frac{zf'(z)}{f(z)} - \frac{zP'(z)}{P(z)} - p = \frac{zg'(z)}{g(z)} - p.$$

As $g \in \mathcal{S}^*(A,B;p;\alpha)$, therefore

$$\frac{zf'(z)}{f(z)} - \frac{zP'(z)}{P(z)} - p \prec \frac{(A-B)(p-\alpha)z}{1+Bz},$$

where $\dfrac{(A-B)(p-\alpha)z}{1+Bz}$ is convex, univalent in E. For $|s| \le 1$, $|t| \le 1$ $(s \neq t)$,

$$h(z) = \int_0^z \left(\frac{s}{1-su} - \frac{t}{1-tu}\right) du$$

is convex univalent in E. Using Lemma 3, we have

$$\left(\frac{zf'(z)}{f(z)} - \frac{zP'(z)}{P(z)} - p\right) * h(z) \prec \frac{(A-B)(p-\alpha)z}{1+Bz} * h(z).$$

For any function $q(z)$ analytic in E with $q(0) = 0$, we obtain

$$(q * h)(z) = \int_{tz}^{sz} q(u)\frac{du}{u}, z \in E.$$

Therefore, we have

$$\int_{tz}^{sz} \left(\frac{uf'(u)}{f(u)} - \frac{uP'(u)}{P(u)} - p\right)\frac{du}{u} \prec (A-B)(p-\alpha)\int_{tz}^{sz} \frac{du}{1+Bu},$$

which follows the result. On the same lines, we can easily prove the result for $B = 0$. ■

4. Results for the class $\mathcal{CS}_1^*(A, B; C, D; p; \beta; \alpha)$

Theorem 7. *Let $f \in \mathcal{CS}_1^*(A, B; C, D; p; \beta; \alpha)$, then for $n \geq 1$,*

$$
\begin{aligned}
(23)\quad |a_n| \leq\ & \frac{p}{n[(n-p)!]}\Pi_{j=0}^{n-(p+1)}|(B-A)(p-\alpha)+Bj| \\
&+\frac{(C-D)(p-\beta)}{p} \\
&\times\left[1+\sum_{m=p+1}^{n-1}\frac{p}{m[(m-p)!]}\Pi_{j=0}^{m-(p+1)}|(B-A)(p-\alpha)+Bj|\right].
\end{aligned}
$$

The result is sharp.

Proof. The proof is obvious by folowing the procedure of Theorem 1 and applying Lemma 7. Equality holds in (23) for the functions f_p defined by

$$
\begin{aligned}
(24)\qquad f_p(z) =\ & p\left[\int_0^z z^{p-1}(1-B\delta_6 z)^{\frac{(A-B)(p-\alpha)}{B}}dz\right] \\
&\times\left[\frac{p+\{pD+(C-D)(p-\beta)\}\delta_7 z}{1+D\delta_7 z}\right],
\end{aligned}
$$

where $|\delta_6| = |\delta_7| = 1, B \neq 0$. ■

Remark 4. (i) On putting $\alpha = 0, \beta = 0$ in Theorem 7, we can easily obtain the result for the class $\mathcal{CS}_1^*(A, B; C, D; p)$.

(ii) On putting $A = 1, B = -1, \alpha = 0, \beta = 0, p = 1$ in Theorem 7, we can easily obtain the result established by Mehrok et al. [14].

Theorem 8. *If $f \in \mathcal{CS}_1^*(A, B; C, D; p; \beta; \alpha)$, then for $|z| = r, 0 < r < 1$, we have for $B \neq 0$,*

$$
\begin{aligned}
(25)\quad & \left[\int_0^r t^{p-1}(1-Bt)^{\frac{(A-B)(p-\alpha)}{B}}dt\right]\left[\frac{p-\{pD+(C-D)(p-\beta)\}r}{1-Dr}\right] \\
&\leq |f(z)| \\
&\leq \left[\int_0^r t^{p-1}(1+Bt)^{\frac{(A-B)(p-\alpha)}{B}}dt\right]\left[\frac{p+\{pD+(C-D)(p-\beta)\}r}{1+Dr}\right];
\end{aligned}
$$

for $B = 0$,

$$
\begin{aligned}
(26)\qquad & \left[\int_0^r t^{p-1}e^{-A(p-\alpha)t}dt\right]\left[\frac{p-\{pD+(C-D)(p-\beta)\}r}{1-Dr}\right] \\
&\leq |f(z)| \\
&\leq \left[\int_0^r t^{p-1}e^{A(p-\alpha)t}dt\right]\left[\frac{p+\{pD+(C-D)(p-\beta)\}r}{1+Dr}\right];
\end{aligned}
$$

Estimates are sharp.

Proof. Following the procedure of Theorem 2 and using Lemma 8, the proof of Theorem 8 is obvious. Sharpness follows for the functions f_p defined as

$$(27)\qquad f_p(z) = \begin{cases} \frac{1}{p}\left[\int_0^z z^{p-1}(1+B\delta_8 z)^{\frac{(A-B)(p-\alpha)}{B}}dz\right] \\ \qquad\qquad \times \left[\frac{p+\{pD+(C-D)(p-\beta)\}\delta_9 z}{1+D\delta_9 z}\right] & \text{if } B \neq 0, \\ \frac{1}{p}\left[\int_0^z z^{p-1}e^{A(p-\alpha)\delta_{10}z}dz\right] \\ \qquad\qquad \times \left[\frac{p+\{pD+(C-D)(p-\beta)\}\delta_9 z}{1+D\delta_9 z}\right] & \text{if } B = 0, \end{cases}$$

where $|\delta_8| = |\delta_9| = |\delta_{10}| = 1$. ∎

Remark 5. (i) On putting $\alpha = 0, \beta = 0$ in Theorem 8, we can easily obtain the result for the class $\mathcal{CS}_1^*(A, B; C, D; p)$.

(ii) For $A = 1, B = -1, \alpha = 0, \beta = 0, p = 1$, Theorem 8 leads to the result proved by Mehrok et al. [14].

Theorem 9. *If $f \in \mathcal{CS}_1^*(A, B; C, D; p; \beta; \alpha)$, then*

$$(28)\quad \left|arg\frac{f(z)}{z^p}\right| \leq \begin{cases} \frac{1}{p}\left[\frac{A(p-\alpha)}{B}\sin^{-1}(Br)\right. \\ \qquad \left.+\sin^{-1}\left(\frac{(C-D)(p-\beta)r}{p-[pD+(C-D)(p-\beta)]Dr^2}\right)\right] & \textit{if } B \neq 0, \\ \frac{1}{p}\left[A(p-\alpha)r\right. \\ \qquad \left.+\sin^{-1}\left(\frac{(C-D)(p-\beta)r}{p-[pD+(C-D)(p-\beta)]Dr^2}\right)\right] & \textit{if } B = 0. \end{cases}$$

The results are sharp.

Proof. Using the result that, for $h \in \mathcal{K}(A, B; p; \alpha)$,

$$\left|arg\frac{h(z)}{z^p}\right| \leq \begin{cases} \frac{A(p-\alpha)}{B}sin^{-1}(Br) & \text{if } B \neq 0, \\ A(p-\alpha)r & \text{if } B = 0, \end{cases}$$

and following the procedure of Theorem 3, the proof is obvious. Results are sharp for the function defined in (27). ∎

Remark 6. (i) On putting $\alpha = 0, \beta = 0$ in Theorem 9, we can easily obtain the result for the class $\mathcal{CS}_1^*(A, B; C, D; p)$.

(ii) For $A = 1, B = -1, \alpha = 0, \beta = 0, p = 1$, Theorem 9 leads to the result proved by Mehrok et al. [14].

Theorem 10. *If* $f \in \mathcal{CS}_1^*(A,B;C,D;p;\beta;\alpha)$, *then*

$$|a_{p+1}| \leq \frac{p(p-\alpha)(A-B)}{p+1} + \frac{(p-\beta)(C-D)}{p} \tag{29}$$

and

$$|a_{p+2}| \leq \begin{cases} \frac{p(p-\alpha)(A-B)}{2(p+2)} + (C-D)(p-\beta)\left[\frac{(A-B)(p-\alpha)}{p+1} + \frac{1}{p}\right] \\ \qquad \text{if } |(A-B)(p-\alpha)-B| \leq 1, \\ \frac{p(p-\alpha)(A-B)}{2(p+2)}[|(A-B)(p-\alpha)p-B|] \\ \qquad + (C-D)(p-\beta)\left[\frac{(A-B)(p-\alpha)}{p+1} + \frac{1}{p}\right] \\ \qquad \text{if } |(A-B)(p-\alpha)-B| > 1. \end{cases} \tag{30}$$

The bounds are sharp.

Proof. It was proved by Aouf [2] that, for $h \in \mathcal{K}(A,B;p;\alpha)$,

$$|b_{p+1}| \leq \frac{p(A-B)(p-\alpha)}{p+1}$$

and

$$|b_{p+2}| \leq \frac{p(A-B)(p-\alpha)}{2(p+2)} max\{1, |(A-B)(p-\alpha)p-B|\}.$$

Following the procedure of Theorem 4 and using the above results, the proof of Theorem 10 is obvious. The results are sharp for the function defined in (24). The extremal function for (29) and first inequality of (30) is given by

$$\begin{aligned} f_3(z) &= z^p + \left[\frac{p(p-\alpha)(A-B)}{p+1} + \frac{(p-\beta)(C-D)}{p}\right] z^{p+1} \\ &+ \left[\frac{p(p-\alpha)(A-B)}{2(p+2)}\right. \\ &+ \left.(C-D)(p-\beta)\left[\frac{(A-B)(p-\alpha)}{p+1} + \frac{1}{p}\right]\right] z^{p+2}\ldots \end{aligned}$$

The extremal function for (29) and second inequality of (30) is given by

$$\begin{aligned} f_3(z) &= z^p + \left[\frac{p(p-\alpha)(A-B)}{p+1} + \frac{(p-\beta)(C-D)}{p}\right] z^{p+1} \\ &+ \left[\frac{p(p-\alpha)(A-B)}{2(p+2)}[|(A-B)(p-\alpha)p-B|]\right. \\ &+ \left.(C-D)(p-\beta)\left[\frac{(A-B)(p-\alpha)}{p+1} + \frac{1}{p}\right]\right] z^{p+2}\ldots \end{aligned}$$

■

Theorem 11. *Let* $-1 \leq D_2 = D_1 < C_1 \leq C_2 \leq 1$ *and* $0 \leq \beta_2 \leq \beta_1 < p$, *then*

$$\mathcal{CS}_1^*(A, B; C_1, D_1; p; \beta_1; \alpha) \subset \mathcal{CS}_1^*(A, B; C_2, D_2; p; \beta_2; \alpha).$$

Proof. Following the procedure of Theorem 5 and using Lemma 2, the proof is obvious. ■

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Gagandeep Singh
Department of Mathematics
Khalsa College
Amritsar, Punjab, India
e-mail: kamboj.gagandeep@yahoo.in

Gurcharanjit Singh
G.N.D.U. College
Chungh (Tarn-Taran), Punjab, India
e-mail: dhillongs82@yahoo.com

Received on 29.11.2021 and, in revised form, on 31.05.2022.