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EXISTENCE AND UNIQUENESS OF POSITIVE PERIODIC SOLUTION FOR NONLINEAR SECOND ORDER DELAY DIFFERENTIAL EQUATIONS

Abstract. This work is concerned with the following nonlinear second order delay differential equations

$$x''(t)+p(t)x'(t)+q(t)x(t) = f(t, x(t), x(t-\tau(t)), x'(t-\tau(t))), t \in \mathbb{R},$$

which includes many key second order delay differential equations that arise in nonlinear analysis and its applications. We use Perov's fixed point theorem to prove the existence and uniqueness of periodic solutions for second order delay differential equations. Our results are obtained under general assumptions.

Key words: Perov's fixed point, second order delay differential equations, periodic solution.

AMS Mathematics Subject Classification: 47H30, 45D05, 45G10.

1. Introduction

The nonlinear second order delay differential equations arise in the modeling of many phenomena in physics, population dynamics and industrial robotics (see, for example [8, 9]). The second order delay differential equations have been considered by many authors (see [2, 3, 10, 12] and the references therein). For example, Y. Liu and W. Ge ([10]) studied the following second order nonlinear Duffing equation with delay and variable coefficients:

$$x'' + p(t)x'(t) + q(t)x(t) = \lambda h(t)f(t, x(t - \tau(t))) + r(t), t \in \mathbb{R}.$$

Moreover, Wang et al.[12] have studied the second order nonlinear delay differential equation with periodic coefficients

$$x'' + p(t)x'(t) + q(t)x(t) = r(t)x'(t - \tau(t)) + f(t, x(t), x(t - \tau(t))), t \in \mathbb{R},$$

by using Krasnoselskii's fixed point theorem.

A. Ardjouni and A. Djoudi [3] have studied the existence of periodic solutions for the following second-order nonlinear neutral differential equation with variable delay

$$\frac{d}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = \frac{d}{dt}g(t, x(t-\tau(t))) + f(t, x(t), x(t-\tau(t))), t \in \mathbb{R},$$

by using the hybrid fixed point theorem of Krasnoselskii.
In the current paper, we consider the following more general form of nonlinear second delay differential equations

$$x''(t) + p(t)x'(t) + q(t)x(t) = f(t, x(t), x(t-\tau(t)), x'(t-\tau(t))), t \in \mathbb{R}. \tag{1}$$

In all the mentioned paper, the derivative x' does not appear in the nonlinear functions. Our purpose here is to use Perov's fixed point in a suitable Banach space to show the existence and uniqueness of a periodic solution for Equation (1), under fairly simple conditions and $x'(t-\tau(t))$ appears in the nonlinear functions.
Perov's fixed point theorem is a generalization of Banach's theorem and it has been used by many authors (see, for example [1, 5, 6, 7]).
The rest of the paper is organized as follows: In Section 2, we give some preliminary results in generalized metric spaces. In Section 3, we give our main result. In the last section, we illustrate our main result by an example.

2. Some preliminaries results in generalized metric spaces

In this section, we recall the following notations and results in generalized metric spaces.

Definition 1 ([11]). *Let X be a nonempty set and $d : X \times X \longrightarrow \mathbb{R}^n$ be a mapping such that for all $x, y, z \in X$, one has :*
1) $d(x, y) \geq 0_{\mathbb{R}^n}$ and $d(x, y) = 0_{\mathbb{R}^n} \Longleftrightarrow x = y$,
2) $d(x, y) = d(y, x)$,
3) $d(x, y) \leq d(x, z) + d(z, y)$,
where for $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ from $\mathbb{R}^n$, we have $x \leq y \Longleftrightarrow x_i \leq y_i$, for any $i = \overline{1, n}$.
Then d is called a generalized metric and (X, d) is a generalized metric space.

Definition 2 ([11]). *If (E, d) is a complete generalized metric space and $T : E \to E$ which satisfies the inequality*

$$d(Tx, Ty) \leq Ad(x, y) \text{ for all } x, y \in E,$$

where A is a matrix convergent to zero (the norms of it's eigenvalues are in the interval $[0, 1)$). We say that T is a Picard operator or generalized contraction.

We recall the following Perov's fixed point theorem.

Theorem 1 ([11]). *Let (E, d) be a complete generalized metric space. If $T : E \to E$ is a map for which there exists a matrix $A \in M_n(\mathbb{R})$ such that*

$$d(Tx, Ty) \leq Ad(x, y),\ \forall x, y \in E$$

and the norms of the eigenvalues of A are in the interval $[0, 1)$, then T has a unique fixed point $x^* \in E$ and the sequence of successive approximations $x_m = T^m(x_0)$ converges to x^* for any $x_0 \in E$. Moreover, the following estimation holds

$$d(x_m, x^*) \leq A^m(I_n - A)^{-1}d(x_0, x_1),\ \forall m \in \mathbb{N}^*.$$

We consider the following functional spaces

$$P(T) = \{x \in C(\mathbb{R}) : x(t + T) = x(t),\ \forall t \in \mathbb{R}\}$$
$$P^1(T) = \left\{x \in C^1(\mathbb{R}) : x(t + T) = x(t),\ \forall t \in \mathbb{R}\right\}$$
$$K^+(T) = \left\{x \in P^1(T) :\ x(t) \geq 0,\ \forall t \in \mathbb{R}\right\}$$

and denote by E the product space $E = K^+(T) \times P(T)$ which is a generalized metric space with the generalized metric $d_C : E \times E \to \mathbb{R}^2$, defined by

$$d_C((x_1, y_1), (x_2, y_2)) = \left(\|x_1 - x_2\| + \left\|x_1' - x_2'\right\|, \|y_1 - y_2\|\right)$$

where $\|u\| = \max\{|u(t)| : t \in [0, T]\}$ for any $u \in P(T)$.

Lemma 1 ([5]). *(E, d_C) is a complete generalized metric space.*

Equation (1) will be studied under the following assumptions:

(i) $f \in C(\mathbb{R} \times (\mathbb{R}^+)^2 \times \mathbb{R}, \mathbb{R})$ and there exists $T > 0$ such that

$$f(t + T, x, y, z) = f(t, x, y, z), \forall(t, x, y, z) \in \mathbb{R} \times (\mathbb{R}^+)^2 \times \mathbb{R}.$$

(ii) There exist $\alpha, \beta, \gamma \geq 0$ such that

$$|f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2)| \leq \alpha |u_1 - u_2| + \beta |v_1 - v_2| + \gamma |w_1 - w_2|,$$

$\forall t \in \mathbb{R}$, $\forall(u_1,\ u_2, v_1, v_2) \in (\mathbb{R}^+)^4$, $\forall(w_1, w_2) \in \mathbb{R}^2$.

(iii) $p, q : \mathbb{R} \longrightarrow \mathbb{R}^+, \tau : \mathbb{R} \longrightarrow \mathbb{R}$ are all continuous $T-$periodic functions, $\int_0^T p(s) > 0$, $\int_0^T q(s) > 0$, and $\tau'(t) \neq 1$, for all $t \in [0, T]$.

3. Auxiliary lemmas

Before stating the main result in the next section, we need the following lemmas,

Lemma 2 ([10])**.** *Suppose that (iii) holds and*

$$R_1 \frac{\left(\exp\left(\int\limits_0^T p(u)du\right) - 1\right)}{Q_1 T} \geq 1,$$

where

$$R_1 = \max_{t\in[0,T]} \left| \int\limits_t^{t+T} \frac{\exp\left(\int\limits_t^s p(u)du\right)}{\exp\left(\int\limits_0^T p(u)du\right) - 1} q(s)ds \right|,$$

$$Q_1 = \left(1 + \exp\left(\int\limits_0^T p(u)du\right)\right)^2 R_1^2.$$

Then there are continuous $T-$periodic functions a and b such that $b(t) > 0$, $\int\limits_0^T a(u)du > 0$ and

$$a(t) + b(t) = p(t), b'(t) + a(t)b(t) = q(t), t \in \mathbb{R}.$$

Lemma 3 ([12])**.** *Suppose the conditions of Lemma 2 hold and $\psi \in P(T)$. Then the equation*

$$x'' + p(x)x'(t) + q(x)x(t) = \psi(t)$$

has a $T-$periodic solution. Moreover, the periodic solution can be expressed by

$$x(t) = \int\limits_t^{t+T} G(t,s)\psi(s)ds$$

where

$$G(t,s) = \frac{\int\limits_t^s \exp\left(\int\limits_t^u b(v)dv + \int\limits_u^s a(v)dv\right)du + \int\limits_s^{t+T} \exp\left(\int\limits_t^u b(v)dv + \int\limits_u^{s+T} a(v)dv\right)du}{\left(\exp\left(\int\limits_0^T a(u)du\right) - 1\right)\left(\exp\left(\int\limits_0^T b(u)du\right) - 1\right)}.$$

Corollary 1 ([12]). *The Green's function $G(t,s)$ satisfies the following properties:*

$$G(t,t+T)=G(t,t),G(t+T,s+T)=G(t,s),$$
$$\frac{\partial G(t,s)}{\partial t}=-b(t)G(t,s)+F(t,s),$$

where $F(t,s)=\dfrac{\exp\left(\int_t^s a(v)dv\right)}{\exp\left(\int_0^T b(v)dv\right)-1}$.

Lemma 4 ([12]). *Let* $A=\int_0^T p(u)du, B=T^2\exp\left(\frac{1}{T}\int_0^T \ln(q(u))du\right)$. *If* $A^2\geq 4B$, *then we have*

$$\min\left\{\int_0^T a(u)du,\int_0^T b(u)du\right\}\geq\frac{1}{2}\left(A-\sqrt{A^2-4B}\right):=l$$
$$\max\left\{\int_0^T a(u)du,\int_0^T b(u)du\right\}\leq\frac{1}{2}\left(A+\sqrt{A^2-4B}\right):=L$$

Corollary 2 ([12]). *The function $G(t,s)$ satisfies*

$$m=:\frac{T}{(e^L-1)^2}\leq G(t,s)\leq\frac{T\exp\left(\int_0^T p(u)du\right)}{(e^l-1)^2}:=M.$$

4. Main result

It is easy to check, under the above assumptions, that x is a solution of (1) in K^+ (see, [4]) if and only if x is the solution of the following integral equation

$$x(t)=\int_t^{t+T}G(t,s)[f(s,x(s),x(s-\tau(s)),x'(s-\tau(s)))]ds. \tag{2}$$

Under the hypotheses (i), (ii), (iii) and the previous lemmas and corollaries, we will make use of Perov's fixed point theorem to prove the following main result.

Theorem 2. *If the hypotheses (i), (ii) and (iii) hold, and if*

$$T\left(\gamma\theta + (\alpha+\beta)(M+\theta)\right) < 1, \tag{3}$$

such that $\theta = \|b\|M + \frac{\exp(L)}{l-1}$, *then, the second order delay differential equation (2) has a unique positive periodic solution in* $K^+(T)$.

Proof. If we differentiate the equation (2) with respect to t and denoting $x'(t) = y(t)$, we obtain, by using Corollary 1, for all $t \in \mathbb{R}$,

$$y(t) = \int_t^{t+T} \frac{\partial G(t,s)}{\partial t}[f(s, x(s), x(s-\tau(s)), y(s-\tau(s)))]ds,$$

which leads to,

$$\begin{cases} x(t) = \int_t^{t+T} G(t,s)[f(s, x(s), x(s-\tau(s)), y(s-\tau(s)))]ds, \\ y(t) = \int_t^{t+T} \frac{\partial G(t,s)}{\partial t}[f(s, x(s), x(s-\tau(s)), y(s-\tau(s)))]ds. \end{cases}$$

Let $A : E \to C(\mathbb{R}) \times C(\mathbb{R})$ be the map defined by the following expression

$$A(x,y)(t) = \begin{pmatrix} A_1(x,y)(t) \\ \\ A_2(x,y)(t) \end{pmatrix},$$

where,

$$A_1(x,y)(t) = \int_t^{t+T} G(t,s)[f(s, x(s), x(s-\tau(s)), y(s-\tau(s)))]ds,$$

and,

$$A_2(x,y)(t) = \int_t^{t+T} \frac{\partial G(t,s)}{\partial t}[f(s, x(s), x(s-\tau(s)), y(s-\tau(s)))]ds. \tag{4}$$

The rest of the proof is divided into the following claims.

Claim 1 : The operator A transform E into itself.
It is clear, from Conditions (i) and Corollary 1, that $A_1(E) \subset C^1(\mathbb{R})$.
Moreover, from Conditions (i), (iii) and Corollary 1, it follows that $\forall t \in \mathbb{R}, \forall (x,y) \in E$, $A_1(x,y)(t) \geq 0$ and

$$A_1(x,y)(t+T) = \int_{t+T}^{t+2T} G(t+T,s)[f(s, x(s), x(s-\tau(s)), y(s-\tau(s)))]ds$$

$$= \int_{t}^{t+T} G(t+T, s+T)$$
$$\times [f(s+T, x(s+T), x(s+T-\tau(s+T)), y(s+T-\tau(s+T)))]ds$$
$$= A_1(x,y)(t).$$

Hence, $A_1(E) \subset K^+(T)$. Similarly, we have,

$$A_2(x,y)(t+T) = \int_{t+T}^{t+2T} \frac{\partial G(t+T,s)}{\partial t}[f(s, x(s), x(s-\tau(s)), y(s-\tau(s)))]ds$$
$$= \int_{t}^{t+T} \frac{\partial G(t+T, s+T)}{\partial t}$$
$$\times [f(s+T, x(s+T), x(s+T-\tau(s+T)), y(s+T-\tau(s+T)))]ds$$
$$= A_2(x,y)(t), \ \forall t \in \mathbb{R}, \forall (x,y) \in E.$$

We deduce that, $A(E) \subset E$.

Claim 2 : The operator A is a generalized contraction.
From Condition (ii), we have

$$|A_1(x_1,y_1)(t) - A_1(x_2,y_2)(t)| + \left|A_1'(x_1,y_1)(t) - A_1'(x_2,y_2)(t)\right|$$
$$\leq \int_{t}^{t+T} G(t,s)\left[\alpha\left|x_1(s)-x_2(s)\right| + \beta\left|x_1(s)-x_2(s)\right| + \gamma\left|y_1(s)-y_2(s)\right|\right]ds$$
$$+ \int_{t}^{t+T} \frac{\partial G(t,s)}{\partial t}\left[\alpha\left|x_1(s)-x_2(s)\right| + \beta\left|x_1(s)-x_2(s)\right| + \gamma\left|y_1(s)-y_2(s)\right|\right]ds$$
$$\leq TM(\alpha+\beta)\|x_1-x_2\| + TM\gamma\|y_1-y_2\| + \left(\|b\|M + \frac{\exp(L)}{l-1}\right)$$
$$\times T[(\alpha+\beta)\|x_1-x_2\| + \gamma\left(\|b\|M + \frac{\exp(L)}{l-1}\right)(\|y_1-y_2\|)]$$
$$\leq T(\alpha+\beta)\left(M + \|b\|M + \frac{\exp(L)}{l-1}\right)\left(\|x_1-x_2\| + \|x_1'-x_2'\|\right)$$
$$+ \ T\gamma\left(M + \|b\|M + \frac{\exp(L)}{l-1}\right)\|y_1-y_2\|$$

Similarly, we have

$$
\begin{aligned}
&|A_2(x_1, y_1)(t) - A_2(x_2, y_2)(t)| \\
&\leq \int_t^{t+T} \frac{\partial G(t,s)}{\partial t} \left[\alpha \left|x_1(s) - x_2(s)\right| + \beta \left|x_1(s) - x_2(s)\right| + \gamma \left|y_1(s) - y_2(s)\right|\right] ds \\
&\leq T \left(\underbrace{\|b\| M + \frac{\exp(L)}{l-1}}_{=\theta} \right) \left[(\alpha+\beta)\|x_1 - x_2\| + \gamma\|y_1 - y_2\|\right] \\
&\leq T(\alpha+\beta)\theta \left(\|x_1 - x_2\| + \|x_1' - x_2'\|\right) + T\gamma\theta\|y_1 - y_2\|]
\end{aligned}
$$

We deduce that,

$$
d_C\left(A\left(x_1, y_1\right), A\left(x_2, y_2\right)\right) \leq K \begin{pmatrix} \|x_1 - x_2\| + \|x_1' - x_2'\| \\ \|y_1 - y_2\| \end{pmatrix},
$$

where the matrix K is given by,

$$
K = \begin{pmatrix} T(\alpha+\beta)(M+\theta) & T\gamma(M+\theta) \\ \\ T(\alpha+\beta)\theta & T\gamma\theta \end{pmatrix}.
$$

The eigenvalues of this matrix are:

$$
\begin{cases} \lambda_1 = T[\gamma\theta + (\alpha+\beta)(M+\theta)] \\ \lambda_2 = 0 \end{cases}
$$

Since the norms of the eigenvalues are in the interval $[0,1)$, then, by Perov's fixed point theorem, the operator T has a unique solution $x^* = (x_*, y_*) \in K^+(\omega) \times P(\omega)$, which implies that $x_* \in C^1(\mathbb{R})$, and for all $t \in \mathbb{R}$

$$
(x_*)'(t) = \int_t^{t+T} \frac{\partial G(t,s)}{\partial t} [f(s, x_*(s), x_*(s-\tau(s)), y_*(s-\tau(s)))] ds
$$

Hence, by using (4), for all $t \in \mathbb{R}$

$$
((x_*)' - y_*)(t) = 0.
$$

We deduce, that $(x_*)' = y_*$ and x_* is the unique solution of (2). ■

The following proposition gives an estimation of the error between the exact solution and the approximate solution of (2).

Proposition 1. *Under the assumptions of Theorem 2, the solution of the equation (2), which is obtained by the successive approximations method starting from any $x^0 = (x_0, y_0) \in E$, verifies the following estimation:*

$$d_C(x^m, x^*) \leq \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \times d_C(x^1, x^0),$$

where

$$(5) \qquad \begin{cases} e_1 = \frac{(M+\gamma)\lambda_1^{m-1}T(\alpha+\beta)}{\lambda_1-1} \\ e_2 = \frac{\gamma\lambda_1^{m-1}T(\alpha+\beta)}{1-\lambda_1} \\ e_3 = \frac{(\gamma+M)\lambda_1^{m-1}T\theta[2T(M+\gamma)(\alpha+\beta)-1]}{\lambda_1-1} \\ e_4 = \frac{\lambda_1^{m-1}T\theta\gamma[T(1+M+\gamma)(\alpha+\beta)-1]}{\lambda_1-1} \end{cases}$$

Proof. From Theorem 1, by the conditions of Theorem 2, one has that

$$d_C(x^m, x^*) \leq A^m(I-A)^{-1}d_C(x^1, x^0), \ \forall m \in \mathbb{N}^*.$$

We have,

$$A^m = \begin{pmatrix} (M+\gamma)\lambda_1^{m-1}T(\alpha+\beta) & (M+\gamma)\lambda_1^{m-1}T\theta \\ \gamma\lambda_1^{m-1}T(\alpha+\beta) & \lambda_1^{m-1}T\theta\gamma \end{pmatrix},$$

And we find

$$A^m = \lambda_1^{m-1}T \begin{pmatrix} (M+\gamma)(\alpha+\beta) & (M+\gamma)\theta \\ \gamma(\alpha+\beta) & \theta\gamma \end{pmatrix},$$

And we have

$$(I-A)^{-1} = \frac{1}{\lambda_1 - 1} \begin{pmatrix} -1+T\theta\gamma & -T\theta(M+\gamma) \\ -T(\alpha+\beta)\gamma & -1+T(\alpha+\beta)(M+\gamma) \end{pmatrix},$$

Which implies that,

$$A^m(I-A)^{-1} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix},$$

where $e_i, i = 1, ..., 4$ are given by (5). ■

To illustrate this result, we have the following example.

Example 1. Consider the following second-order delay differential equation:

$$(6) \quad x''(t) + p(t)x'(t) + q(t)x(t) = \frac{1}{\lambda}x(t) + \frac{1}{\lambda}x(t-\tau(t)) + \frac{10}{\lambda}x'(t-\tau(t)),$$

$t \in \mathbb{R}$, $\lambda \geq 10^3$, where $p(t) = \frac{1}{2}, q(t) = \frac{1}{16}, \tau(t) = 1 + \sin(6t)$.
Hence, by using the notations of Theorem 2, we have $T = \frac{\pi}{3}$,
$\alpha = \beta = \frac{1}{\lambda}, \gamma = \frac{10}{\lambda}$, where $\lambda = 10^3$ is a positive number. We may see that the conditions of Lemma 2 hold, and $a(t) = b(t) = \frac{1}{4}$, $F(t,s) = \frac{\exp\left(\frac{1}{4}(s-t)\right)}{\exp\left(\frac{1}{4}T\right)-1}$,

$$G(t,s) = \frac{(s-t)\exp\left(\frac{1}{4}(s-t)\right) + (t + \frac{\pi}{3} - s)\exp\left(\frac{1}{4}(s + \frac{\pi}{3} - t)\right)}{\left(\exp(\frac{\pi}{12})\right)^2}.$$

By using the notations of Lemma 4, we have $A = \frac{\pi}{6}, B = \frac{\pi^2}{144}$.
Which implies that $l = L = \frac{\pi}{12}$, $M = \frac{\pi\exp\left(\frac{\pi}{6}\right)}{\left(\exp(\frac{\pi}{12})-1\right)^2}$ and $\|F\|_\infty = \frac{\exp\left(\frac{\pi}{12}\right)}{\exp\left(\frac{\pi}{12}\right)-1}$.
We find $\theta = 19.149, M = 59.25$ and by using the previous values of $T, M, \alpha, \beta, \gamma$ therefore, the inequality in Theorem 2 takes the form

$$T\left(\gamma\theta + (\alpha+\beta)(M+\theta)\right) < 1.$$

Then by Theorem 2, we conclude that the second-order delay differential equation (6) has a periodic solution .

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