F A S C I C U L I   M A T H E M A T I C I

Nr 67 2024

DOI: 10.21008/j.0044-4413.2024.0004

A. Halder, B. Das and S. Debnath

ON $\mathcal{I}$-CONVERGENCE OF COMPLEX UNCERTAIN SEQUENCES

Abstract. In this paper, we introduce convergence concepts namely, $\mathcal{I}-$convergence almost surely, $\mathcal{I}-$convergence in measure, $\mathcal{I}-$convergence in mean, $\mathcal{I}-$convergence in distribution in complex uncertain theory using an ideal $\mathcal{I}$. Also investigate some relationship among them.

Key words: uncertainty theory; complex uncertain variable; $\mathcal{I}-$convergence.

AMS Mathematics Subject Classification: 60B10, 40A35, 40G15.

1. Introduction

In the challenges of everyday life, the notion of uncertainty is crucial. In both mathematical and real-world models, the classical measure meeting nonnegativity and countable additivity is frequently utilised. But the measure used in real-world applications lacks the countable additivity property. The uncertain measure was developed by Liu [16] and it is a set function that satisfies the axioms of normality, monotonicity, self-duality, and countable subadditivity. Complex uncertain variables, which are measurable functions from an uncertainty space to a complex number, was first described by Peng [22]. Then many researchers have also done a lot of theoretical work based on complex uncertain variables, such as Chen et. al. [1], Das et. al. [3], Debnath and Das [4, 5], Khan et.al.[12], Kisi[14], Roy et. al. [23], Saha et. al. [24], Tripathy and Nath [21, 29].

In order to extend the notion of convergence, statistical convergence of sequences was introduced in 1951 by Fast [7], Steinhaus [27] independently. Later it was studied by Fridy [9], Conor [2], Freedman [8], Esi et. al. [6], Kadak and Mohiuddine[10], Mohiuddine et. al.[17, 19], Savas et. al. [25, 26] and many others. The notion of an $\mathcal{I}$-convergence is a generalization of the statistical convergence. The concept of $\mathcal{I}$-convergence was introduced by Kostyrko et. al. [15]. Over the last twenty years a lot of work has been done on this convergence and associated topics and it has turned out to be

one of the most active research. Some results connected with the notion of the $\mathcal{I}$-convergence can be found in [11, 13, 18, 20, 28].
Our main goal, as stated above, is to propose the several types of convergence of complex uncertain sequence by using ideal. In addition, we have also attempted to form some relationships between them.

2. Definitions and preliminaries

We shall give some basic definitions and results of uncertainty theory in this section, which will be used in the subsequent sections :

Definition 1 ([16]). *Let $\mathfrak{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:*
Axiom 1 (Normality): $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2 (Duality): $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathfrak{L}$;
Axiom 3 (Subadditivity): For every countable sequence of $\{\Lambda_j\} \in \mathfrak{L}$, we have

$$\mathcal{M}\left\{ \bigcup_{j=1}^{\infty} \Lambda_j \right\} \leq \textstyle\sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathfrak{L}, \mathcal{M})$ is called an uncertainty space and each element Λ in $\mathfrak{L}$ is called an event. In order to obtain an uncertain measure of compound event, a product uncertain measure is defined by Liu [16]as:

$$\mathcal{M}\left\{ \prod_{k=1}^{\infty} \Lambda_k \right\} = \textstyle\bigwedge_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\}.$$

Definition 2 ([22]). *A complex uncertain variable is a measurable function ζ from an uncertainty space $(\Gamma, \mathfrak{L}, \mathcal{M})$ to the set of complex numbers such that $\{\zeta \in \mathfrak{B}\} = \{\gamma \in \Gamma : \zeta(\gamma) \in \mathfrak{B}\}$ is an event for any Borel set $\mathfrak{B}$ of complex numbers.*

Definition 3 ([22]). *An uncertainty distribution Φ of a complex uncertain variable $\zeta = \xi + i\eta$ is defined by*

$$\Phi(z) = \mathcal{M}\{\zeta \leq z\} = \mathcal{M}\{\xi \leq x, \eta \leq y\} = \mathcal{M}\{\xi \leq x\} \wedge \mathcal{M}\{\eta \leq y\},$$

for any complex $z = x + iy$.

Definition 4 ([22]). *Let $\zeta = \xi + i\eta$ be a complex uncertain variable. If the expected value of ξ and η i.e., $E[\xi]$ and $E[\eta]$ exist, then the expected value of ζ is defined by*

$$E[\zeta] = E[\xi] + iE[\eta].$$

Definition 5 ([1]). *A complex uncertain sequence (ζ_n) is said to be convergent almost surely (a.s) to ζ if for every $\varepsilon > 0$ there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\lim_{n\to\infty} \| \zeta_n(\gamma) - \zeta(\gamma) \| = 0,$$

for every $\gamma \in \Lambda$.

Definition 6 ([1]). *A complex uncertain sequence (ζ_n) is said to be convergent in measure to ζ if*

$$\lim_{n\to\infty} \mathcal{M}\Big(\| \zeta_n - \zeta \| \geq \varepsilon \Big) = 0,$$

for every $\varepsilon > 0$.

Definition 7 ([1]). *A complex uncertain sequence (ζ_n) is said to be convergent in mean to ζ if*

$$\lim_{n\to\infty} E\Big[\| \zeta_n - \zeta \| \Big] = 0.$$

Definition 8 ([1]). *Let $\Phi, \Phi_1, \Phi_2, ...$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, ...$ respectively. Then the sequence (ζ_n) converges in distribution to ζ if*

$$\lim_{n\to\infty} \| \Phi_n(z) - \Phi(z) \| = 0,$$

for all z at which $\Phi(z)$ is continuous.

Definition 9 ([15]). *A non-void class $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if $\mathcal{I}$ is additive (i.e., $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$) and hereditary (i.e., $A \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$). An ideal $\mathcal{I}$ is said to be non-trivial if $\mathcal{I} \neq 2^{\mathbb{N}}$. A non-trivial ideal $\mathcal{I}$ is said to be admissible if $\mathcal{I}$ contains every finite subset of $\mathbb{N}$.*

Example 1. (i) $\mathcal{I}_f :=$ The set of all finite subsets of $\mathbb{N}$ forms an non trivial admissible ideal.

(ii) $\mathcal{I}_d :=$ The set of all subsets of $\mathbb{N}$ whose natural density is zero forms an non trivial admissible ideal.

Definition 10 ([15]). *A sequence $x = (x_n)$ is said to be $\mathcal{I}$ convergent if there exists $L \in \mathbb{R}$ such that for all $\varepsilon > 0$, the set $\{n \in \mathbb{N} : | x_n - L | \geq \varepsilon\} \in \mathcal{I}$. The usual convergence of sequences is a special case of $\mathcal{I}$-convergence ($\mathcal{I}$=$\mathcal{I}_f$-the ideal of all finite subsets of $\mathbb{N}$). The statistical convergence of sequences is also the special case of $\mathcal{I}$-convergence. In this case $\mathcal{I}$=$\mathcal{I}_d$ $= \{A \subseteq \mathbb{N} : \lim\limits_{n\to\infty} \frac{|A\cap\{1,2,...,n\}|}{n} = 0\}$, where $| A |$ being the cardinality of the set A. For more examples on $\mathcal{I}$-convergence see [15].*

2. Main results

Definition 11. *A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-convergent almost surely ($\mathcal{I}$.a.s.) to ζ if for every $\varepsilon > 0$, there exists an event Λ with $\mathcal{M}(\Lambda) = 1$ such that*

$$\{n \in \mathbb{N} : \| \zeta_n(\gamma) - \zeta(\gamma) \| \geq \varepsilon\} \in \mathcal{I},$$

for every $\gamma \in \Lambda$. In this case we write $\zeta_n \xrightarrow{\mathcal{I}.a.s.} \zeta$. The almost surely convergence of complex uncertain sequences is a special case of $\mathcal{I}$-convergence almost surely ($\mathcal{I}$=$\mathcal{I}_f$). The statistical convergence almost surely of complex uncertain sequences is also the special case of $\mathcal{I}$-convergence almost surely. In this case $\mathcal{I}$=$\mathcal{I}_d$.

Definition 12. *A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-convergent in measure to ζ if for every $\varepsilon, \delta > 0$*

$$\Big\{n \in \mathbb{N} : \mathcal{M}\Big(\| \zeta_n(\gamma) - \zeta(\gamma) \| \geq \varepsilon\Big) \geq \delta\Big\} \in \mathcal{I}.$$

Definition 13. *A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-convergent in mean to ζ if for every $\varepsilon > 0$,*

$$\Big\{n \in \mathbb{N} : E\Big(\| \zeta_n(\gamma) - \zeta(\gamma) \| \Big) \geq \varepsilon\Big\} \in \mathcal{I}.$$

Definition 14. *Let $\Phi, \Phi_1, \Phi_2, ...$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, ...$ respectively. Then the complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-convergent in distribution to ζ if for every $\varepsilon > 0$*

$$\Big\{n \in \mathbb{N} : \| \Phi_n(z) - \Phi(z) \| \geq \varepsilon\Big\} \in \mathcal{I},$$

for all z at which $\Phi(z)$ is continuous.

Definition 15. *A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-convergent uniformly almost surely ($\mathcal{I}$.u.a.s.) to ζ if for every $\varepsilon > 0$, $\exists\ \delta > 0$ and a sequence (X_n) of events such that*

$$\Big\{n \in \mathbb{N} : | \mathcal{M}(X_n) | \geq \varepsilon\Big\} \in \mathcal{I}$$
$$\implies \Big\{n \in \mathbb{N} : \| \zeta_n(\gamma) - \zeta(\gamma) \| \geq \delta\Big\} \in \mathcal{I}.$$

Theorem 1. *The complex uncertain sequence (ζ_n) where $\zeta_n = \xi_n + i\eta_n$ is $\mathcal{I}$-convergent almost surely to $\zeta = \xi + i\eta$ if and only if the uncertain sequence (ξ_n) and (η_n) are $\mathcal{I}$-convergent almost surely to ξ and η, respectively.*

Proof. Let the uncertain sequence (ξ_n) and (η_n) are $\mathcal{I}$-convergent almost surely to ξ and η, respectively. Then from the definition of $\mathcal{I}$-convergent almost surely of uncertain sequences, it follows that for any small $\varepsilon > 0$,

$$\left\{n \in \mathbb{N} :\mid \xi_n(\gamma) - \xi(\gamma) \mid\geq \tfrac{\varepsilon}{\sqrt{2}}\right\} \in \mathcal{I}$$

and

$$\left\{n \in \mathbb{N} :\mid \eta_n(\gamma) - \eta(\gamma) \mid\geq \tfrac{\varepsilon}{\sqrt{2}}\right\} \in \mathcal{I}.$$

Note that $\| \zeta_n - \zeta \|= \sqrt{|\xi_n - \xi|^2 + |\eta_n - \eta|^2}$.
Thus we have

$$\{\| \zeta_n - \zeta \|\geq \varepsilon\} \subset \{|\xi_n - \xi| \geq \tfrac{\varepsilon}{\sqrt{2}}\} \cup \{|\eta_n - \eta| \geq \tfrac{\varepsilon}{\sqrt{2}}\}.$$

Therefore

$$\left\{n \in \mathbb{N} :\| \zeta_n(\gamma) - \zeta(\gamma) \|\geq \varepsilon\right\}$$

$$\subset \{n \in \mathbb{N} : |\xi_n(\gamma) - \xi(\gamma)| \geq \tfrac{\varepsilon}{\sqrt{2}}\} \cup \{n \in \mathbb{N} : |\eta_n(\gamma) - \eta(\gamma)| \geq \tfrac{\varepsilon}{\sqrt{2}}\} \in \mathcal{I}.$$

Hence

$$\{n \in \mathbb{N} :\| \zeta_n(\gamma) - \zeta(\gamma) \|\geq \varepsilon\} \in \mathcal{I}.$$

Conversely let, the complex uncertain sequence be (ζ_n) is $\mathcal{I}$-convergent almost surely to ζ. Then from the definition of $\mathcal{I}$-convergent almost surely of uncertain sequences, it follows that for any small $\varepsilon > 0$,

$$\{n \in \mathbb{N} :\| \zeta_n(\gamma) - \zeta(\gamma) \|\geq \varepsilon\} \in \mathcal{I}.$$

Note that

$$|\xi_n - \xi| \leq |(\xi_n - \xi) + i(\eta_n - \eta)| = |(\xi_n + i\eta_n) - (\xi + i\eta)| =\| \zeta_n - \zeta \| .$$

Thus we have

$$\{n \in \mathbb{N} :\mid \xi_n - \xi \mid\geq \varepsilon\} \subseteq \{n \in \mathbb{N} :\| \zeta_n - \zeta \|\geq \varepsilon\}.$$

Therefore

$$\{n \in \mathbb{N} :\mid \xi_n(\gamma) - \xi(\gamma) \mid\geq \varepsilon\} \subseteq \{n \in \mathbb{N} :\| \zeta_n(\gamma) - \zeta(\gamma) \|\geq \varepsilon\} \in \mathcal{I}.$$

Hence

$$\{n \in \mathbb{N} :\mid \xi_n(\gamma) - \xi(\gamma) \mid\geq \varepsilon\} \in \mathcal{I}.$$

Similarly,

$$\{n \in \mathbb{N} :\mid \eta_n(\gamma) - \eta(\gamma) \mid\geq \varepsilon\} \in \mathcal{I}.$$

This completes the proof. ∎

Theorem 2. *The complex uncertain sequence* (ζ_n) *where* $\zeta_n = \xi_n + i\eta_n$ *is* $\mathcal{I}$*-convergent in measure to* $\zeta = \xi + i\eta$ *if and only if the uncertain sequence* (ξ_n) *and* (η_n) *are* $\mathcal{I}$*-convergent in measure to* ξ *and* η*, respectively.*

Proof. Let the uncertain sequence (ξ_n) and (η_n) are $\mathcal{I}$-convergent in measure to ξ and η, respectively. Then from the definition of $\mathcal{I}$-convergent in measure of uncertain sequences, it follows that for any small $\varepsilon, \delta > 0$,

$\{n \in \mathbb{N} : \mathcal{M}(\| \xi_n - \xi \| \geq \frac{\varepsilon}{\sqrt{2}}) \geq \frac{\delta}{2}\} \in \mathcal{I}$ and $\{n \in \mathbb{N} : \mathcal{M}(\| \eta_n - \eta \| \geq \frac{\varepsilon}{\sqrt{2}}) \geq \frac{\delta}{2}\} \in \mathcal{I}$. Note that $\| \zeta_n - \zeta \| = \sqrt{|\xi_n - \xi|^2 + |\eta_n - \eta|^2}$.
Thus we have $\{\| \zeta_n - \zeta \| \geq \varepsilon\} \subset \{|\xi_n - \xi| \geq \frac{\varepsilon}{\sqrt{2}}\} \cup \{|\eta_n - \eta| \geq \frac{\varepsilon}{\sqrt{2}}\}$.
$\implies \mathcal{M}\{\| \zeta_n - \zeta \| \geq \varepsilon\} \leq \mathcal{M}\{|\xi_n - \xi| \geq \frac{\varepsilon}{\sqrt{2}}\} + \mathcal{M}\{|\eta_n - \eta| \geq \frac{\varepsilon}{\sqrt{2}}\}$.
Therefore $\Big\{n \in \mathbb{N} : \mathcal{M}\{\| \zeta_n - \zeta \| \geq \varepsilon\} \geq \delta\Big\} \subseteq \{n \in \mathbb{N} : \mathcal{M}(\| \xi_n - \xi \| \geq \frac{\varepsilon}{\sqrt{2}}) \geq \frac{\delta}{2}\} \cup \{n \in \mathbb{N} : \mathcal{M}(\| \eta_n - \eta \| \geq \frac{\varepsilon}{\sqrt{2}}) \geq \frac{\delta}{2}\} \in \mathcal{I}$.
Hence $\{n \in \mathbb{N} : \mathcal{M}(\| \zeta_n - \zeta \| \geq \varepsilon) \geq \delta\} \in \mathcal{I}$.
Conversely, let the complex uncertain sequence (ζ_n) is $\mathcal{I}$-convergent in measure to ζ. Then from the definition of $\mathcal{I}$-convergent in measure of complex uncertain sequences, it follows that for any small $\varepsilon, \delta > 0$, $\{n \in \mathbb{N} : \mathcal{M}(\| \zeta_n - \zeta \| \geq \varepsilon) \geq \delta\} \in \mathcal{I}$.
Note that $|\xi_n - \xi| \leq |(\xi_n - \xi) + i(\eta_n - \eta)| = |(\xi_n + i\eta_n) - (\xi + i\eta)| = \| \zeta_n - \zeta \|$.
Thus we have $\{| \xi_n - \xi | \geq \varepsilon\} \subseteq \{\| \zeta_n - \zeta \| \geq \varepsilon\}$
$\implies \mathcal{M}\{| \xi_n - \xi | \geq \varepsilon\} \leq \mathcal{M}\{\| \zeta_n - \zeta \| \geq \varepsilon\}$.
Therefore

$$\Big\{n \in \mathbb{N} : \mathcal{M}(| \xi_n - \xi \| \geq \varepsilon) \geq \delta\Big\} \subseteq \Big\{n \in \mathbb{N} : \mathcal{M}\{\| \zeta_n - \zeta \| \geq \varepsilon\} \geq \delta\Big\} \in \mathcal{I}.$$

Hence $\{n \in \mathbb{N} : \mathcal{M}(| \xi_n - \xi | \geq \varepsilon) \geq \delta\} \in \mathcal{I}$.
Similarly $\{n \in \mathbb{N} : \mathcal{M}(| \eta_n - \eta | \geq \varepsilon) \geq \delta\} \in \mathcal{I}$.
This completes the proof. ∎

Theorem 3. *The complex uncertain sequence* (ζ_n) *where* $\zeta_n = \xi_n + i\eta_n$ *is* $\mathcal{I}$*-convergent in distribution to* $\zeta = \xi + i\eta$ *if the uncertain sequence* (ξ_n) *and* (η_n) *are* $\mathcal{I}$*-convergent in distribution to* ξ *and* η*, respectively.*

Proof. Let $\Phi(z), \Phi_1(z), \Phi_2(z), ...$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, ...$ respectively and $\phi(x), \phi_n(x), \phi(y), \phi_n(y)$ be the uncertainty distribution of uncertain variables ξ, ξ_n, η, η_n respectively. Let (ξ_n) and (η_n) be $\mathcal{I}$-convergent in distribution to ξ and η, respectively.
Then from the definition of $\mathcal{I}$-convergent in distribution of uncertain sequences, it follows that for every $\varepsilon > 0$,

$\{n \in \mathbb{N} :| \phi_n(x) - \phi(x) | \geq \frac{\varepsilon}{2}\} \in \mathcal{I}$, for all x at which $\phi(x)$ is continuous and $\{n \in \mathbb{N} :| \phi_n(y) - \phi(y) | \geq \frac{\varepsilon}{2}\} \in \mathcal{I}$, for all y at which $\phi(y)$ is continuous.

$$\begin{aligned}\text{Now } \ ||\Phi_n(z)-\Phi(z)|| &= |\mathcal{M}\{\xi_n \le x, \eta_n \le y\} - \mathcal{M}\{\xi \le x, \eta \le y\}| \\ &= |\mathcal{M}\{\xi_n \le x\} \wedge \mathcal{M}\{\eta_n \le y\} - \mathcal{M}\{\xi \le x\} \wedge \mathcal{M}\{\eta \le y\}| \\ &= |\phi_n(x) \wedge \phi_n(y) - \phi(x) \wedge \phi(y)| \\ &= |\min\{\phi_n(x), \phi_n(y)\} - \min\{\phi(x), \phi(y)\}| \\ &= \Big|\tfrac{\phi_n(x)+\phi_n(y)+|\phi_n(x)-\phi_n(y)|}{2} - \tfrac{\phi(x)+\phi(y)+|\phi(x)-\phi(y)|}{2}\Big|.\end{aligned}$$

Then it can be easily shown that, $||\Phi_n(z)-\Phi(z)|| \le |\phi_n(x)-\phi(x)|+|\phi_n(y)-\phi(y)|$.

Therefore $\Big\{n \in \mathbb{N} : ||\Phi_n(z)-\Phi(z)|| \ge \varepsilon\Big\}$

$$\subseteq \Big\{n \in \mathbb{N} : |\phi_n(x)-\phi(x)| \ge \tfrac{\varepsilon}{2}\Big\} \cup \Big\{n \in \mathbb{N} : |\phi_n(y)-\phi(y)| \ge \tfrac{\varepsilon}{2}\Big\} \in \mathcal{I}.$$

Hence $\Big\{n \in \mathbb{N} : ||\Phi_n(z)-\Phi(z)|| \ge \varepsilon\Big\} \in \mathcal{I}$, for all z at which $\Phi(z)$ is continuous. This completes the proof. ■

Theorem 4. *If a complex uncertain sequence (ζ_n) is $\mathcal{I}$-convergent in mean to ζ, then it is $\mathcal{I}$-convergent in measure to ζ.*

Proof. Let the complex uncertain sequence (ζ_n) is $\mathcal{I}$-convergent in mean to ζ. Then from the definition of $\mathcal{I}$-convergent in mean of complex uncertain sequences, it follows that for every $\delta > 0$,

$$\Big\{n \in \mathbb{N} : E\Big[\ || \zeta_n(\gamma) - \zeta(\gamma) ||\ \Big] \ge \delta\Big\} \in \mathcal{I}.$$

Using Markov inequality we can see that for given $\varepsilon \ge 1, \delta > 0$, we have

$$\mathcal{M}\{|| \zeta_n(\gamma) - \zeta(\gamma) || \ge \varepsilon\} \le \frac{E\Big[||\zeta_n(\gamma)-\zeta(\gamma)||\Big]}{\varepsilon} \le E\Big[\ || \zeta_n(\gamma) - \zeta(\gamma) ||\ \Big].$$

Therefore $\Big\{n \in \mathbb{N} : \mathcal{M}(|| \zeta_n(\gamma) - \zeta(\gamma) || \ge \varepsilon) \ge \delta\Big\}$

$$\subseteq \Big\{n \in \mathbb{N} : E\Big[\ || \zeta_n(\gamma) - \zeta(\gamma) ||\ \Big] \ge \delta\Big\} \in \mathcal{I}.$$

Hence $\Big\{n \in \mathbb{N} : \mathcal{M}(|| \zeta_n(\gamma) - \zeta(\gamma) || \ge \varepsilon) \ge \delta\Big\} \in \mathcal{I}$.

Thus (ζ_n) is $\mathcal{I}$-convergent in measure to ζ and the theorem is proved. ■

Remark 1. Converse of the above theorem is not true. i.e. $\mathcal{I}-$convergent in measure does not imply $\mathcal{I}-$convergent in mean.

Example 2. Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, ...\}$ with

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \frac{1}{n}, & if \ \sup\limits_{\gamma_n \in \Lambda} \frac{1}{n} < \frac{1}{2} \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \frac{1}{n}, & if \ \sup\limits_{\gamma_n \in \Lambda^c} \frac{1}{n} < \frac{1}{2} \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Also we defined the complex uncertain variables ζ_n by

$$\zeta_n(\gamma) = \begin{cases} 0, & \text{otherwise} \\ ni, & if\ \gamma = \gamma_n \end{cases}$$

for $n \in \mathbb{N}$ and $\zeta \equiv 0$. Take $\mathcal{I}$=$\mathcal{I}_d$. For $\varepsilon, \delta > 0$ and we have

$$\begin{aligned} &\{n \in \mathbb{N} : \mathcal{M}(\| \zeta_n - \zeta \| \geq \varepsilon) \geq \delta\} \\ &\quad = \{n \in \mathbb{N} : \mathcal{M}(\gamma : \| \zeta_n(\gamma) - \zeta(\gamma) \| \geq \varepsilon) \geq \delta\} \\ &\quad = \{n \in \mathbb{N} : \mathcal{M}\{\gamma_n\} \geq \delta\} \in \mathcal{I}. \end{aligned}$$

The sequence (ζ_n) is therefore $\mathcal{I}$-convergent in measure to ζ. However, the uncertainty distribution of uncertain variable (ζ_n) is as follows for each n,

$$\Phi_n(z) = \begin{cases} 0, & \text{if } \ x < 0, y < \infty, \\ 0, & \text{if } \ x > 0, y < 0, \\ 1 - \frac{1}{n}, & \text{if } \ x \geq 0, \ 0 \leq y < n, \\ 1, & \text{if } \ x \geq 0, \ y \geq n. \end{cases}$$

So for each n, we have

$$\begin{aligned} &\Big\{ n \in \mathbb{N} : E\Big[\ \| \zeta_n(\gamma) - \zeta(\gamma) \| \ \Big] \geq \varepsilon \Big\} \\ &\qquad = \Big\{ n \in \mathbb{N} : \Big[\int_0^\infty \int_0^n 1 - \Big(1 - \tfrac{1}{n} \Big) \, dydx \Big] \geq \varepsilon \Big\} \notin \mathcal{I}. \end{aligned}$$

Therefore the sequence (ζ_n) does not $\mathcal{I}-$convergent in mean to ζ.

Theorem 5. *Assume that a complex uncertain sequence (ζ_n) with real part (ξ_n) and imaginary part (η_n) are $\mathcal{I}$-convergent in measure to ξ and η, respectively. Then the complex uncertain sequence (ζ_n) is $\mathcal{I}$-convergent in distribution to $\zeta = \xi + i\eta$.*

Proof. Let $z = x + iy$ be a given continuity point of the complex uncertainty distribution Φ. On the other hand, for any $\alpha > x, \beta > y$, we have

$$\begin{aligned} \{\xi_n \leq x, \eta_n \leq y\} = &\{\xi_n \leq x, \eta_n \leq y, \xi \leq \alpha, \eta \leq \beta\} \\ &\cup \{\xi_n \leq x, \eta_n \leq y, \xi > \alpha, \eta > \beta\} \\ &\cup \{\xi_n \leq x, \eta_n \leq y, \xi \leq \alpha, \eta > \beta\} \\ &\cup \{\xi_n \leq x, \eta_n \leq y, \xi > \alpha, \eta \leq \beta\} \\ &\subset \{\xi \leq \alpha, \eta \leq \beta\} \\ &\cup \{| \xi_n - \xi | \geq \alpha - x\} \cup \{| \eta_n - \eta | \geq \beta - y\}. \end{aligned}$$

It follows from the subadditivity axiom that

$$\Phi_n(z) = \Phi_n(x+iy) \leq \Phi(\alpha+i\beta)+\mathcal{M}\{\mid \xi_n-\xi \mid\geq \alpha-x\}+\mathcal{M}\{\mid \eta_n-\eta \mid\geq \beta-y\}.$$

Since (ξ_n) and (η_n) are $\mathcal{I}$-convergent in measure to ξ and η, respectively. So for any small $\varepsilon, \delta > 0$ we have

$$\{n \in \mathbb{N} : \mathcal{M}(\| \xi_n - \xi \|\geq \alpha - x) \geq \delta\} \in \mathcal{I}$$
$$\text{and } \{n \in \mathbb{N} : \mathcal{M}(\| \eta_n - \eta \|\geq \beta - y) \geq \delta\} \in \mathcal{I}.$$

Thus we obtain $\mathcal{I} - \limsup\limits_{n\to\infty} \Phi_n(z) \leq \Phi(\alpha + i\beta)$ for any $\alpha > x, \beta > y$. Letting $\alpha + i\beta \to x + iy$, we get

$$\mathcal{I} - \limsup_{n\to\infty} \Phi_n(z) \leq \Phi(z) \tag{1}$$

On the other hand, for any $\gamma < x, \kappa < y$ we have,

$$\begin{aligned}\{\xi \leq \gamma, \eta \leq \kappa\} =& \{\xi_n \leq x, \eta_n \leq y, \xi \leq \gamma, \eta \leq \kappa\}\\ &\cup \{\xi_n > x, \eta_n > y, \xi \leq \gamma, \eta \leq \kappa\}\\ &\cup \{\xi_n > x, \eta_n \leq y, \xi \leq \gamma, \eta \leq \kappa\}\\ &\cup \{\xi_n \leq x, \eta_n > y, \xi \leq \gamma, \eta \leq \kappa\}\\ &\subset \{\xi_n \leq x, \eta_n \leq y\} \cup \{\mid \xi_n - \xi \mid\geq x - \gamma\}\\ &\cup \{\mid \eta_n - \eta \mid\geq y - \kappa\}.\end{aligned}$$

It follows from the subadditivity axiom that
$\Phi(\gamma + i\kappa) \leq \Phi_n(x + iy) + \mathcal{M}\{\mid \xi_n - \xi \mid\geq x - \gamma\} + \mathcal{M}\{\mid \eta_n - \eta \mid\geq y - \kappa\}$.
Since (ξ_n) and (η_n) are $\mathcal{I}$-convergent in measure to ξ and η, respectively. So for any small $\varepsilon > 0$ we have

$$\{n \in \mathbb{N} : \mathcal{M}(\| \xi_n - \xi \|\geq x - \gamma) \geq \delta\} \in \mathcal{I}$$
$$\text{and } \{n \in \mathbb{N} : \mathcal{M}(\| \eta_n - \eta \|\geq y - \kappa) \geq \delta\} \in \mathcal{I}.$$

Thus we obtain $\Phi(\gamma + i\kappa) \leq \liminf_{n\to\infty} \Phi_n(x + iy)$ for any $\gamma < x, \kappa < y$. Letting $\gamma + i\kappa \to x + iy$, we get

$$\Phi(z) \leq \mathcal{I} - \liminf_{n\to\infty} \Phi_n(z) \tag{2}$$

It follows from (1) and (2) that $\Phi_n(z) \to \Phi(z)$ as $n \to \infty$ .i.e., the complex uncertain sequence (ζ_n) is $\mathcal{I}-$convergent in distribution to $\zeta = \xi + i\eta$. ■

Remark 2. Converse of the above theorem is not necessarily true, i.e. $\mathcal{I}-$convergent in distribution does not imply $\mathcal{I}-$convergent in measure.

Example 3. Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}(\gamma_1) = \mathcal{M}(\gamma_1) = \frac{1}{2}$. We define a complex uncertain variable as

$$\zeta(\gamma) = \begin{cases} i, & \text{if } \gamma = \gamma_1, \\ -i, & \text{if } \gamma = \gamma_2. \end{cases}$$

We also define $\zeta_n = -\zeta$ for $n = 1, 2, ...$ Take $\mathcal{I}$=$\mathcal{I}_d$.
Then the sequence (ζ_n) and ζ have the same distribution as:

$$\Phi_n(z) = \Phi_n(x + iy) = \begin{cases} 0, & \text{if } \ x < 0, -\infty < y < +\infty, \\ 0, & \text{if } \ x \geq 0, y < -1, \\ \frac{1}{2}, & \text{if } \ x \geq 0, -1 \leq y < 1, \\ 1, & \text{if } \ x \geq 0, y \geq 1. \end{cases}$$

So the sequence (ζ_n) is $\mathcal{I}-$convergent in distribution to ζ.
However, for a given $\varepsilon, \delta > 0$, we have

$$\{n \in \mathbb{N} : \mathcal{M}(\gamma :\| \zeta_n(\gamma) - \zeta(\gamma) \| \geq \varepsilon) \geq \delta\} \notin \mathcal{I}.$$

Thus the sequence (ζ_n) is not $\mathcal{I}-$convergent in measure to ζ. In addition, since $\zeta_n = -\zeta$ for $n = 1, 2, ...$, the sequence (ζ_n) is not $\mathcal{I}-$convergent a.s. to ζ.

Theorem 6. *$\mathcal{I}-$convergent in measure does not imply $\mathcal{I}-$convergent a.s.*

Example 4. Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. For any positive integer n, there exists an integer r such that $n = 2^r + k$ where k is an integer between 0 and $2^r - 1$. Then we define a complex uncertain variable by

$$\zeta_n(\gamma) = \begin{cases} i, & \text{if } \ \frac{k}{2^m} \leq \gamma \leq \frac{k+1}{2^m}; \\ 0, & \ \text{otherwise}; \end{cases}$$

for $n = 1, 2, ...$ and $\zeta \equiv 0$. Take $\mathcal{I}$=$\mathcal{I}_d$. However for small $\varepsilon, \delta > 0$ and $n \geq 2$, we have

$$\begin{aligned} &\{n \in \mathbb{N} : \mathcal{M}(\gamma :\| \zeta_n(\gamma) - \zeta(\gamma) \| \geq \varepsilon) \geq \delta\} \\ &= \{n \in \mathbb{N} : \mathcal{M}(\gamma_n) \geq \delta\} \in \mathcal{I}. \end{aligned}$$

Thus, the sequence (ζ_n) is $\mathcal{I}$-convergent in measure to ζ. In addition for every $\varepsilon > 0$, we have

$$\{n \in \mathbb{N} : E(\| \zeta_n - \zeta \| \geq \varepsilon)\} \in \mathcal{I}.$$

Hence the sequence (ζ_n) is $\mathcal{I}$-convergent in mean to ζ. But, for any $\gamma \in [0, 1]$, there is an infinite number of intervals of the form $[\frac{k}{2^m}, \frac{k+1}{2^m}]$ containing γ. Thus $(\zeta_n(\gamma))$ is not $\mathcal{I}$-convergent to 0 i.e., the sequence (ζ_n) is not $\mathcal{I}$-convergent a.s. to ζ.

Theorem 7. *$\mathcal{I}-$convergent a.s. does not imply $\mathcal{I}-$convergent in mean.*

Proposition 1. *Let $\zeta, \zeta_1, \zeta_2, \ldots$ be complex uncertain variables. Then (ζ_n) is $\mathcal{I}-$convergent a.s. to ζ if and only if for any $\varepsilon, \delta > 0$, we have*

$$\{n \in \mathbb{N} : \mathcal{M}\Big(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \| \zeta_n - \zeta \| \geq \varepsilon \Big) \geq \delta\} \in \mathcal{I}.$$

Proof. From the definition of $\mathcal{I}-$convergent a.s., we have there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that, for every $\varepsilon > 0$

$$\{n \in \mathbb{N} : \| \zeta_n(\gamma) - \zeta(\gamma) \| \geq \varepsilon\} \in \mathcal{I}.$$

Then for any $\varepsilon > 0$, there exists k such that $\| \zeta_n - \zeta \| < \varepsilon$ where $n > k$ and for any $\gamma \in \Lambda$, that is equivalent to

$$\{n \in \mathbb{N} : \mathcal{M}\Big(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \| \zeta_n - \zeta \| < \varepsilon \Big) \geq 1\} \in \mathcal{I}.$$

It follows from the duality axiom of uncertain measure that

$$\{n \in \mathbb{N} : \mathcal{M}\Big(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \| \zeta_n - \zeta \| \geq \varepsilon \Big) \geq \delta\} \in \mathcal{I}.$$

■

Proposition 2. *If the complex uncertain sequence (ζ_n) is $\mathcal{I}-$convergent uniformly a.s. to ζ, then (ζ_n) is $\mathcal{I}-$convergent a.s. to ζ.*

Proof. A sequence (ζ_n) is $\mathcal{I}-$convergent uniformly a.s. to ζ, then by definition we can say,

$$\Big\{n \in \mathbb{N} : \mathcal{M}\Big(\bigcup_{n=k}^{\infty} \| \zeta_k - \zeta \| \geq \varepsilon \Big) \geq \delta \Big\} \in \mathcal{I}.$$

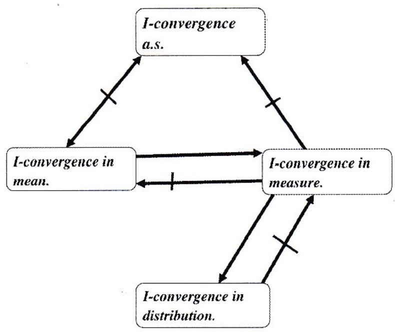
Since $\mathcal{M}\Big(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \Big\{ \| \zeta_n - \zeta \| \geq \varepsilon \Big\} \Big) \leq \mathcal{M}\Big(\bigcup_{n=k}^{\infty} \Big\{ \| \zeta_n - \zeta \| \geq \varepsilon \Big\} \Big)$, so we have $\Big\{ n \in \mathbb{N} : \mathcal{M}\Big(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \Big\{ \| \zeta_n - \zeta \| \geq \varepsilon \Big\} \Big) \geq \delta \Big\}$

$$\subseteq \Big\{ n \in \mathbb{N} : \mathcal{M}\Big(\bigcup_{n=k}^{\infty} \Big\{ \| \zeta_n - \zeta \| \geq \varepsilon \Big\} \Big) \geq \delta \Big\}.$$

Thus we get $\Big\{ n \in \mathbb{N} : \mathcal{M}\Big(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \Big\{ \| \zeta_n - \zeta \| \geq \varepsilon \Big\} \Big) \geq \delta \Big\} \in \mathcal{I}$.

By Proposition (1), (ζ_n) $\mathcal{I}-$convergent a.s. to ζ. ■

3. Diagrammatic representation



Then, the interrelationships among these concepts are depicted in the figure above.

4. Conclusions

The concept of statistical convergence of complex uncertain variables have been studied by Tripathy and Nath [29]. In this article for the first time, we introduce the concept $\mathcal{I}$-convergence, which is the generalization of statistical convergence of complex uncertain variables. These results unify and generalizes the existing results. It may attract the future researcher's in this direction.

Acknowledgment. The first author is grateful to the **Council of Scientific and Industrial Research, India** for their fellowships funding under the **CSIR-JRF** scheme **(File No: 09/0714(11674)/2021-EMR-I)** during the preparation of this paper.

References

[1] Chen X., Ning X., Wang X., Convergence of complex uncertain sequences, *J. Intell. Fuzzy Syst.,* 30(6)(2016), 3357-3366.

[2] Conor J.S., The statistical and strong p-Cesaro convergence of sequences, *Analysis,* 8(1988), 47-63.

[3] Das B., Tripathy B.C., Debnath P., Bhattacharya B., Almost convergence of complex uncertain double sequences, *Filomat,* 35(1)(2021), 61-78.

[4] Debnath S., Das B., Statistical convergence of order α for complex uncertain sequences, *J. Uncertain Syst.,* 14(2)(2021), doi:10.1142/S1752890921500124.

[5] On Rough Convergence of Complex Uncertain Sequences J. Uncertain Syst. 14(4)(2021), doi:10.1142/S1752890921500215

[6] Esi A.,Debnath S., Saha S., Asymptotically double λ_2-statistically equivalent sequences of interval numbers, *Mathematica,* 62(85)(2020), 39-46.

[7] FAST H., Sur la convergence statistique, *Colloq. Math.,* 2(3-4)(1951), 241-244.

[8] FREEDMAN A.R.,SEMBER J.J., RAPHAEL M., Some Cesàro-type summability spaces, *Proc. Lond. Math. Soc.,* 3(3)(1978), 508-520.

[9] FRIDY J.A., On statistically convergence, *Analysis,* 5(1985), 301-313.

[10] KADAK U., MOHIUDDINE S.A., Generalized statistically almost convergence based on the difference operator which includes the (p, q)-gamma function and related approximation theorems, *Results Math.,* (2018), 73:9.

[11] KHAN V.A., KHAN I.A., HAZARIKA B., RAHMAN Z., Strongly $\mathcal{I}$-deferred Cesàro summablity and μ-deferred $\mathcal{I}$-statistically convergence in amenable semigroups, *Filomat,* 36(14)(2022), 4839-4856.

[12] KHAN V.A., HAZARIKA B., KHAN I.A., RAHMAN Z., A study on $\mathcal{I}$-deferred strongly Cesàro summable and μ-deferred $\mathcal{I}$-statistical convergence for complex uncertain sequences, *Filomat,* 36(20)(2022), 7001-7020.

[13] KHAN V.A., HAZARIKA B., KHAN I.A., TUBA U., $\mathcal{I}$-deferred strongly Cesàro summable and μ-deferred $\mathcal{I}$-statistically convergent sequence spaces, *Ricerche Mat.,* (2021), doi:10.1007/s11587-021-00619-8.

[14] KIŞI Ö, $S_\lambda(\mathcal{I})$-convergence of complex uncertain sequences, *Mat. Stud.,* 51(2)(2019), 183-194.

[15] KOSTYRKO P., MAČAJ M., SLEZIAK M., ŠALÁT T., $\mathcal{I}$-convergence, *Real Anal. Exchange,* 26(2000/2001), 669-686.

[16] LIU B., *Uncertainty Theory,* 4th edition, Springer-Verlag, Berlin, doi:10.1007/978-3-662-44354-5.

[17] MOHIUDDINE S.A., ASIRI A., HAZARIKA B., Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems, *Int. J. Gen. Syst.,* 48(5)(2019), 492-506.

[18] MOHIUDDINE S.A., HAZARIKA B., ALGHAMDI M.A., Ideal relatively uniform convergence with Korovkin and Voronovskaya types approximation theorem, *Filomat,* 33(14)(2019), 4549-4560.

[19] MOHIUDDINE S.A., ALAMRI B.A.S., Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems, *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. RACSAM,* 113(3)(2019), 1955-1973.

[20] MURSALEEN M., DEBNATH S., RAKSHIT D., $\mathcal{I}$-statistical limit superior and $\mathcal{I}$-statistical limit inferior, *Filomat,* 31(7)(2018), 2103-2108.

[21] NATH P.K., TRIPATHY B.C., $\mathcal{I}^{\mathcal{K}}$-convergence, *Ann. Univ. Craiova-Math. Comput. Sci. Ser.,* 46(1)(2019), 139-149.

[22] PENG Z., Complex uncertain variable, *Doctoral Dissertation, Tsinghua University, ...*

[23] ROY S., TRIPATHY B.C., SAHA S., Some results on p-distance and sequence of complex uncertian variables, *Commun. Korean. Math. Soc.,* 35(3)(2020), 907-916.

[24] SAHA S., TRIPATHY B.C., ROY S., On almost convergence of complex uncertain sequences, *New Math. Nat. Comput.,* 16(3)(2020), 573-580.

[25] SAVAS E., DEBNATH S., On lacunary statistically ϕ-convergence, *Note Di Matematica,* 39(2)(2019), 111-120.

[26] Savas E., Debnath S., Rakshit D., On $\mathcal{I}$-statistically rough convergence, *Publications de l'Institut Mathmatique,* 105(119)(2019), 145-150.

[27] Steinhaus H., Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.,* 2(1)(1951), 73-74.

[28] Tripathy B.C., Tripathy B.K., On $\mathcal{I}$-convergent double sequences, *Soochow J. Math.,* 31(4)(2005), 549-560.

[29] Tripathy B.C., Nath P.K., Statistical convergence of complex uncertain sequences, *New Math. Nat. Comput.,* 13(3)(2017), 359-374.

Amit Halder
Department of Mathematics
Tripura University (A Central University)
Suryamaninagar-799022, Agartala,India
e-mail: mit2905halder@gmail.com

Bijoy Das
Department of Mathematics
Tripura University (A Central University)
Suryamaninagar-799022, Agartala,India
ee-mail: bijoy.mathematics@tripurauniv.in

Shyamal Debnath
Department of Mathematics
Tripura University (A Central University)
Suryamaninagar-799022, Agartala,India
e-mail: shyamalnitamath@gmail.com

Received on 28.12.2022 and, in revised form, on 21.09.2023.