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Sung Guen Kim

# THE NORMING SETS OF MULTILINEAR FORMS ON $\mathbb{R}^2$ WITH A ROTATED SUPREMUM NORM

Abstract. Let $n \in \mathbb{N}$, $n \geq 2$. An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of $T \in \mathcal{L}(^nE)$ if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|$, where $\mathcal{L}(^nE)$ denotes the space of all continuous $n$-linear form on $E$. For $T \in \mathcal{L}(^nE)$, we define

$$\text{Norm}(T) = \Big\{(x_1, \ldots, x_n) \in E^n : (x_1, \ldots, x_n) \text{ is a norming point of } T\Big\}.$$

Norm$(T)$ is called the *norming set* of $T$.
Let $0 \leq \theta < \frac{\pi}{4}$ and $\ell^2_{(\infty,\theta)} = \mathbb{R}^2$ with the rotated supremum norm

$$\|(x, y)\|_{(\infty,\theta)} = \max\Big\{|x\cos\theta + y\sin\theta|,\ |x\sin\theta - y\cos\theta|\Big\}.$$

In this paper, we characterize Norm$(T)$ for every $T \in \mathcal{L}(^m\ell^2_{(\infty,\theta)})$ for $m \geq 2$.

Key words: norming points, multilinear forms.

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## 1. Introduction

In 1961 Bishop and Phelps [2] showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jiménez-Sevilla and Payá [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n \in \mathbb{N}$, $n \geq 2$. We write $S_E$ for the unit sphere of a Banach space $E$. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous $n$-linear forms on $E$ endowed with the norm $\|T\| = \sup_{(x_1,\cdots,x_n)\in S_E\times\cdots\times S_E} |T(x_1,\cdots,x_n)|$. $T \in \mathcal{L}(^nE)$ is symmetric if $T(x_{\sigma(1)},\ldots,x_{\sigma(n)}) = T(x_1,\ldots,x_n)$ for every $x_1,\ldots,x_n \in E$ and permutation $\sigma$ on $\{1,\ldots,n\}$. $\mathcal{L}_s(^nE)$ denote the closed subspace of all continuous symmetric $n$-linear forms on $E$. An element $(x_1,\ldots,x_n) \in E^n$ is called a *norming point* of $T$ if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1,\ldots,x_n)| = \|T\|$.

For $T \in \mathcal{L}(^nE)$, we define

$$\text{Norm}(T) = \Big\{(x_1,\ldots,x_n) \in E^n : (x_1,\ldots,x_n) \text{ is a norming point of } T\Big\}.$$

Norm$(T)$ is called the *norming set* of $T$. Notice that $(x_1,\ldots,x_n) \in \text{Norm}(T)$ if and only if $(\epsilon_1 x_1,\ldots,\epsilon_n x_n) \in \text{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1,\ldots,n)$. Indeed, if $(x_1,\ldots,x_n) \in \text{Norm}(T)$, then

$$|T(\epsilon_1 x_1,\ldots,\epsilon_n x_n)| = |\epsilon_1\cdots\epsilon_n T(x_1,\ldots,x_n)| = |T(x_1,\ldots,x_n)| = \|T\|,$$

which shows that $(\epsilon_1 x_1,\ldots,\epsilon_n x_n) \in \text{Norm}(T)$. If $(\epsilon_1 x_1,\ldots,\epsilon_n x_n) \in \text{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1,\ldots,n)$, then

$$(x_1,\ldots,x_n) = \Big(\epsilon_1(\epsilon_1 x_1),\ldots,\epsilon_n(\epsilon_n x_n)\Big) \in \text{Norm}(T).$$

The following examples show that Norm$(T) = \emptyset$ or an infinite set.

**Example.** $(a)$ Let

$$T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i y_i \in \mathcal{L}_s(^2c_0).$$

We claim that Norm$(T) = \emptyset$. Obviously, $\|T\| = 1$. Assume that Norm$(T) \neq \emptyset$. Let $\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) \in \text{Norm}(T)$. Then,

$$1 = \Big|T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big)\Big| \leq \sum_{i=1}^{\infty} \frac{1}{2^i}|x_i|\ |y_i| \leq \sum_{i=1}^{\infty}\frac{1}{2^i} = 1,$$

which shows that $|x_i| = |y_i| = 1$ for all $i \in \mathbb{N}$. Hence, $(x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}} \notin c_0$. This is a contradiction. Therefore, Norm$(T) = \emptyset$.

$(b)$ Let

$$T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) = x_1 y_1 \in \mathcal{L}_s(^2c_0).$$

Then,

$$\text{Norm}(T) = \Big\{ \Big( (\pm 1, x_2, x_3, \ldots),\ (\pm 1, y_2, y_3, \ldots) \Big) \in c_0 \times c_0 : |x_j| \le 1, \\ |y_j| \le 1 \text{ for } j \ge 2 \Big\}.$$

A mapping $P : E \to \mathbb{R}$ is a continuous $n$-homogeneous polynomial if there exists a continuous $n$-linear form $L$ on the product $E \times \cdots \times E$ such that $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^n E)$ the Banach space of all continuous $n$-homogeneous polynomials from $E$ into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$.

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^n E)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}(^n E)$, we define

$$\text{Norm}(P) = \Big\{ x \in E : x \text{ is a norming point of } P \Big\}.$$

Norm$(P)$ is called the *norming set* of $P$. Notice that Norm$(P) = \emptyset$ or a finite set or an infinite set.

Kim [7] classified Norm$(P)$ for every $P \in \mathcal{P}(^2 \ell_\infty^2)$, where $\ell_\infty^2 = \mathbb{R}^2$ with the supremum norm.

If Norm$(T) \neq \emptyset$, $T \in \mathcal{L}(^n E)$ is called a *norm attaining* $n$-linear form and if Norm$(P) \neq \emptyset$, $P \in \mathcal{P}(^n E)$ is called a *norm attaining* $n$-homogeneoue polynomial.(See [3])

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

It seems to be natural and interesting to study about Norm$(T)$ for $T \in \mathcal{L}(^n E)$. For $m \in \mathbb{N}$, let $\ell_p^m := \mathbb{R}^m$ with the the $\ell_p$-norm $(1 \le p \le \infty)$. Notice that if $E = \ell_1^m$ or $\ell_\infty^2$ and $T \in \mathcal{L}(^n E)$, Norm$(T) \neq \emptyset$ since $S_E$ is compact. Kim [6, 8, 9, 11] classified Norm$(T)$ for every $T \in \mathcal{L}_s(^2 \ell_\infty^2), \mathcal{L}(^2 \ell_\infty^2), \mathcal{L}(^2 \ell_1^2)$, $\mathcal{L}_s(^2 \ell_1^3)$ or $\mathcal{L}_s(^3 \ell_1^2)$. Kim [12] classified Norm$(T)$ for every $T \in \mathcal{L}(^2 \mathbb{R}^2_{h(w)})$, where $\mathbb{R}^2_{h(w)}$ denotes the planewith the hexagonal norm with weight $0 < w < 1$ $\|(x, y)\|_{h(w)} = \max \Big\{ |y|,\ |x| + (1 - w)|y| \Big\}$.

Let $0 \le \theta < \frac{\pi}{4}$ and $\ell^2_{(\infty,\theta)} = \mathbb{R}^2$ with the rotated supremum norm

$$\|(x, y)\|_{(\infty,\theta)} = \max \Big\{ |x \cos\theta + y \sin\theta|, |x \sin\theta - y \cos\theta| \Big\}.$$

Kim [10] introduced the rotated supremum norm on the plane and classified the extreme, exposed and smooth points of the unit ball of $\mathcal{L}(^2 \ell^2_{(\infty,\theta)})$, respectively.

In this paper, we characterize Norm$(T)$ for every $T \in \mathcal{L}(^m \ell^2_{(\infty,\theta)})$ for $m \ge 2$.

## 2. Results

**Lemma A** ([11]). *Let $n, m \geq 2$. Let $T \in \mathcal{L}(^m\ell_1^n)$ with*

$$T\Big(\big(x_1^{(1)}, \ldots, x_n^{(1)}\big), \ldots, \big(x_1^{(m)}, \ldots, x_n^{(m)}\big)\Big) = \sum_{1 \leq i_k \leq n,\ 1 \leq k \leq m} a_{i_1 \cdots i_m}\ x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}$$

*for some $a_{i_1 \cdots i_m} \in \mathbb{R}$. Then*

$$\|T\| = \max\Big\{|a_{i_1 \cdots i_m}| : 1 \leq i_k \leq n,\ 1 \leq k \leq m\Big\}.$$

**Lemma B.** *Let $0 \leq \theta < \frac{\pi}{4}$. Then, $\ell_1^2$ and $\ell_{(\infty,\theta)}^2$ are isometric by the mapping $\phi : \ell_1^2 \rightarrow \ell_{(\infty,\theta)}^2$ such that*

$$\phi(x, y) = \Big((x + y)cos\theta + (y - x)sin\theta,\ (x - y)cos\theta + (x + y)sin\theta\Big).$$

**Proof.** It follows that for $(x, y) \in \ell_1^2$,

$$\begin{aligned}
&\Big\|\Big((x + y)cos\theta + (y - x)sin\theta,\ (x - y)cos\theta + (x + y)sin\theta\Big)\Big\|_{(\infty,\theta)} \\
&= \max\Big\{\big|\cos\theta\big((x + y)\cos\theta + (y - x)\sin\theta\big) + \sin\theta\big((x - y)\cos\theta \\
&\qquad + (x + y)\sin\theta\big)\big|, \big|\sin\theta\big((x + y)\cos\theta + (y - x)\sin\theta\big) \\
&\qquad - \cos\theta\big((x - y)\cos\theta + (x + y)\sin\theta\big)\big|\Big\} \\
&= \max\Big\{|x + y|,\ |x - y|\Big\} = |x| + |y| = \|(x, y)\|_1.
\end{aligned}$$

■

The following shows relations between the norming sets of $\mathcal{L}(^m\ell_{(\infty,\theta)}^2)$ and $\mathcal{L}(^m\ell_1^2)$.

**Theorem C.** *Let $m \geq 2$. Let $T \in \mathcal{L}(^m\ell_{(\infty,\theta)}^2)$ with*

$$T\Big((x_1^{(1)}, x_2^{(1)}), \cdots, (x_1^{(m)}, x_2^{(m)})\Big) = \sum_{1 \leq k \leq m,\ i_k = 1,2} a_{i_1 \cdots i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}.$$

*Let $W_1 = (cos\theta - sin\theta,\ cos\theta + sin\theta),\ W_2 = (cos\theta + sin\theta,\ - cos\theta + sin\theta)$.*
*We define $S_T \in \mathcal{L}(^m\ell_1^2)$ by*

$$\begin{aligned}
S_T\Big((t_1^{(1)}, t_2^{(1)}), \cdots, (t_1^{(m)}, t_2^{(m)})\Big) &= T\Big(t_1^{(1)} W_1 + t_2^{(1)} W_2, \cdots, t_1^{(m)} W_1 + t_2^{(m)} W_2\Big) \\
&:= \sum_{1 \leq k \leq m,\ i_k = 1,2} A_{i_1 \cdots i_m} t_{i_1}^{(1)} \cdots t_{i_m}^{(m)}.
\end{aligned}$$

*The following assertions hold:*

(*a*) $\|T\| = \max\Big\{|A_{i_1\cdots i_m}| : i_k = 1,2 \text{ and } 1 \le k \le m\Big\}$;

(*b*) $\mathrm{Norm}(T) = \Big\{\Big(t_1^{(1)}W_1 + t_2^{(1)}W_2, \ldots, t_1^{(m)}W_1 + t_2^{(m)}W_2\Big) : \Big((t_1^{(1)}, t_2^{(1)}), \ldots, (t_1^{(m)}, t_2^{(m)})\Big) \in \mathrm{Norm}(S_T)\Big\}$.

**Proof.** (*a*). We claim that $\|T\| = \|S_T\|_{\mathcal{L}(^m\ell_1^2)}$.

Let $(x_1^{(j)}, x_2^{(j)}) \in S_{\ell^2_{(\infty,\theta)}}$ for $1 \le j \le m$. Let

$$t_1^{(j)} = \frac{1}{2}\Big((\cos\theta - \sin\theta)x_1^{(j)} + (\cos\theta + \sin\theta)x_2^{(j)}\Big),$$
$$t_2^{(j)} = \frac{1}{2}\Big((\cos\theta + \sin\theta)x_1^{(j)} + (-\cos\theta + \sin\theta)x_2^{(j)}\Big).$$

for $1 \le j \le m$. By Lemma B, $(t_1^{(j)}, t_2^{(j)}) \in S_{\ell_1^2}$ and $(x_1^{(j)}, x_2^{(j)}) = t_1^{(j)}W_1 + t_2^{(j)}W_2$ for $1 \le j \le m$.

It follows that

$$\begin{aligned}&\Big|T\Big((x_1^{(1)}, x_2^{(1)}), \cdots, (x_1^{(m)}, x_2^{(m)})\Big)\Big|\\ &\quad= \Big|T\Big(t_1^{(1)}W_1 + t_2^{(1)}W_2, \cdots, t_1^{(m)}W_1 + t_2^{(m)}W_2\Big)\Big|\\ &\quad= \Big|S_T\Big((t_1^{(1)}, t_2^{(1)}), \cdots, (t_1^{(m)}, t_2^{(m)})\Big)\Big| \le \|S_T\|_{\mathcal{L}(^m\ell_1^2)}.\end{aligned}$$

Thus, $\|T\| \le \|S\|_{\mathcal{L}(^m\ell_1^2)}$.

Let $(t_1^{(j)}, t_2^{(j)}) \in S_{\ell_1^2}$ for $1 \le j \le m$. It follows that

$$\begin{aligned}&\Big|S_T\Big((t_1^{(1)}, t_2^{(1)}), \cdots, (t_1^{(m)}, t_2^{(m)})\Big)\Big|\\ &\quad= \Big|T\Big(t_1^{(1)}W_1 + t_2^{(1)}W_2, \cdots, t_1^{(m)}W_1 + t_2^{(m)}W_2\Big)\Big|\\ &\quad\le \|T\| \prod_{1\le j\le m} \Big\|t_1^{(j)}W_1 + t_2^{(j)}W_2\Big\|_{(\infty,\theta)}\\ &\quad= \|T\| \prod_{1\le j\le m} |t_1^{(j)}| + |t_2^{(j)}| = \|T\|.\end{aligned}$$

Thus, $\|S_T\|_{\mathcal{L}(^m\ell_1^2)} \le \|T\|$. Thus, the claim holds. By Lemma A, (a) is proved.

(*b*). Let

$$\mathcal{M} = \Big\{\Big(t_1^{(1)}W_1 + t_2^{(1)}W_2, \ldots, t_1^{(m)}W_1 + t_2^{(m)}W_2\Big) : \Big((t_1^{(1)}, t_2^{(1)}), \ldots, (t_1^{(m)}, t_2^{(m)})\Big) \in \mathrm{Norm}(S)\Big\}.$$

By Lemma B, $\mathcal{M} \subseteq (S_{\ell^2_{(\infty,\theta)}})^m := S_{\ell^2_{(\infty,\theta)}} \times \cdots \times S_{\ell^2_{(\infty,\theta)}}$. We will show that $\text{Norm}(T) = \mathcal{M}$.

($\subseteq$). Let $\Big((x_1^{(1)}, x_2^{(1)}), \cdots, (x_1^{(m)}, x_2^{(m)})\Big) \in \text{Norm}(T)$. It follows that

$$\begin{aligned} \|S_T\|_{\mathcal{L}(^m\ell_1^2)} &= \|T\| = \Big|T\Big((x_1^{(1)}, x_2^{(1)}), \cdots, (x_1^{(m)}, x_2^{(m)})\Big)\Big| \\ &= \Big|T\Big(t_1^{(1)}W_1 + t_2^{(1)}W_2, \ldots, t_1^{(m)}W_1 + t_2^{(m)}W_2\Big)\Big| \\ &= \Big|S_T\Big((t_1^{(1)}, t_2^{(1)}), \cdots, (t_1^{(m)}, t_2^{(m)})\Big)\Big|. \end{aligned}$$

Thus, $\Big((t_1^{(1)}, t_2^{(1)}), \cdots, (t_1^{(m)}, t_2^{(m)})\Big) \in \text{Norm}(S_T)$. Therefore,

$$\Big((x_1^{(1)}, x_2^{(1)}), \cdots, (x_1^{(m)}, x_2^{(m)})\Big) = \Big(t_1^{(1)}W_1 + t_2^{(1)}W_2, \ldots, t_1^{(m)}W_1 + t_2^{(m)}W_2\Big) \in \mathcal{M}.$$

($\supseteq$). Let $\Big(t_1^{(1)}W_1 + t_2^{(1)}W_2, \ldots, t_1^{(m)}W_1 + t_2^{(m)}W_2\Big) \in \mathcal{M}$. By the definition of $\mathcal{M}$, $\Big((t_1^{(1)}, t_2^{(1)}), \cdots, (t_1^{(m)}, t_2^{(m)})\Big) \in \text{Norm}(S_T)$.

It follows that

$$\begin{aligned} \|T\| &= \|S_T\|_{\mathcal{L}(^m\ell_1^2)} = \Big|S_T\Big((t_1^{(1)}, t_2^{(1)}), \cdots, (t_1^{(m)}, t_2^{(m)})\Big)\Big| \\ &= \Big|T\Big(t_1^{(1)}W_1 + t_2^{(1)}W_2, \ldots, t_1^{(m)}W_1 + t_2^{(m)}W_2\Big)\Big|. \end{aligned}$$

Since $t_1^{(j)}W_1 + t_2^{(j)}W_2 \in S_{\ell^2_{9\infty,\theta)}}$ for $1 \leq j \leq m$,

$$\Big(t_1^{(1)}W_1 + t_2^{(1)}W_2, \ldots, t_1^{(m)}W_1 + t_2^{(m)}W_2\Big) \in \text{Norm}(T).$$

This completes the proof. ■

**Theorem D.** *Let $m \geq 2,\ 0 \leq \theta < \frac{\pi}{4}$ and $T \in \mathcal{L}(^m\ell^2_{(\infty,\theta)})$ with*

$$T\Big((x_1^{(1)}, x_2^{(1)}), \cdots, (x_1^{(m)}, x_2^{(m)})\Big) = \sum_{1\leq k\leq m,\ i_k=1,2} a_{i_1\cdots i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}$$

*for some $a_{i_1\cdots i_m} \in \mathbb{R}$. Let $S_T \in \mathcal{L}(^m\ell_1^2)$ be the one in Theorem C. Write $T = (a_{i_1\cdots i_m})^t_{i_k=1,2}$ and $S_T = (A_{i_1\cdots i_m})^t_{i_k=1,2}$. Let $M_{m(\theta)}$ be the $2^m \times 2^m$-matrix such that $M_{m(\theta)}\ T = S_T$ for every $T \in \mathcal{L}(^m\ell^2_{(\infty,\theta)})$. Then $M_{m(\theta)}$ is invertible and $M^{-1}_{m(\theta)} = \frac{1}{2^m}\ M_{m(\theta)}$.*

**Proof.** Note that for every $T \in \mathcal{L}(^m\ell^2_{(\infty,\theta)})$,

$$\begin{aligned}
&\Big(M_{m(\theta)}T\Big)\Big((x_1^{(1)}, x_2^{(1)}), \cdots, (x_1^{(m)}, x_2^{(m)})\Big) \\
&= T\Big(\big((\cos\theta - \sin\theta)x_1^{(1)} + (\cos\theta + \sin\theta)x_2^{(1)}, \\
&\qquad (\cos\theta + \sin\theta)x_1^{(1)} + (-\cos\theta + \sin\theta)x_2^{(1)}\big), \\
&\qquad \cdots, \big((\cos\theta - \sin\theta)x_1^{(m)} + (\cos\theta + \sin\theta)x_2^{(m)}, \\
&\qquad (\cos\theta + \sin\theta)x_1^{(m)} + (-\cos\theta + \sin\theta)x_2^{(m)}\ \big)\Big).
\end{aligned}$$

*Claim.* $M^2_{m(\theta)} = 2^m$ Id, where Id is the identity matrix.

It follows that for every $T \in \mathcal{L}(^m\ell^2_{(\infty,\theta)})$,

$$\begin{aligned}
&M_{m(\theta)}\Big[\Big(M_{m(\theta)}\ T\Big)\Big((x_1^{(1)}, x_2^{(1)}), \cdots, (x_1^{(m)}, x_2^{(m)})\Big)\Big] \\
&= M_{m(\theta)}\ T\Big(\big((\cos\theta - \sin\theta)x_1^{(1)} + (\cos\theta + \sin\theta)x_2^{(1)}, \\
&\qquad (\cos\theta + \sin\theta)x_1^{(1)} + (-\cos\theta + \sin\theta)x_2^{(1)}\big), \\
&\qquad \cdots, \big((\cos\theta - \sin\theta)x_1^{(m)} + (\cos\theta + \sin\theta)x_2^{(m)}, \\
&\qquad (\cos\theta + \sin\theta)x_1^{(m)} + (-\cos\theta + \sin\theta)x_2^{(m)}\big)\Big) \\
&= T\Big(\Big((\cos\theta - \sin\theta)\Big[(\cos\theta - \sin\theta)x_1^{(1)} + (\cos\theta + \sin\theta)x_2^{(1)}\Big] \\
&\qquad + (\cos\theta + \sin\theta)\Big[(\cos\theta + \sin\theta)x_1^{(1)} + (-\cos\theta + \sin\theta)x_2^{(1)}\Big], \\
&\quad (\cos\theta + \sin\theta)\Big[(\cos\theta - \sin\theta)x_1^{(1)} + (\cos\theta + \sin\theta)x_2^{(1)}\Big] \\
&\qquad + (-\cos\theta + \sin\theta)\Big[(\cos\theta + \sin\theta)x_1^{(1)} + (-\cos\theta + \sin\theta)x_2^{(1)}\Big]\Big), \\
&\quad \cdots, \Big((\cos\theta - \sin\theta)\Big[(\cos\theta - \sin\theta)x_1^{(m)} + (\cos\theta + \sin\theta)x_2^{(m)}\Big] \\
&\qquad + (\cos\theta + \sin\theta)\Big[(\cos\theta + \sin\theta)x_1^{(m)} + (-\cos\theta + \sin\theta)x_2^{(m)}\Big], \\
&\quad (\cos\theta + \sin\theta)\Big[(\cos\theta - \sin\theta)x_1^{(m)} + (\cos\theta + \sin\theta)x_2^{(m)}\Big] \\
&\qquad + (-\cos\theta + \sin\theta)\Big[(\cos\theta + \sin\theta)x_1^{(m)} + (-\cos\theta + \sin\theta)x_2^{(m)}\Big]\Big)\Big) \\
&= T\Big((2x_1^{(1)}, 2x_2^{(1)}), \cdots, (2x_1^{(m)}, 2x_2^{(m)})\Big) \\
&= 2^m T\Big((x_1^{(1)}, x_2^{(1)}), \cdots, (x_1^{(m)}, x_2^{(m)})\Big),
\end{aligned}$$

which implies that $M^2_{m(\theta)}T = 2^m\ T$ for every $T \in \mathcal{L}(^m\ell^2_{(\infty,\theta)})$. Thus, the Claim holds. Since

$$[\det(M_{m(\theta)})]^2 = \det(M^2_{m(\theta)}) = \det(2^m \text{ Id}) = 2^{m^2},$$

$\det(M_{m(\theta)}) \neq 0$. Therefore, $M_{m(\theta)}$ is invertible and $M^{-1}_{m(\theta)} = \frac{1}{2^m} M_{m(\theta)}$. This completes the proof. ∎

**Corollary E.** *Let $m \geq 2$. Let $T \in \mathcal{L}({}^3\ell^2_{(\infty,\theta)})$ with*

$$T\Big((x_1^{(1)}, x_2^{(1)}), (x_1^{(2)}, x_2^{(2)}), (x_1^{(3)}, x_2^{(3)})\Big) = \sum_{1\leq k\leq 3,\ i_k=1,2} a_{i_1i_2i_3} x_{i_1}^{(1)} x_{i_2}^{(2)} x_{i_3}^{(3)}$$

*for some $a_{i_1i_2i_3} \in \mathbb{R}$. Then,*

$$\begin{aligned}
\|T\| &= \max\Big\{\Big|(\cos\theta - \sin\theta)^3 a_{111} + (\cos\theta + \sin\theta)^3 a_{222} \\
&+ \cos 2\theta(\cos\theta + \sin\theta)(a_{122} + a_{212} + a_{221}) \\
&+ \cos 2\theta(\cos\theta - \sin\theta)(a_{211} + a_{121} + a_{112})\Big|, \\
&\Big|(\cos\theta + \sin\theta)^3 a_{111} - (\cos\theta - \sin\theta)^3 a_{222} \\
&+ \cos 2\theta(\cos\theta - \sin\theta)(a_{122} + a_{212} + a_{221}) \\
&- \cos 2\theta(\cos\theta + \sin\theta)(a_{211} + a_{121} + a_{112})\Big|, \\
&\Big|\cos 2\theta(\cos\theta + \sin\theta)a_{111} + \cos 2\theta(\cos\theta - \sin\theta)a_{222} \\
&+ (\cos\theta - \sin\theta)^3 a_{122} - \cos 2\theta(\cos\theta + \sin\theta)(a_{212} + a_{221}) \\
&+ (\cos\theta + \sin\theta)^3 a_{211} - \cos 2\theta(\cos\theta - \sin\theta)(a_{121} + a_{112})\Big|, \\
&\Big|\cos 2\theta(\cos\theta + \sin\theta)a_{111} + \cos 2\theta(\cos\theta - \sin\theta)a_{222} \\
&- \cos 2\theta(\cos\theta + \sin\theta)a_{122} + (\cos\theta - \sin\theta)^3 a_{212} \\
&- \cos 2\theta(\cos\theta + \sin\theta)a_{221} - \cos 2\theta(\cos\theta - \sin\theta)a_{211} \\
&+ (\cos\theta + \sin\theta)^3 a_{121} - \cos 2\theta(\cos\theta - \sin\theta)a_{112}\Big|, \\
&\Big|\cos 2\theta(\cos\theta + \sin\theta)a_{111} + \cos 2\theta(\cos\theta - \sin\theta)a_{222} \\
&- \cos 2\theta(\cos\theta + \sin\theta)(a_{122} + a_{212}) + (\cos\theta - \sin\theta)^3 a_{221} \\
&- \cos 2\theta(\cos\theta - \sin\theta)(a_{211} + a_{121}) + (\cos\theta + \sin\theta)^3 a_{112}\Big|, \\
&\Big|\cos 2\theta(\cos\theta - \sin\theta)a_{111} - \cos 2\theta(\cos\theta + \sin\theta)a_{222} \\
&+ (\cos\theta + \sin\theta)^3 a_{122} - \cos 2\theta(\cos\theta - \sin\theta)(a_{212} + a_{221}) \\
&- (\cos\theta - \sin\theta)^3 a_{211} + \cos 2\theta(\cos\theta + \sin\theta)(a_{121} + a_{112})\Big|, \\
&\Big|\cos 2\theta(\cos\theta - \sin\theta)a_{111} - \cos 2\theta(\cos\theta + \sin\theta)a_{222} \\
&- \cos 2\theta(\cos\theta - \sin\theta)a_{122} + (\cos\theta + \sin\theta)^3 a_{212}
\end{aligned}$$

$$
\begin{aligned}
&- \cos 2\theta(\cos\theta - \sin\theta)a_{221} + \cos 2\theta(\cos\theta + \sin\theta)a_{211} \\
&-(\cos\theta - \sin\theta)^3 a_{121} + \cos 2\theta(\cos\theta + \sin\theta)a_{112}\Big|, \\
&\quad \Big| \cos 2\theta(\cos\theta - \sin\theta)a_{111} - \cos 2\theta(\cos\theta + \sin\theta)a_{222} \\
&- \cos 2\theta(\cos\theta - \sin\theta)(a_{122} + a_{212}) + (\cos\theta + \sin\theta)^3 a_{221} \\
&+ \cos 2\theta(\cos\theta + \sin\theta)(a_{211} + a_{121}) - (\cos\theta - \sin\theta)^3 a_{112}\Big| \Big\}.
\end{aligned}
$$

**Proof.** Let $S_T \in \mathcal{L}(^3\ell_1^2)$ be such that

$$
\begin{aligned}
S_T&\Big((t_1^{(1)}, t_2^{(1)}), (t_1^{(2)}, t_2^{(2)}), (t_1^{(3)}, t_2^{(3)})\Big) \\
&= T\Big(t_1^{(1)}W_1 + t_2^{(1)}W_2, \cdots, t_1^{(m)}W_1 + t_2^{(m)}W_2\Big).
\end{aligned}
$$

By Theorem C, $\|T\| = \|S_T\|_{\mathcal{L}(^3\ell_1^2)}$. Expanding $S_T$, we conclude the proof. ■

Let $T = (a_{111}, a_{222}, a_{122}, a_{212}, a_{221}, a_{211}, a_{121}, a_{112})^t \in \mathcal{L}(^3\ell^2_{(\infty,\theta)})$ and let

$$
\begin{aligned}
A_{111} &= (\cos\theta - \sin\theta)^3 a_{111} + (\cos\theta + \sin\theta)^3 a_{222} \\
&\quad + \cos 2\theta(\cos\theta + \sin\theta)(a_{122} + a_{212} + a_{221}) \\
&\quad + \cos 2\theta(\cos\theta - \sin\theta)(a_{211} + a_{121} + a_{112}), \\
A_{222} &= (\cos\theta + \sin\theta)^3 a_{111} - (\cos\theta - \sin\theta)^3 a_{222} \\
&\quad + \cos 2\theta(\cos\theta - \sin\theta)(a_{122} + a_{212} + a_{221}) \\
&\quad - \cos 2\theta(\cos\theta + \sin\theta)(a_{211} + a_{121} + a_{112}), \\
A_{122} &= \cos 2\theta(\cos\theta + \sin\theta)a_{111} + \cos 2\theta(\cos\theta - \sin\theta)a_{222} \\
&\quad + (\cos\theta - \sin\theta)^3 a_{122} - \cos 2\theta(\cos\theta + \sin\theta)(a_{212} + a_{221}) \\
&\quad + (\cos\theta + \sin\theta)^3 a_{211} - \cos 2\theta(\cos\theta - \sin\theta)(a_{121} + a_{112}), \\
A_{212} &= \cos 2\theta(\cos\theta + \sin\theta)a_{111} + \cos 2\theta(\cos\theta - \sin\theta)a_{222} \\
&\quad - \cos 2\theta(\cos\theta + \sin\theta)a_{122} + (\cos\theta - \sin\theta)^3 a_{212} \\
&\quad - \cos 2\theta(\cos\theta + \sin\theta)a_{221} - \cos 2\theta(\cos\theta - \sin\theta)a_{211} \\
&\quad + (\cos\theta + \sin\theta)^3 a_{121} - \cos 2\theta(\cos\theta - \sin\theta)a_{112}, \\
A_{221} &= \cos 2\theta(\cos\theta + sin\theta)a_{111} + \cos 2\theta(\cos\theta - \sin\theta)a_{222} \\
&\quad - \cos 2\theta(\cos\theta + \sin\theta)(a_{122} + a_{212}) + (\cos\theta - \sin\theta)^3 a_{221} \\
&\quad - \cos 2\theta(\cos\theta - \sin\theta)(a_{211} + a_{121}) + (\cos\theta + \sin\theta)^3 a_{112}, \\
A_{211} &= \cos 2\theta(\cos\theta - \sin\theta)a_{111} - \cos 2\theta(\cos\theta + \sin\theta)a_{222} \\
&\quad + (\cos\theta + \sin\theta)^3 a_{122} - \cos 2\theta(\cos\theta - \sin\theta)(a_{212} + a_{221}) \\
&\quad - (\cos\theta - \sin\theta)^3 a_{211} + \cos 2\theta(\cos\theta + \sin\theta)(a_{121} + a_{112}),
\end{aligned}
$$

$$\begin{aligned}A_{121} &= \cos 2\theta(\cos\theta-\sin\theta)a_{111}-\cos 2\theta(\cos\theta+\sin\theta)a_{222}\\ &- \cos 2\theta(\cos\theta-\sin\theta)a_{122}+(\cos\theta+\sin\theta)^3a_{212}\\ &- \cos 2\theta(\cos\theta-\sin\theta)a_{221}+\cos 2\theta(\cos\theta+\sin\theta)a_{211}\\ &- (\cos\theta-\sin\theta)^3a_{121}+\cos 2\theta(\cos\theta+\sin\theta)a_{112},\\ A_{112} &= \cos 2\theta(\cos\theta-\sin\theta)a_{111}-\cos 2\theta(\cos\theta+\sin\theta)a_{222}\\ &- \cos 2\theta(\cos\theta-\sin\theta)(a_{122}+a_{212})+(\cos\theta+\sin\theta)^3a_{221}\\ &+ \cos 2\theta(\cos\theta+\sin\theta)(a_{211}+a_{121})-(\cos\theta sin\theta)^3a_{112}.\end{aligned}$$

Then,

$$\begin{aligned}&S_T\Big((t_1^{(1)},t_2^{(1)}),(t_1^{(2)},t_2^{(2)}),(t_1^{(3)},t_2^{(3)})\Big)\\ &\qquad\qquad = \sum_{1\leq k\leq 3,\ i_k=1,2} A_{i_1i_2i_3}t_{i_1}^{(1)}t_{i_2}^{(2)}t_{i_3}^{(3)} \in \mathcal{L}(^3\ell_1^2).\end{aligned}$$

By Corollary E, $\|S_T\| = \max\Big\{|A_{i_1i_2i_3}| : i_k = 1,2 \text{ and } 1\leq k\leq 3\Big\}$.

By Theorem D, $M_{3(\theta)}$ is invertible and that $T = M_{3(\theta)}^{-1}\ S_T = \frac{1}{8}M_{3(\theta)}\ S_T$.

**Example.** Let $\mathcal{W}\subseteq (S_{\ell^2_{(\infty,\theta)}})^n$. We denote

$$\begin{aligned}\mathrm{Sym}(\mathcal{W}) \ = \ &\Big\{\big((x_{\sigma(1)},y_{\sigma(1)}),\ldots,(x_{\sigma(n)},y_{\sigma(n)})\big)\\ &: X = \big((x_1,y_1),\ldots,(x_n,y_n)\big)\in\mathcal{W},\\ &\sigma \text{ is a permutation on } \{1,\ldots,n\}\Big\}.\end{aligned}$$

Let $T = (a_{111},a_{222},a_{122},a_{212},a_{221},a_{211},a_{121},a_{112})^t \in \mathcal{L}(^3\ell^2_{(\infty,\theta)})$ such that $A_{111} = A_{222} = -A_{112} = -A_{121} = -A_{211} = -A_{122} = -A_{212} = -A_{221} = 1$. Notice that

$$\begin{aligned}\mathrm{Norm}(T) \ = \ \mathrm{Sym}\Big(\Big\{&\big(\pm(tW_1-(1-t)W_2),\pm W_1,\pm W_1\big),\\ &\big(\pm(tW_1-(1-t)W_2),\pm W_2,\pm W_2\big),\\ &\big(\pm(tW_1+(1-t)W_2),\pm W_1,\pm W_2\big) : 0\leq t\leq 1\Big\}\Big).\end{aligned}$$

The following characterizes the norming sets of $\mathcal{L}(^n\ell^2_{(\infty,\theta)})$.

**Theorem F.** *Let $n\in\mathbb{N}$ and $T\in\mathcal{L}(^n\ell^2_{(\infty,\theta)})$ with $\|T\|=1$. Then,*

$$\mathrm{Norm}(T) = \bigcup_{k=1}^{n}\big(A_k^+\cup A_k^-\cup B_{k,1}\cup B_{k,2}\big),$$

*where* $W_1 = (\cos\theta - \sin\theta, \cos\theta + \sin\theta)$, $W_2 = (\cos\theta + \sin\theta, -\cos\theta + \sin\theta)$,

$$\begin{aligned} A_k^+ &= \Big\{\Big( \pm X_1, \ldots, \pm X_{k-1}, \pm(tW_1 + (1-t)W_2), \pm X_{k+1}, \ldots, \pm X_n\Big) \\ &\quad \in \big(S_{\ell^2_{(\infty,\theta)}}\big)^n : T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, \pm X_n\big) \\ &\quad \times T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, X_n\big) = 1, 0 \le t \le 1\Big\}, \\ A_k^- &= \Big\{\Big( \pm X_1, \ldots, \pm X_{k-1}, \pm(tW_1 - (1-t)W_2), \pm X_{k+1}, \ldots, \pm X_n\Big) \\ &\quad \in \big(S_{\ell^2_{(\infty,\theta)}}\big)^n : T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, \pm X_n\big) \\ &\quad \times T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, X_n\big) = -1, 0 \le t \le 1\Big\}, \\ B_{k,1} &= \Big\{\Big( \pm X_1, \ldots, \pm X_{k-1}, \pm W_1, \pm X_{k+1}, \ldots, \pm X_n\Big) \in \big(S_{\ell^2_{(\infty,\theta)}}\big)^n : \\ 1 &= \Big|T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, \pm X_n\big)\Big| \\ &> \Big|T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, X_n\big)\Big|\Big\}, \\ B_{k,2} &= \Big\{\Big( \pm X_1, \ldots, \pm X_{k-1}, \pm W_2, \pm X_{k+1}, \ldots, \pm X_n\Big) \in \big(S_{\ell^2_{(\infty,\theta)}}\big)^n : \\ 1 &= \Big|T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, \pm X_n\big)\Big| \\ &> \Big|T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, X_n\big)\Big|\Big\}. \end{aligned}$$

**Proof.** Let $\mathcal{F}_k = A_k^+ \cup A_k^- \cup B_{k,1} \cup B_{k,2}$ for $k = 1, \ldots, n$.

$(\subseteq)$. Let $\Big(X_1, \ldots, X_n\Big) \in \mathrm{Norm}(T)$. Let $1 \le k \le n$ be fixed. Then $X_k = \lambda_1^{(k)} W_1 + \lambda_2^{(k)} W_2$ for some $\lambda_1^{(k)}, \lambda_2^{(k)} \in \mathbb{R}$ with $|\lambda_1^{(k)}| + |\lambda_2^{(k)}| = 1$.

**Case 1**.

$$\begin{aligned} &T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, \pm X_n\big) \\ &\qquad \times T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, X_n\big) = 1. \end{aligned}$$

Since $\|T\| = 1$, we have

$$\begin{aligned} 1 &= T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, \pm X_n\big) \\ &= T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, X_n\big) \end{aligned}$$

or

$$\begin{aligned} -1 &= T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, \pm X_n\big) \\ &= T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, X_n\big). \end{aligned}$$

*Claim 1.* $X_k \in \big\{ \pm (tW_1 + (1-t)W_2) : 0 \le t \le 1 \big\}$.

By $n$-linearity of $T$, it follows that

$$\begin{aligned} 1 &= T\big(X_1,\dots,X_n\big) = T\big(X_1,\dots,X_{k-1},(\lambda_1^{(k)}W_1+\lambda_2^{(k)}W_2),X_{k+1},\dots,X_n\big) \\ &= \big|\lambda_1^{(k)}T\big(X_1,\dots,X_{k-1},W_1,X_{k+1},\dots,X_n\big) \\ &\quad + \lambda_2^{(k)}T\big(X_1,\dots,X_{k-1},W_2,X_{k+1},\dots,X_n\big)\big| \\ &= |\lambda_1^{(k)}+\lambda_2^{(k)}| \le |\lambda_1^{(k)}|+|\lambda_2^{(k)}| = 1. \end{aligned}$$

Thus, $|\lambda_1^{(k)}+\lambda_2^{(k)}| = |\lambda_1^{(k)}|+|\lambda_2^{(k)}| = 1$. Hence, $\mathrm{sign}(\lambda_1^{(k)}) = \mathrm{sign}(\lambda_1^{(k)})$. Thus,

$$\begin{aligned} X_k &\in \big\{|\lambda_1^{(k)}|W_1+|\lambda_2^{(k)}|W_2, -(|\lambda_1^{(k)}|W_1+|\lambda_2^{(k)}|W_2)\big\} \\ &\subseteq \big\{ \pm (te_1+(1-t)e_2) : 0 \le t \le 1\big\}. \end{aligned}$$

Therefore, $X \in A_k^+ \subseteq \mathcal{F}_k \subseteq \bigcup_{j=1}^n \mathcal{F}_j$.

**Case 2**.

$$\begin{aligned} &T\big(X_1,\dots,X_{k-1},W_1,X_{k+1},\dots,\pm X_n\big) \\ &\qquad \times T\big(X_1,\dots,X_{k-1},W_2,X_{k+1},\dots,X_n\big) = -1. \end{aligned}$$

Since $\|T\| = 1$, we have

$$\begin{aligned} 1 &= T\big(X_1,\dots,X_{k-1},W_1,X_{k+1},\dots,\pm X_n\big) \\ &= -T\big(X_1,\dots,X_{k-1},W_2,X_{k+1},\dots,X_n\big) \end{aligned}$$

or

$$\begin{aligned} -1 &= T\big(X_1,\dots,X_{k-1},W_1,X_{k+1},\dots,\pm X_n\big) \\ &= -T\big(X_1,\dots,X_{k-1},W_2,X_{k+1},\dots,X_n\big). \end{aligned}$$

*Claim 2.* $X_k \in \big\{ \pm (tW_1 - (1-t)W_2) : 0 \le t \le 1 \big\}$.

It follows that

$$\begin{aligned} 1 &= T\big(X_1,\dots,X_n\big) \\ &= T\big(X_1,\dots,X_{k-1},(\lambda_1^{(k)}W_1+\lambda_2^{(k)}W_2),X_{k+1},\dots,X_n\big) \\ &= \big|\lambda_1 T\big(X_1,\dots,X_{k-1},W_1,X_{k+1},\dots,X_n\big) \\ &\quad + \lambda_2 T\big(X_1,\dots,X_{k-1},W_2,X_{k+1},\dots,X_n\big)\big| \\ &= |\lambda_1^{(k)}-\lambda_2^{(k)}| \le |\lambda_1^{(k)}|+|\lambda_2^{(k)}| = 1. \end{aligned}$$

Thus, $|\lambda_1^{(k)} - \lambda_2^{(k)}| = |\lambda_1^{(k)}| + |\lambda_2^{(k)}| = 1$. Hence, $\mathrm{sign}(\lambda_1^{(k)}) = -\mathrm{sign}(\lambda_2^{(k)})$. Thus,

$$\begin{aligned} X_k &\in \big\{|\lambda_1^{(k)}|W_1 - |\lambda_2^{(k)}|W_2, -(|\lambda_1^{(k)}|W_1 - |\lambda_2^{(k)}|W_2)\big\} \\ &\subseteq \big\{ \pm (tW_1 - (1-t)W_2) : 0 \le t \le 1 \big\}. \end{aligned}$$

Therefore, $X \in A_k^- \subseteq \mathcal{F}_k \subseteq \bigcup_{j=1}^n \mathcal{F}_j$.

**Case 3**.

$$\begin{aligned} 1 &= \big|T\big(X_1, \dots, X_{k-1}, W_1, X_{k+1}, \dots, \pm X_n\big)\big| \\ &> \big|T\big(X_1, \dots, X_{k-1}, W_2, X_{k+1}, \dots, X_n\big)\big|. \end{aligned}$$

*Claim 3.* $\lambda_2^{(k)} = 0$.

Assume that $\lambda_2^{(k)} \neq 0$. It follows that

$$\begin{aligned} 1 &= \big|T\big(X_1, \dots, X_n\big)\big| \\ &= \big|T\big(X_1, \dots, X_{k-1}, (\lambda_1^{(k)}W_1 + \lambda_2^{(k)}W_2), X_{k+1}, \dots, X_n\big)\big| \\ &\le |\lambda_1^{(k)}|\, \big|T\big(X_1, \dots, X_{k-1}, W_1, X_{k+1}, \dots, X_n\big)\big| \\ &\quad + |\lambda_2^{(k)}|\, \big|T\big(X_1, \dots, X_{k-1}, W_2, X_{k+1}, \dots, X_n\big)\big| \\ &< |\lambda_1^{(k)}|\big|T\big(X_1, \dots, X_{k-1}, W_1, X_{k+1}, \dots, X_n\big)\big| + |\lambda_2^{(k)}| \\ &\le |\lambda_1^{(k)}| + |\lambda_2^{(k)}| = 1, \end{aligned}$$

which is a contradiction. Thus, $\lambda_2^{(k)} = 0$ and so $X_k = W_1$. Therefore, $X \in B_{k,1} \subseteq \mathcal{F}_k \subseteq \bigcup_{j=1}^n \mathcal{F}_j$.

**Case 4**.

$$\begin{aligned} 1 &= \big|T\big(X_1, \dots, X_{k-1}, W_2, X_{k+1}, \dots, \pm X_n\big)\big| \\ &> \big|T\big(X_1, \dots, X_{k-1}, W_1, X_{k+1}, \dots, X_n\big)\big|. \end{aligned}$$

*Claim 4.* $\lambda_1^{(k)} = 0$.

Assume that $\lambda_1^{(k)} \neq 0$. By $(*)$, it follows that

$$\begin{aligned} 1 &= \big|T\big(X_1, \dots, X_n\big)\big| \\ &= \big|T\big(X_1, \dots, X_{k-1}, (\lambda_1^{(k)}W_1 + \lambda_2^{(k)}W_2), X_{k+1}, \dots, X_n\big)\big| \\ &\le |\lambda_1^{(k)}|\, \big|T\big(X_1, \dots, X_{k-1}, W_1, X_{k+1}, \dots, X_n\big)\big| \\ &\quad + |\lambda_2^{(k)}|\, \big|T\big(X_1, \dots, X_{k-1}, W_2, X_{k+1}, \dots, X_n\big)\big| \\ &< |\lambda_1^{(k)}| + |\lambda_2^{(k)}|\big|T\big(X_1, \dots, X_{k-1}, W_2, X_{k+1}, \dots, X_n\big)\big| \\ &\le |\lambda_1^{(k)}| + |\lambda_2^{(k)}| \le 1, \end{aligned}$$

which is a contradiction. Thus, $\lambda_1 = 0$ and so $X_k = W_2$. Therefore, $X \in B_{k,2} \subseteq \mathcal{F}_k \subseteq \bigcup_{j=1}^n \mathcal{F}_j$.

($\supseteq$). We will show that $\mathcal{F}_k \subseteq \mathrm{Norm}(T)$ for every $1 \le k \le n$.

Let $1 \le k \le n$ be fixed and $Y = (Y_1, \dots, Y_n) \in \mathcal{F}_k$.

Suppose that $Y \in A_k^+$.

Then $Y_k = \pm(t_k W_1 + (1-t_k)W_2)$ for some $0 \le t_k \le 1$ and

$$T\big(Y_1, \dots, Y_{k-1}, W_1, Y_{k+1}, \dots, \pm Y_n\big) \times T\big(Y_1, \dots, Y_{k-1}, W_2, Y_{k+1}, \dots, Y_n\big) = 1.$$

It follows that

$$\begin{aligned} 1 &= T\big(Y_1, \dots, Y_n\big) \\ &= T\big(Y_1, \dots, Y_{k-1}, (t_k W_1 + (1-t_k)W_2), Y_{k+1}, \dots, Y_n\big) \\ &= \big|t_k T\big(Y_1, \dots, Y_{k-1}, W_1, Y_{k+1}, \dots, Y_n\big) \\ &\quad + (1-t_k)T\big(Y_1, \dots, Y_{k-1}, W_2, Y_{k+1}, \dots, Y_n\big)\big| \\ &= |t_k + (1-t_k)| = 1. \end{aligned}$$

Thus, $Y \in \mathrm{Norm}(T)$.

Suppose that $Y \in A_k^-$.

Then $Y_k = \pm(t_k W_1 - (1-t_k)W_2)$ for some $0 \le t_k \le 1$ and

$$T\big(Y_1, \dots, Y_{k-1}, W_1, Y_{k+1}, \dots, \pm Y_n\big) \times T\big(Y_1, \dots, Y_{k-1}, W_2, Y_{k+1}, \dots, Y_n\big) = -1.$$

It follows that

$$\begin{aligned} 1 &= T\big(Y_1, \dots, Y_n\big) \\ &= T\big(Y_1, \dots, Y_{k-1}, (t_k W_1 - (1-t_k)W_2), Y_{k+1}, \dots, Y_n\big) \\ &= \big|t_k T\big(Y_1, \dots, Y_{k-1}, W_1, Y_{k+1}, \dots, Y_n\big) \\ &\quad -(1-t_k)T\big(Y_1, \dots, Y_{k-1}, W_2, Y_{k+1}, \dots, Y_n\big)\big| \\ &= |t_k + (1-t_k)| = 1. \end{aligned}$$

Thus, $Y \in \mathrm{Norm}(T)$.

Suppose that $Y \in B_{k,1}$. Then $Y_k = \pm W_1$ and

$$\big|T\big(Y_1, \dots, Y_n\big)\big| = \big|T\big(Y_1, \dots, Y_{k-1}, W_1, Y_{k+1}, \dots, \pm Y_n\big)\big| = 1.$$

Thus, $Y \in \mathrm{Norm}(T)$.

Suppose that $Y \in B_{k,2}$. Then $Y_k = \pm W_2$ and

$$\big|T\big(Y_1, \dots, Y_n\big)\big| = \big|T\big(Y_1, \dots, Y_{k-1}, W_2, Y_{k+1}, \dots, \pm Y_n\big)\big| = 1.$$

Thus, $Y \in \mathrm{Norm}(T)$. We complete the proof. ∎

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SUNG GUEN KIM
DEPARTMENT OF MATHEMATIC
KYUNGPOOK NATIONAL UNIVERSITY
DAEGU 702-701, REPUBLIC OF KOREA
*e-mail:* sgk317@knu.ac.kr


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