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STUDY OF CURVATURE PROPERTIES OF KENMOTSU MANIFOLDS CONCERN TO A TYPE OF SEMI-SYMMETRIC NON-METRIC CONNECTION

Abstract. In the present paper we study Riemannian curvature tensor, projective, Wely conformal and con-harmonic curvature tensors on Kenmotsu manifolds along with a type of semi-symmetric non-metric connection. Also, we deduce some results for cyclic and η-parallel Ricci tensors. In the end, we give an example to validate some of the obtained results.

Key words: Kenmotsu manifold, semi-symmetric non-metric connection, projective curvature tensor, Weyl conformal curvature tensor, and con-harmonic curvature tensor.

AMS Mathematics Subject Classification: 53B15, 53C05, 53C25, 58A07.

1. Introduction

In 1958, W.M. Boothby and H.C. Wang [3], studied the topological properties of an odd-dimensional differentiable manifold. Sasaki [14], did tremendous work in the field of contact geometry. To characterize the properties of an odd-dimensional differentiable manifold equipped with contact structures he used the tensor calculus. Such manifolds were known as the contact manifolds. The several classes of contact manifolds had been characterized by many researchers who studied their properties with different connections. In this series, Kenmotsu [10], introduced the concept of the Kenmotsu manifold by considering a class of contact metric manifolds satisfying certain tensorial relations. Also, he proved that a semi-symmetric Kenmotsu manifold $(C(L, M).C) = 0$, where $L, M \in \chi(\mathbb{M})$, is a manifold of constant curvature -1, where C refers to the Riemannian curvature tensor. If $\nabla C = 0$, then the manifold $\mathbb{M}$ is called locally symmetric. The Kenmotsu manifolds have been studied by many researchers, for instance, we refer to [2, 4, 8, 13, 16] and the references therein.

After the study of the Riemannian Curvature tensor, the projective curvature tensor, con-circular curvature tensor, Weyl conformal curvature tensor, con-harmonic curvature tensor, and many others are extensively studied by the geometers.

A. Hayden introduced a metric connection on a Riemannian manifold, in [9]. A linear connection ∇ is said to be a metric on a manifold $\mathbb{M}$ if $\nabla g = 0$; otherwise, it is non-metric. In 1970, a semi-symmetric metric connection on the Riemannian manifold was introduced by Yano[18]. Agashe and Chafle [1], Sengupta [15], Chaubey [5, 6], and many others [11, 12, 17] studied various and important properties of semi-symmetric metric and non-metric connections on several differentiable manifolds and also defined some new type of connections on Riemannian manifold.
Chaubey [7] studied a new type of semi-symmetric non-metric connection in 2019. He observed that under certain conditions Riemannian manifold will be projectively invariant with respect to this connection.

Motivated by the above studies, we have studied some curvature properties of semi-symmetric non-metric connection defined on the Kenmotsu manifolds.

2. Preliminaries

An odd-dimensional differentiable manifold with the almost contact Riemannian structure (ϕ, ξ, η, g) is an almost contact metric manifold if,

$$\phi^2 L = -L + \eta(L)\xi, \ \ \eta(\xi) = 1, \ \ \phi\xi = 0, \ \ \eta \circ \phi = 0, \tag{1}$$

$$g(L, M) = g(\phi L, \phi M) + \eta(L)\eta(M), \tag{2}$$

$$g(L, \phi M) = -g(\phi L, M), g(L, \xi) = \eta(L), \tag{3}$$

for all $L, M \in \mathbb{M}$, where ϕ is a $(1, 1)-$ tensor field, ξ is a vector field and η is a 1-form. An almost contact metric manifold is said to be a Kenmotsu manifold if,

$$(\nabla_L \phi)M = -g(L, \phi M)\xi - \eta(M)\phi L, \tag{4}$$

$$(\nabla_L \eta)M = g(L, M) - \eta(L)\eta(M), \tag{5}$$

$$\nabla_L \xi = X - \eta(M)\xi, \tag{6}$$

where ∇ is the Levi-Civita connection of Riemannian metric g. An odd-dimensional say $(n = 2m+1)$, Kenmotsu manifold is normal but not Sasakian.

Also, every Kenmotsu manifold is non-compact and $div\xi = n-1$. Moreover, the Riemannian curvature tensor C, Ricci tensor S , and Ricci operator Q are given by,

$$C(L,M)\xi = \eta(L)M - \eta(M)L, \tag{7}$$

$$C(\xi,L)M = \eta(M)L - g(L,M)\xi, \tag{8}$$

$$C(\xi,L)\xi = L - \eta(L)\xi, \tag{9}$$

$$S(L,\xi) = -(n-1)\eta(L), \tag{10}$$

$$Q\xi = -(n-1)\xi. \tag{11}$$

Any Riemannian manifold $\mathbb{M}$ is a generalized Ricci-recurrent manifold if its Ricci tensor satisfies the condition similar to,

$$(\nabla_L S)(M,N) = \alpha(L)S(M,N) + \beta(L)g(M,N), \tag{12}$$

for some 1-forms α and β. In particular, if $\beta = 0$ and $\alpha \neq 0$, then $\mathbb{M}$ is Ricci-recurrent.
Also, $\mathbb{M}$ has a cyclic Ricci tensor, if

$$\sum_{cyclic} (\nabla_L S)(M,N) = 0, \tag{13}$$

and, $\mathbb{M}$ has η-parallel Ricci tensor if,

$$(\nabla_L S)(\phi M,\phi N) = 0, \tag{14}$$

$\forall L,M,N \in \chi(\mathbb{M})$.

The Projective curvature tensor $\mathbb{P}(L,M)N$ for an n-dimensional Riemannian manifold concerning the Levi-Civita connection ∇ is given by,

$$\mathbb{P}(L,M)N = C(L,M)N - \frac{1}{n-1}\{S(M,N)L - S(L,N)M\}. \tag{15}$$

The con-harmonic curvature tensor $\mathbb{N}(L,M)N$ for an n-dimensional Riemannian manifold concerning the Levi-Civita connection ∇ is given by,

$$\begin{aligned}\mathbb{N}(L,M)N \;=\; & C(L,M)N - \frac{1}{n-2}\{S(M,N)L - S(L,N)M \\ & + g(M,N)QL - g(L,N)QM\}.\end{aligned} \tag{16}$$

The Weyl conformal curvature tensor $\mathbb{K}(L,M)N$ for an n-dimensional Riemannian manifold concerning the Levi-Civita connection ∇ is given by,

$$\begin{aligned}\mathbb{K}(L,M)N &= C(L,M)N - \frac{1}{n-2}\{S(M,N)L - S(L,N)M \\ &\quad + g(M,N)QL - g(L,N)QM\} \\ &\quad + \frac{s}{(n-1)(n-2)}\{g(M,N)L - g(L,N)M\},\end{aligned} \tag{17}$$

where C is the curvature tensor, S is the Ricci curvature tensor, Q is the Ricci operator and s is scalar curvature for an n-dimensional Riemannian manifold for all $L, M, N \in \chi(\mathbb{M})$.

3. A type of semi-symmetric non-metric connection

In 2019, S.K Chaubey and A. Yildiz introduced a new type of semi-symmetric non-metric connection in [7]. The linear connection $\tilde{\nabla}$ on a Riemannian manifold $(\mathbb{M}, g)$ is given by,

$$\tilde{\nabla}_L M = \nabla_L M + \frac{1}{2}\{\eta(M)L - \eta(L)M\} \tag{18}$$

is a semi-symmetric non-metric connection, where ∇ is the Levi-Civita connection of Riemannian metric g. The torsion tensor $\tilde{T}$ on $\mathbb{M}$ concerning $\tilde{\nabla}$ satisfies the equation,

$$\tilde{T}(L,M) = \eta(M)L - \eta(L)M, \tag{19}$$

where η is 1-form associated with the vector field ξ and satisfies

$$\eta(L) = g(L, \xi) \tag{20}$$

and the metric g holds the relation

$$(\tilde{\nabla}_L g)(M,N) = \frac{1}{2}\{2\eta(L)g(M,N) - \eta(M)g(L,N) - \eta(N)g(L,M)\}. \tag{21}$$

The curvature tensor $\tilde{C}$ corresponding to $\tilde{\nabla}$ on an n-dimensional Riemannian manifold is given by[7],

$$\begin{aligned}\tilde{C}(L,M,N) &= C(L,M,N) + \frac{1}{2}\{\Theta(L,N)M - \Theta(M,N)L \\ &\quad - (\Theta(L,M) - \Theta(M,L))N\},\end{aligned} \tag{22}$$

where $\Theta(L,M) = g(BL,M) = (\nabla_L \eta)M - \frac{1}{2}\eta(L)\eta(M)$ and $BL = \nabla_L \xi - \frac{1}{2}\eta(L)\xi$. The Ricci curvature $\tilde{S}$ is given by

$$\tilde{S}(M,N) = S(M,N) - \frac{(n-1)}{2}\Theta(M,N). \tag{23}$$

The Ricci operator $\tilde{Q}$ is defined as,

$$\tilde{Q}(M) = QM - \frac{(n-1)}{2} BM. \tag{24}$$

And the scalar curvature $\tilde{s}$ is defined as,

$$\tilde{s} = s - \frac{(n-1)}{2} trace(B). \tag{25}$$

4. Kenmotsu Manifolds admitting $\hat{\nabla}$

Let $(\mathbb{M}, g)$ be a Kenmotsu manifold of dimension $n = 2m + 1$, admitting the connection $\hat{\nabla}$, then from equations (18) and (6) we get,

$$\hat{\nabla}_L M = \nabla_L M + \frac{1}{2}\{\eta(M)L - \eta(L)M\}. \tag{26}$$

Also,

$$\hat{\nabla}_L \xi = \frac{3}{2}(L - \eta(L)\xi). \tag{27}$$

Let C and $\hat{C}$ be the curvature tensors of Levi-Civita connection ∇ and semi-symmetric non-metric connection $\hat{\nabla}$ respectively, then

$$C(L, M)N = \nabla_L \nabla_M N - \nabla_N \nabla_L M - \nabla_{[L,M]} N, \tag{28}$$

$$\hat{C}(L, M)N = \hat{\nabla}_L \hat{\nabla}_M N - \hat{\nabla}_M \hat{\nabla}_L N - \hat{\nabla}_{[L,M]} N. \tag{29}$$

Then, by using the equations (5), (6), (28) and (29), the curvature tensor of $(\mathbb{M}, g)$ concerning the semi-symmetric non-metric connection $\hat{\nabla}$ is given by,

$$\begin{aligned} \hat{C}(L, M)N &= C(L, M)N + \frac{3}{4}(\eta(M)L - \eta(L)M)\eta(N) \\ &\quad + \frac{1}{2}(g(L, N)M - g(M, N)L). \end{aligned} \tag{30}$$

Taking the inner product of (30) with W, we have,

$$\begin{aligned} \hat{C}(L, M, N, W) &= C(L, M, N, W) + \frac{3}{4}(\eta(M)g(L, W) \\ &\quad - \eta(L)g(M, W))\eta(N) + \frac{1}{2}(g(L, N)g(M, W) \\ &\quad - g(M, N)g(L, W)), \end{aligned} \tag{31}$$

where $\hat{C}(L, M, N, W) = g(\hat{C}(L, M)N, W)$.

Proposition 1. *From equation* (30) *and* (31) *we can deduce the following curvature identities,*

(*i*) $\hat{C}(L, M)\xi = \frac{3}{4}(\eta(L)M - \eta(M)L)$.
(*ii*) $\hat{C}(\xi, M)\xi = \frac{1}{4}(M - \eta(M)\xi)$.
(*iii*) $\hat{C}(\xi, M)N = \frac{3}{2}(\eta(N)M - g(M, N)\xi) + \frac{3}{4}(\eta(M)\xi - M)\eta(N)$.
(*iv*) $\hat{C}(L, M, N, W) + \hat{C}(L, M, N, W) = 0$.
(*v*) $\hat{C}(L, M, N, W) + \hat{C}(L, M, W, N) = 0$.

Let $\{e_i\}$ be the orthonormal basis of tangent space at each point of the manifold $(\mathbb{M}, g)$ then contracting the equation (30) by L we get,

$$\hat{S}(M, N) = S(M, N) - \frac{(n-1)}{2}g(M, N) + \frac{3(n-1)}{4}\eta(M)\eta(N). \tag{32}$$

The Ricci operator Q is defined by,

$$S(L, M) = g(QL, M)$$

Hence, the Ricci operator $\hat{Q}$ for the Kenmotsu manifold $(\mathbb{M}, g)$ concerning the semi-symmetric non-metric connection $\hat{\nabla}$ is given by,

$$\hat{S}(L, M) = g(\hat{Q}L, M). \tag{33}$$

Now, from equations (32) and (33), we get

$$\hat{Q}M = QM - \frac{(n-1)}{2}M + \frac{3(n-1)}{4}\eta(M)\xi. \tag{34}$$

Also, the constant curvature $\hat{s}$ is given by,

$$\hat{s} = s - \frac{(2n-3)(n-1)}{4}. \tag{35}$$

Proposition 2. *From equation* (33) *and* (34), *we can deduce the following,*

(*i*) $\hat{S}(M, \xi) = -\frac{3(n-1)}{4}\eta(M)$,
(*ii*) $\hat{S}(M, N)$ *is symmetric,*
(*iii*) $\hat{Q}\xi = \frac{3(n-1)}{4}\xi$.

Theorem 1. *Let* $(\mathbb{M}, g)$ *be a locally symmetric Kenmotsu manifold along with* $\hat{\nabla}$. *Then* $\mathbb{M}$ *is an* η*-Einstein manifold concerning* $\hat{\nabla}$. *Also, the scalar curvature for* ∇ *on* $\mathbb{M}$ *is* $(1-n)$.

Proof. Let $(\mathbb{M}, g)$ be a locally symmetric Kenmotsu manifold admitting $\hat{\nabla}$, then

$$(\hat{\nabla}_L \hat{C})(M, N)W = 0. \tag{36}$$

Now, contracting the equation (36) concerning to M, we get

$$(\hat{\nabla}_L \hat{S})(N, W) = \hat{\nabla}_L \hat{S}(N, W) - \hat{S}(\hat{\nabla}_L N, W) - \hat{S}(N, \hat{\nabla}_L W) = 0. \tag{37}$$

Taking $W = \xi$ in equation (37), we get

$$(\hat{\nabla}_L \hat{S})(N, \xi) = \hat{\nabla}_L \hat{S}(N, \xi) - \hat{S}(\hat{\nabla}_L N, \xi) - \hat{S}(N, \hat{\nabla}_L \xi) = 0. \tag{38}$$

Using equations (32), (37), and (38) we have

$$\hat{S}(N, L) = -\frac{1}{4}(n-1)\eta(L)\eta(N) - \frac{1}{2}(n-1)g(L, N). \tag{39}$$

Hence, $(\mathbb{M}, g)$ is an η-Einstein manifold concerning $\hat{\nabla}$. Further, put the value from equation (32) we get,

$$S(N, L) = -(n-1)\eta(L)\eta(N). \tag{40}$$

Now, contract the equation (40) over L and N, and we get

$$s = -(n-1).$$

Hence, the theorem. ■

Theorem 2. *Let $(\mathbb{M}, g)$ be a connected Kenmotsu manifold with an η-parallel Ricci tensor concerning $\hat{\nabla}$. Then $(\mathbb{M}, g)$ has constant scalar curvature concerning $\hat{\nabla}$ if and only if the scalar curvature for $(\mathbb{M}, g)$ concerning ∇ is $3(1-n)$.*

Proof. Let $(\mathbb{M}, g)$ be a connected Kenmotsu manifold with an η-parallel Ricci tensor concerning $\hat{\nabla}$, then we have

$$(\hat{\nabla}_L \hat{S})(\phi M, \phi N) = 0, \tag{41}$$

$\forall L, M, N \in \chi(\mathbb{M})$. Using the equations (32), and (41) we get,

$$\begin{aligned}(\hat{\nabla}_L \hat{S})(M, N) \ = \ & -S(L, N)\eta(M) - S(L, M)\eta(N) \\ & + \frac{1}{2}\eta(L)S(M, N) - \frac{(n-1)}{2}\eta(L)g(M, N).\end{aligned} \tag{42}$$

Put $M = N = e_i$ and sum up for $i = 1, 2, ..., n$, in the equation (42), we obtain

$$\hat{d}(\hat{s}(L)) = \eta(L)\{\frac{3(n-1)+s}{2}\}. \tag{43}$$

So, if $s = 3(1-n)$, then $\hat{d}(\hat{s}(X)) = 0$. On integrating the equation (43) we get $\hat{s} = constant$. Hence, the theorem. ■

Theorem 3. *Let* $(\mathbb{M}, g)$ *be an* n*-dimensional generalized Ricci-recurrent Kenmotsu manifold concerning* $\hat{\nabla}$*, then* $\beta = \frac{3}{2}(n-1)\alpha$.

Proof. Let $(\mathbb{M}, g)$ be an n-dimensional generalized Ricci-recurrent Kenmotsu manifold concerning $\hat{\nabla}$, then we have

$$(\hat{\nabla}_L \hat{S})(M, N) = \alpha(L)\hat{S}(M, N) + \beta(L)g(M, N). \tag{44}$$

As we know,

$$(\hat{\nabla}_L \hat{S})(M, N) = \hat{\nabla}_L \hat{S}(M, N) - \hat{S}(\hat{\nabla}_L M, N) - \hat{S}(M, \hat{\nabla}_L N). \tag{45}$$

Now, using the equations (44) and (45), and put $M = N = \xi$, we get,

$$\beta(L) = \frac{3}{2}(n-1)\alpha(L), \tag{46}$$

which is equivalent to, $\beta = \frac{3}{2}(n-1)\alpha$. Hence proved. ■

Corollary 1. *Let* $(\mathbb{M}, g)$ *be an* n*-dimensional generalized Ricci-recurrent Kenmotsu manifold concerning* $\hat{\nabla}$ *and if* $(\mathbb{M}, g)$ *has cyclic Ricci tensor then,* $\alpha(\xi)\{\hat{S} + \frac{3}{2}(n-1)g\} = 0$.

Proof. If $(\mathbb{M}, g)$ has a cyclic Ricci tensor, then we have

$$\sum_{cyclic} (\hat{\nabla}_L \hat{S})(M, N) = 0. \tag{47}$$

As, $(\mathbb{M}, g)$ is a generalized Ricci-recurrent Kenmotsu manifold, so equation (47) reduces to,

$$\sum_{cyclic} \alpha(L)\hat{S}(M, N) + \beta(L)g(M, N) = 0. \tag{48}$$

Now, put $L = \xi$ in the equation (48) we get the required result,

$$\alpha(\xi)\{\hat{S} + \frac{3}{2}(n-1)g\} = 0. \tag{49}$$

■

Corollary 2. *Let* $(\mathbb{M}, g)$ *be an* n*-dimensional generalized Ricci-recurrent Kenmotsu manifold concerning* $\hat{\nabla}$ *and if* $(\mathbb{M}, g)$ *has cyclic Ricci tensor then,* $\alpha(\xi)\{S + (n-1)g + \frac{3(n-1)}{2}\eta \otimes \eta\} = 0$.

Proof. Put the value from the equation (32) in the equation (49), and we get

$$\alpha(\xi)\{S+(n-1)g+\frac{3(n-1)}{2}\eta\otimes\eta\}=0.$$

Hence, proved. ■

5. Some other curvature tensors on Kenmotsu manifolds admitting $\hat{\nabla}$

In this section, we obtain some results for the projective curvature tensor, con-circular curvature tensor, Weyl conformal curvature tensor, and con-harmonic curvature tensor on an n-dimensional Kenmotsu manifold admitting $\hat{\nabla}$.

Theorem 4. *The projective curvature tensor for an n-dimensional Kenmotsu manifold $(\mathbb{M},g)$ concerning $\hat{\nabla}$ and ∇ coincides for $\forall n>1$.*

Proof. The projective curvature tensor $\hat{\mathbb{P}}(L,M)N$ for Kenmotsu manifold $(\mathbb{M},g)$ concerning $\hat{\nabla}$ is given by,

$$\hat{\mathbb{P}}(L,M)N=\hat{C}(L,M)N-\frac{1}{n-1}\{\hat{S}(M,N)L-\hat{S}(L,N)M\}. \tag{50}$$

Now, put the values from equation (15),(30) in equation (50), we get

$$\hat{\mathbb{P}}(L,M)N=\mathbb{P}(L,M)N$$

Hence, the theorem. ■

Corollary 3. *An n-dimensional Kenmotsu manifold $(\mathbb{M},g)$ concerning $\hat{\nabla}$ is ξ-projectively flat if and only if it is ξ-projectively flat concerning ∇.*

Proof. A manifold is said to be ξ-projectively flat concerning ∇, if $\mathbb{P}(L,M)\xi=0$. Now, put $N=\xi$, in equation (50), we get

$$\hat{\mathbb{P}}(L,M)\xi=0$$

Hence, proved. ■

Theorem 5. *The Weyl conformal curvature tensor for an n-dimensional Kenmotsu manifold $((\mathbb{M},g)$ concerning $\hat{\nabla}$ and ∇ is coincided for $\forall n>1$. Moreover, $\hat{\mathbb{K}}$ will be ξ-conformally flat if and only if the conformal curvature tensor concerning ∇ is ξ-conformally flat.*

Proof. The Weyl conformal curvature tensor $\hat{\mathbb{K}}(L,M)N$ for Kenmotsu manifold $(\mathbb{M},g)$ concerning $\hat{\nabla}$ is given by,

$$\begin{aligned}\hat{\mathbb{K}}(L,M)N \;=\; & \hat{C}(L,M)N-\frac{1}{2}\{\hat{S}(M,N)L-\hat{S}(L,N)M \\ & +g(M,N)\hat{Q}L-g(L,N)\hat{Q}M\} \\ & +\frac{\hat{s}}{(n-1)(n-2)}\{g(M,N)L-g(L,N)M\}.\end{aligned} \tag{51}$$

Now, put the values from the equations (17), (30) in the equation (51), we get

$$\hat{\mathbb{K}}(L,M)N=\mathbb{K}(L,M)N.$$

A manifold is said to be ξ-conformally flat concerning to ∇, if $\mathbb{K}(L,M)\xi=0$.

Now, put $N=\xi$, in equation (51), we get

$$\hat{\mathbb{K}}(L,M)\xi=0.$$

Hence, the theorem. ■

6. Example

In this section, we reconstruct an example of a 5-dimensional Kenmotsu manifold concerning the connection $\hat{\nabla}$.

Example 1. Let us consider a 5-dimensional manifold $\mathbb{M}=\{(l,m,n,o,p)\in\mathbb{R}^5\}$ where (l,m,n,o,p) are the standard coordinates of $\mathbb{R}^5$. We choose the linearly independent vector fields, at each point of $\mathbb{M}$.

$$r_1=\frac{\partial}{\partial l},\ r_2=r^{-l}\frac{\partial}{\partial m},\ r_3=r^{-l}\frac{\partial}{\partial n},\ r_4=r^{-l}\frac{\partial}{\partial o},\ r_5=r^{-l}\frac{\partial}{\partial p}$$

also, $l\neq 0$.

Let g be the Riemannian metric defined by,

$$g(r_i,r_j)=\begin{cases}1 & \text{if}\quad i=j\\ 0 & \text{if}\quad i\neq j\end{cases}\ ;\quad i,j=1,2,3,4,5.$$

Let η be the 1-form defined by,

$$\eta(A)=g(A,r_1)$$

for any $A\in\chi(\mathbb{M})$. Using the linearity of ϕ and g, we have

$$\phi^2A=-A+\eta(A)r_1 \text{ and } \eta(r_1)=1.$$

Also,

$$g(\phi A, \phi B) = g(A, B) - \eta(A)\eta(B).$$

For any $A, B \in \chi(\mathbb{M})$. Thus $r_1 = \xi$, the structure (ϕ, ξ, η, g) defines an almost Kenmotsu manifold. For Levi-Civita connection ∇, we have

$$[r_1, r_2] = -r_2, [r_1, r_3] = -r_3, [r_1, r_4] = -r_4, [r_1, r_5] = -r_5,$$

$$[r_2, r_3] = [r_2, r_4] = [r_2, r_5] = [r_3, r_4] = [r_3, r_5] = [r_4, r_5] = 0$$

The Riemannian connection ∇ of the metric g is given by Koszul's formula which is given by,

$$\begin{aligned} 2g(\nabla_A B, C) &= Ag(B, C) + Bg(C, A) - Cg(A, B) - g(A, [B, C]) \\ &\quad - g(B, [A, C]) + g(C, [A, B]) \end{aligned}$$

Taking $r_1 = \xi$ and using Koszul's formula, we get

$$\begin{array}{lllll} \nabla_{r_1} r_1 = 0 & \nabla_{r_1} r_2 = 0 & \nabla_{r_1} r_3 = 0 & \nabla_{r_1} r_4 = 0 & \nabla_{r_1} r_5 = 0 \\ \nabla_{r_2} r_1 = r_2 & \nabla_{r_2} r_2 = -r_1 & \nabla_{r_2} r_3 = 0 & \nabla_{r_2} r_4 = 0 & \nabla_{r_2} r_5 = 0, \\ \nabla_{r_3} r_1 = r_3 & \nabla_{r_3} r_2 = 0 & \nabla_{r_3} r_3 = -r_1 & \nabla_{r_3} r_4 = 0 & \nabla_{r_3} r_5 = 0 \\ \nabla_{r_4} r_1 = r_4 & \nabla_{r_4} r_2 = 0 & \nabla_{r_4} r_3 = 0 & \nabla_{r_4} r_4 = -r_1 & \nabla_{r_4} r_5 = 0, \\ \nabla_{r_5} r_1 = r_5 & \nabla_{r_5} r_2 = 0 & \nabla_{r_5} r_3 = 0 & \nabla_{r_5} r_4 = 0 & \nabla_{r_5} r_5 = -r_1. \end{array} \tag{52}$$

From the above values, it is clear that (ϕ, ξ, η, g) is a 5-dimensional Kenmotsu manifold.

Using the results from equation (52), we can obtain the for $\hat{\nabla}$,

$$\begin{array}{lllll} \hat{\nabla}_{r_1} r_1 = 0 & \hat{\nabla}_{r_1} r_2 = -\frac{r_2}{2} & \hat{\nabla}_{r_1} r_3 = -\frac{r_3}{2} & \hat{\nabla}_{r_1} r_4 = -\frac{r_4}{2} & \hat{\nabla}_{r_1} r_5 = -\frac{r_5}{2} \\ \hat{\nabla}_{r_2} r_1 = \frac{3}{2} r_2 & \hat{\nabla}_{r_2} r_2 = -r_1 & \hat{\nabla}_{r_2} r_3 = 0 & \hat{\nabla}_{r_2} r_4 = 0 & \hat{\nabla}_{r_2} r_5 = 0, \\ \hat{\nabla}_{r_3} r_1 = \frac{3}{2} r_3 & \hat{\nabla}_{r_3} r_2 = 0 & \hat{\nabla}_{r_3} r_3 = -r_1 & \hat{\nabla}_{r_3} r_4 = 0 & \hat{\nabla}_{r_3} r_5 = 0 \\ \hat{\nabla}_{r_4} r_1 = \frac{3}{2} r_4 & \hat{\nabla}_{r_4} r_2 = 0 & \hat{\nabla}_{r_4} r_3 = 0 & \hat{\nabla}_{r_4} r_4 = -r_1 & \hat{\nabla}_{r_4} r_5 = 0, \\ \hat{\nabla}_{r_5} r_1 = \frac{3}{2} r_5 & \hat{\nabla}_{r_5} r_2 = 0 & \hat{\nabla}_{r_5} r_3 = 0 & \hat{\nabla}_{r_5} r_4 = 0 & \hat{\nabla}_{r_5} r_5 = -r_1. \end{array}$$

Using the results from equation (52), we can obtain the components of the Riemannian curvature tensors concerning ∇ as follows:

$$\begin{array}{l} C(r_1, r_2) r_1 = r_2, C(r_1, r_2) r_2 = -r_1, C(r_1, r_3) r_1 = r_3, C(r_1, r_3) r_3 = -r_1, \\ C(r_1, r_4) r_1 = r_4, C(r_1, r_4) r_4 = -r_1, C(r_1, r_5) r_1 = r_5, C(r_1, r_5) r_5 = -r_1, \\ C(r_2, r_3) r_2 = r_3, C(r_2, r_3) r_3 = -r_2, C(r_2, r_4) r_2 = r_4, C(r_2, r_4) r_4 = -r_2, \\ C(r_2, r_5) r_2 = r_5, C(r_2, r_5) r_5 = -r_2, C(r_3, r_4) r_3 = r_4, C(r_3, r_4) r_4 = -r_3, \\ C(r_3, r_5) r_3 = r_5, C(r_3, r_5) r_5 = -r_3, C(r_4, r_5) r_4 = r_5, C(r_4, r_5) r_5 = -r_4. \end{array}$$

Hence, the non-vanishing Riemannian curvature tensors concerning $\hat{\nabla}$ as follows:

$$\begin{aligned}
&\hat{C}(r_1,r_2)r_1 = \tfrac{3}{4}r_2, \hat{C}(r_1,r_2)r_2 = -\tfrac{3}{2}r_1, \hat{C}(r_1,r_3)r_1 = \tfrac{3}{4}r_3, \hat{C}(r_1,r_3)r_3 = -\tfrac{3}{2}r_1,\\
&\hat{C}(r_1,r_4)r_1 = \tfrac{3}{4}r_4, \hat{C}(r_1,r_4)r_4 = -\tfrac{3}{2}r_1, \hat{C}(r_1,r_5)r_1 = \tfrac{3}{4}r_5, \hat{C}(r_1,r_5)r_5 = -\tfrac{3}{2}r_1,\\
&\hat{C}(r_2,r_3)r_2 = \tfrac{3}{2}r_3, \hat{C}(r_2,r_3)r_3 = -\tfrac{3}{2}r_2, \hat{C}(r_2,r_4)r_2 = \tfrac{3}{2}r_4, \hat{C}(r_2,r_4)r_4 = -\tfrac{3}{2}r_2,\\
&\hat{C}(r_2,r_5)r_2 = \tfrac{3}{2}r_5, \hat{C}(r_2,r_5)r_5 = -\tfrac{3}{2}r_2, \hat{C}(r_3,r_4)r_3 = \tfrac{3}{2}r_4, \hat{C}(r_3,r_4)r_4 = -\tfrac{3}{2}r_3,\\
&\hat{C}(r_3,r_5)r_3 = \tfrac{3}{2}r_5, \hat{C}(r_3,r_5)r_5 = -\tfrac{3}{2}r_3, \hat{C}(r_4,r_5)r_4 = \tfrac{3}{2}r_5, \hat{C}(r_4,r_5)r_5 = -\tfrac{3}{2}r_4.
\end{aligned}$$

So, the Ricci tensor with respect to ∇ will be,

$$S(r_1,r_1) = S(r_2,r_2) = S(r_3,r_3) = S(r_4,r_4) = S(r_5,r_5) = -4.$$

So, the scalar curvature s with respect to ∇, of the manifold will be,

$$s = -20.$$

Hence, the Ricci curvature for $\hat{\nabla}$ as follows:

$$\hat{S}(r_1,r_1) = -3, \hat{S}(r_2,r_2) = \hat{S}(r_3,r_3) = \hat{S}(r_4,r_4) = \hat{S}(r_5,r_5) = -6.$$

And hence, the scalar curvature of $\hat{\nabla}$ is

$$\hat{s} = -27.$$

From this example, the equations (30), (32), (34) and (35) are verified.

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