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**ON EXISTENCE OF SOLUTIONS OF SINGULAR
CAUCHY-NICOLETTI PROBLEM FOR SYSTEM
OF TWO DIFFERENTIAL EQUATIONS***

In the paper a singular two-point Cauchy-Nicoletti problem for system of two ordinary differential equations is considered. The existence of solutions which graph remains in a properly chosen domain is proved. Moreover, the theorem about uniqueness of solution in this domain is given. The applicability of results is shown on an illustrative example.

Key words: singular Cauchy-Nicoletti problem, uniqueness of solutions.

1. Introduction

Consider the following singular Cauchy-Nicoletti problem for the system of two ordinary differential equations

$$(1) \quad y'_1 = \omega_1(x, y_1, y_2),$$

$$y'_2 = \omega_2(x, y_1, y_2)$$

and

$$(2) \quad y_1(a^+) = A, \quad y_2(b^-) = B$$

where a, b, A, B are constants and $a < b$. We assume that the functions $\omega_i(x, y_1, y_2)$, $i = 1, 2$ are continuous and satisfy Lipschitz condition in the variables y_1, y_2 in a region D_1 (such that intersection $D_1 \cap \{(x, y_1, y_2) : x = x^*; y_1 \in \mathbb{R}, y_2 \in \mathbb{R}\} \neq \emptyset$ for each $x^* \in (a, b)$ but $\{(x, y_1, y_2) \in D_1, (x-a)(x-b) = 0\} = \emptyset$) indicated below. Under these conditions the solutions of the system (1) are in D_1 uniquely determined by their initial data but for $x = a$ or $x = b$ this need not be the case.

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Definition. The solution of the problem (1), (2) we define as a vector-function $y(x) = (y_1(x), y_2(x)) \in C^1(a, b)$ which on (a, b) satisfies the system (1), $(x, y_1(x), y_2(x)) \in D_1$ on (a, b) and $y_1(a^+) = A, y_2(b^-) = B$.

The Cauchy-Nicoletti problem, the generalized Cauchy problems or the boundary value problems for systems of ordinary differential equations have been considered by many authors. Singular problems of such types have been studied e. g. in works [1–10].

In this paper we give sufficient conditions for solvability and uniqueness of the problem (1), (2) in the above mentioned sense. Some estimations of the components of solutions are given too.

2. Main results

We will consider real functions $\alpha_i(x), \beta_i(x), \gamma_i(x), \delta_i(x), i = 1, 2$ which satisfy the following conditions (H 1)–(H 3):

(H 1):

$$\alpha_i(x), \beta_i(x) \in C[a, b], \alpha_i(x) \leq \beta_i(x) \text{ on } [a, b], i = 1, 2,$$

$$\alpha_1(a^+) = \beta_1(a^+) = A, \alpha_2(b^-) = \beta_2(b^-) = B;$$

(H 2):

$$\gamma_1(x), \delta_1(x) \in C(a, b], \gamma_1(x) \leq \delta_1(x) \text{ on } (a, b],$$

$$\gamma_2(x), \delta_2(x) \in C(a, b], \gamma_2(x) \leq \delta_2(x) \text{ on } (a, b]$$

and, moreover, there are finite (possible improper) integrals

$$\int_a^b \gamma_i(t) dt, \int_a^b \delta_i(t) dt, \quad i = 1, 2$$

in the Riemann's sense;

(H3): on $[a, b]$ it holds

$$\alpha_1(x) \leq A + \int_a^x \gamma_1(t) dt \leq A + \int_a^x \delta_1(t) dt \leq \beta_1(x),$$

$$\alpha_2(x) \leq B - \int_x^b \delta_2(t) dt \leq B - \int_x^b \gamma_2(t) dt \leq \beta_2(x).$$

Now, by D_1 and D we denote the domains

$$D_1 = \{(x, y_1, y_2) : x \in (a, b), \alpha_i(x) \leq y_i \leq \beta_i(x), i = 1, 2\}, \quad D = \bar{D}_1.$$

The following two theorems are the main results of the paper.

Theorem 1. Let functions $\alpha_i(x), \beta_i(x), \gamma_i(x), \delta_i(x), i = 1, 2$ satisfy the condition (H 1) – (H 3) and, moreover,

- a) $\omega_i(x, y_1, y_2) \in C(D_1)$,
- b) $\gamma_i(x) \leq \omega_i(x, y_1, y_2) \leq \delta_i(x)$ where $(x, y_1, y_2) \in D_1$,
- c) for arbitrary points $(x, \bar{y}_1, \bar{y}_2), (x, \bar{y}_1, \bar{y}_2) \in D_1$

$$|\omega_i(x, \bar{y}_1, \bar{y}_2) - \omega_i(x, \bar{y}_1, \bar{y}_2)| \leq M(|\bar{y}_1 - \bar{y}_1| + |\bar{y}_2 - \bar{y}_2|)$$

where $0 \leq M = \text{const}$. Then there is a solution of the Cauchy-Nicoletti problem (1), (2).

Theorem 2. Let all assumptions of Theorem 1 hold and, moreover, $M < 1/(b-a)$. Then the solution of the Cauchy-Nicoletti problem (1), (2) is unique.

Proof of Theorem 1. In view of a), b) and (H 2) the problem (1), (2) is equivalent in D_1 with the following system of integral equations

$$(3) \quad \begin{aligned} y_1(x) &= A + \int_a^x \omega_1(t, y_1(t), y_2(t)) dt, \\ y_2(x) &= B - \int_x^b \omega_2(t, y_1(t), y_2(t)) dt \end{aligned}$$

where the Riemann's integrals can be improper in points a or b . Define with the aid of (3) two sequences of functions $\{y_1^m(x)\}, \{y_2^m(x)\}$ on interval (a, b) as follows

$$(4) \quad \begin{aligned} y_1^{m+1}(x) &= A + \int_a^x \omega_1(t, y_1^m(t), y_2^m(t)) dt, \\ y_2^{m+1}(x) &= B - \int_x^b \omega_2(t, y_1^m(t), y_2^m(t)) dt \end{aligned}$$

where $m = 0, 1, 2, \dots$ and

$$y_i^0(x) = \frac{1}{2}(\alpha_i(x) + \beta_i(x)), \quad i = 1, 2.$$

We divide the remaining part of the proof into three parts.

I. By method of induction it may be easily proved (with the aid of (H 1) – (H 3), a) and b)) that all terms of these sequences can be continuously extended on the whole interval $[a, b]$ and, moreover,

$$(5) \quad (x, y_1^m(x), y_2^m(x)) \in D$$

if $x \in [a, b]$, $m = 0, 1, 2, \dots$ We will take into account this fact in the next text.

II. We show by Arczel's theorem that there are subsequences $\{y_i^{m_k}(x)\}$, $i = 1, 2$ of the sequences $\{y_i^m(x)\}$ which converge uniformly on $[a, b]$. It is necessary to prove that all members of these sequences are uniformly bounded and equicontinuous. The uniform boundedness follows from the fact that by previous part $(x, y_1^m(x), y_2^m(x)) \in D$, $m = 0, 1, 2, \dots$ on $[a, b]$ and functions $\alpha_i(x)$, $\beta_i(x)$, $i = 1, 2$ are bounded (by (H 1)) on $[a, b]$. Now we prove the equicontinuity. Define $\psi_i(x) = \max(|\gamma_i(x)|, |\delta_i(x)|)$, $i = 1, 2$, $\varphi_1(x) = \max(|\alpha_1(x) - A|, |\beta_1(x) - A|)$ and $\varphi_2(x) = \max(|\alpha_2(x) - B|, |\beta_2(x) - B|)$. From (4) and (5) we obtain

$$\left| \int_a^x \omega_1(t, y_1^m(t), y_2^m(t)) dt \right| \leq \varphi_1(x), \quad \left| \int_x^b \omega_2(t, y_1^m(t), y_2^m(t)) dt \right| \leq \varphi_2(x)$$

where $x \in [a, b]$, $m = 0, 1, 2, \dots$ Choose arbitrary positive number ε_1 . Then, because $\varphi_1(a) = 0$, there is a $\eta \in (0, b - a]$ such that $\varphi_1(x) < \frac{1}{2}\varepsilon_1$ if $x \in [a, a + \eta_1]$. Let $I_1 = (a + (\eta_1/2), b]$, $M_1 = \sup_{x \in I_1} \psi_1(x)$, $0 < \lambda_1^* < (\varepsilon_1/M_1)$ and $\lambda_1 = \lambda_1(\varepsilon_1) = \min\{\lambda_1^*, (\eta_1/2)\}$. We obtain for $x', x'' \in [a, b]$, $|x' - x''| < \lambda_1$:
 $\alpha)$ if $x', x'' \in I_1$ then

$$\begin{aligned} |y_1^m(x') - y_1^m(x'')| &= \left| \int_{x'}^{x''} \omega_1(t, y_1^{m-1}(t), y_2^{m-1}(t)) dt \right| \leq \\ &\leq \left| \int_{x'}^{x''} \psi_1(t) dt \right| \leq M_1 |x' - x''| \leq M_1 \lambda_1 < \varepsilon_1, \end{aligned}$$

$\beta)$ if $x', x'' \notin I_1$ or $x' \in I_1$, $x'' \notin I_1$ then $x', x'' \in [a, a + \eta_1]$ and

$$\begin{aligned} |y_1^m(x') - y_1^m(x'')| &\leq \left| \int_a^{x'} \omega_1(t, y_1^{m-1}(t), y_2^{m-1}(t)) dt \right| + \\ &+ \left| \int_a^{x''} \omega_1(t, y_1^{m-1}(t), y_2^{m-1}(t)) dt \right| \leq \varphi_1(x') + \varphi_1(x'') < \varepsilon_1. \end{aligned}$$

By analogy we can prove that for each positive number ε_2 there is (because $\varphi_2(b) = 0$) a $\eta_2 \in (0, b - a]$ such that $\varphi_2(x) < \frac{1}{2}\varepsilon_2$ if $x \in [b - \eta_2, b]$. Moreover

for $I_2 = [a, b - (\eta_2/2))$, $M_2 = \sup_{x \in I_2} \psi_2(x)$, $0 < \lambda_2^* < (\varepsilon_2/M_2)$, $\lambda_2 = \lambda_2(\varepsilon_2) = \min \{ \lambda_2^*, (\eta_2/2) \}$ we have $|y_2^m(x') - y_2^m(x'')| < \varepsilon_2$ if $x', x'' \in [a, b]$, $|x' - x''| < \lambda_2$ for both cases α): $x, x'' \in I_2$ and β): $x', x'' \notin I_2$, or $x' \in I_2, x'' \notin I_2$. Therefore the equicontinuity is proved and above-mentioned subsequences exist. We denote the limits of these subsequences as $y_1(x)$ and $y_2(x)$. In the next reasonings we will use, without loss of generality, the previous sequences instead of these subsequences. Because $(x, y_1^m(x), y_2^m(x)) \in D$ for each $m = 0, 1, 2, \dots, x \in [a, b]$ then $(x, y_1(x), y_2(x)) \in D$ on $[a, b]$ too.

III. Prove that the functions $y_1(x), y_2(x)$ satisfy on (a, b) the system (3). For each positive ε there is (in view of uniformly convergence) an index $K = K(\varepsilon)$ such that for $m > K$ and all $x \in [a, b]$ $|y_i(x) - y_i^m(x)| < \varepsilon$, $i = 1, 2$. Then in view of condition c) for $x \in (a, b)$ and $m > K$

$$\left| \int_a^x \omega_1(t, y_1(t), y_2(t)) dt - \int_a^x \omega_1(t, y_1^m(t), y_2^m(t)) dt \right| \leq 2M\varepsilon(b-a)$$

and

$$\left| \int_x^b \omega_2(t, y_1(t), y_2(t)) dt - \int_x^b \omega_2(t, y_1^m(t), y_2^m(t)) dt \right| \leq 2M\varepsilon(b-a).$$

Since ε is arbitrary, left-hand sides must be equal to zero. Therefore the vector-function $y(x) = (y_1(x), y_2(x))$ is a solution of (3) and, consequently, a solution of the problem (1), (2) with mentioned in Definition properties too. The theorem is proved.

Proof of Theorem 2. Let there exist two different solutions $y(x) = (y_1(x), y_2(x))$, $u(x) = (u_1(x), u_2(x))$ of the problem (1), (2). Then by condition c) of Theorem 1

$$(6) \quad \begin{aligned} |y_1(x) - u_1(x)| &\leq M \int_a^x \Delta(t) dt, \\ |y_2(x) - u_2(x)| &\leq M \int_x^b \Delta(t) dt \end{aligned}$$

where $\Delta(x) \equiv |y_1(x) - u_1(x)| + |y_2(x) - u_2(x)|$ and $x \in (a, b)$. Denote $\Delta = \sup_{x \in (a, b)} \Delta(x)$. From (6) it follows

$$\Delta(x) \leq M \left(\int_a^x \Delta(t) dt + \int_x^b \Delta(t) dt \right) = M \int_a^b \Delta(t) dt, \quad x \in (a, b)$$

and we obtain a contradiction, because $0 < \Delta \leq M\Delta(b-a) < \Delta$. The theorem is proved.

3. Applications

Consider the system

$$(7) \quad \begin{aligned} y_1' &= c(x) y_1 + f_1(x, y_1, y_2), \\ y_2' &= d(x) y_2 + f_2(x, y_1, y_2). \end{aligned}$$

We will suppose that $c(x) \in C(0, 1]$ and $d(x) \in C(x) \in C[0, 1)$. Define the functions $\sigma_1(x)$, $\sigma_2(x)$:

$$\sigma_1(x) \equiv \exp\left(\int_{x_1}^x c(t) dt\right)$$

where $x_1 = 0^+$ if $|\int_{0^+} c(t) dt| < \infty$ and $x_1 = 1$ in the opposite case;

$$\sigma_2(x) \equiv \exp\left(\int_{x_2}^x d(t) dt\right)$$

where $x_2 = 1^-$ if $|\int^{1^-} d(t) dt| < \infty$ and $x_2 = 0$ in the opposite case. Let $h_i(x)$, $i = 1, 2$ be nonnegative functions continuous on $(0, 1]$ and $[0, 1]$ respectively such that there exist integrals $\int_{0^+} h_1(t) dt$, $\int^{1^-} h_2(t) dt$. Denote

$$D_2 \equiv \left\{ (x, w_1, w_2) : x \in (a, b), |w_1| \leq \int_{0^+} h_1(t) dt, |w_2| \leq \int_x^{1^-} h_2(t) dt \right\}$$

and

$$g_i(x, w_1, w_2) \equiv \frac{1}{\sigma_i(x)} f_i(x, \sigma_1(x)(K_1 + w_1), \sigma_2(x)(K_2 + w_2)),$$

where $i = 1, 2$ and K_i are real constants.

Theorem 3. Assume that for certain constants K_i and functions h_i , $i = 1, 2$:

- $g_i(x, w_1, w_2) \in C(D_2)$,
- $|g_i(x, w_1, w_2)| \leq h_i(x)$ where $(x, w_1, w_2) \in C(D_2)$,
- for arbitrary $(x, \bar{w}_1, \bar{w}_2)$, $(x, \tilde{w}_1, \tilde{w}_2) \in D_2$

$$|g_i(x, \bar{w}_1, \bar{w}_2) - g_i(x, \tilde{w}_1, \tilde{w}_2)| \leq M(|\bar{w}_1 - \tilde{w}_1| + |\bar{w}_2 - \tilde{w}_2|).$$

Then there is a solution $y(x) = (y_1(x), y_2(x))$ of (7) such that on $(0, 1)$

$$|y_1(x) - K_1 \sigma_1(x)| \leq \sigma_1(x) \int_0^x h_1(t) dt,$$

(8)

$$|y_2(x) - K_2 \sigma_2(x)| \leq \sigma_2(x) \int_x^1 h_2(t) dt.$$

Proof. Consider the system

$$(9) \quad w'_i = g_i(t, w_1, w_2), \quad i = 1, 2$$

which follows from (7) if $y_i = \sigma_i(x)(K_i + w_i)$ and the problem

$$(10) \quad w_1(0^+) = w_2(1^-) = 0.$$

If we put $\alpha_1(x) \equiv -\beta_1(x) \equiv -\int_{0^+}^x h_1(t) dt$, $\alpha_2(x) \equiv -\beta_2(x) \equiv -\int_x^{1^-} h_2(t) dt$, $-\gamma_i(x) \equiv \delta_i(x) \equiv h_i(x)$, $i = 1, 2$ and $a = 0, b = 1, A = B = 0$ then all conditions of Theorem 1 hold for right sides of system (9) in $D_1 \equiv D_2$. Therefore the problem (9), (10) has a solution $w(x) = (w_1(x), w_2(x))$ with graph in D_2 . Then the system (7) have a solution $y(x) = (y_1(x), y_2(x))$ where $y_i(x) = \sigma_i(x)(K_i + w_i(x))$, $i = 1, 2$ and, moreover, inequalities (8) hold. The theorem is proved.

Example. Let the system (7) be of the form

$$y'_1 = \frac{y_1 \delta_1}{x^2} + (\cos y_2) \exp(-x^{-2}),$$

$$y'_2 = \frac{y_2 \delta_2}{(1-x)^2} + (\cos y_1) \exp(-(1-x)^{-2}),$$

where $\delta_i = \text{const}$, $\delta_i \in \{-1, 1\}$, $i = 1, 2$. In this case $c(x) = \delta_1 x^{-2}$, $d(x) = \delta_2 (1-x)^{-2}$, $\sigma_1(x) \equiv \exp(\delta_1(-x^{-1} + 1))$, $\sigma_2(x) \equiv \exp(\delta_2((1-x)^{-1} - 1))$ and

$$g_1(x, w_1, w_2) = \frac{1}{\sigma_1(x)} (\cos[\sigma_2(x)(K_2 + w_2)]) \exp(-x^{-2}),$$

$$g_2(x, w_1, w_2) = \frac{1}{\sigma_2(x)} (\cos[\sigma_1(x)(K_1 + w_1)]) \exp(-(1-x)^{-2}).$$

If we put

$$h_1(x) = \frac{1}{\sigma_1(x)} \exp(-x^{-2}), \quad h_2(x) = \frac{1}{\sigma_2(x)} \exp(-(1-x)^{-2})$$

then all assumptions of Theorem 3 hold for arbitrary chosen constants K_1, K_2 and, consequently, for each fixed constants K_1, K_2 there is a solution

$$y(x, K_1, K_2) = (y_1(x, K_1, K_2), y_2(x, K_1, K_2))$$

for which on $(0, 1)$ inequalities (8) hold.

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