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OSCILLATIONS OF ADVANCED DIFFERENTIAL EQUATIONS

The aim of this paper is to paper is to deduce oscillatory and asymptotic behaviour of the solutions of the nonlinear advanced differential equations

$$L_n y(t) \pm p(t)f(y(\tau(t))) = 0.$$

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We consider the functional differential equations with advanced argument

$$(1^+) \qquad L_n y(t) + p(t)f(y(\tau(t))) = 0,$$

$$(1^-) \qquad L_n y(t) - p(t)f(y(\tau(t))) = 0,$$

where $n \geqslant 2$ for (1^+) and $n \geqslant 3$ for (1^-),

$$L_n y(t) = \left(\frac{1}{r_{n-1}(t)} \cdots \left(\frac{1}{r_1(t)} y'(t)\right)' \cdots\right)'$$

and

(i) $p(t)$, $r_i(t)$ and $\tau(t) \in C([t_0, \infty))$, $p(t) \geq 0$, $r_i(t) > 0$, $1 \leqslant i \leqslant n-1$, $\tau(t) > t$, $\tau(t)$ is increasing;

(ii) $f(x) \in C(\mathbb{R})$, $\mathbb{R} = (-\infty, \infty)$, $xf(x) > 0$ for $x \neq 0$.

We always assume that

$$R_i(t) = \int_{t_0}^{t} r_i(s)\,ds \to \infty \text{ as } t \to \infty \text{ for } 1 \leqslant i \leqslant n-1.$$

It is usual to denote the *quasi*-derivatives of $y(t)$ by

$$L_0 y(t) = y(t), \; L_i y(t) = \frac{1}{r_i(t)}(L_{i-1} y(t))' \quad i = 1, 2, ..., n,$$

where $r_n(t) \equiv 1$.

By a solution of equation (1^+) $((1^-))$ is meant a function $y(t)$, such that $L_i y(t)$, $0 \leq i \leq n$ exist and are continuous on $[T_y, \infty)$ and $y(t)$ satisfies (1^+)

$((1^-))$. We restrict our considerations to those solutions of (1^+) and (1^-), which exist on some ray $[T_y, \infty)$ and satisfy

$$\sup\{|y(t)| : t_1 \leqslant t < \infty\} > 0 \quad \text{for any } t_1 \in [T_y, \infty).$$

Such a solution of (1^+) or (1^-) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. Equation (1^+) is said to be oscillatory if all its solutions are oscillatory.

The asymptotic behaviour of the nonoscillatory solutions of (1^+) and (1^-) is described in the following lemma which is adapted from [3] and [9] and contgains a generalization of a lemma of Kiguradze [5, Lemma 3].

Lemma 1. *Let $y(t)$ be a nonoscillatory solution of (1^+) $((1^-))$ then there exist an integer l, $l \in \{0, 1, \ldots, n\}$ and $t_1 \geqslant t_0$ with $n+l$ odd (even), such that*

$$\text{(2)} \qquad \begin{aligned} & y(t)\, L_i\, y(t) > 0, \quad 0 \leqslant i \leqslant l, \\ & (-1)^{i-l} y(t)\, L_i\, y(t) > 0, \quad l \leqslant i \leqslant n \end{aligned}$$

for all $t \geqslant t_1$.

A function $y(t)$ satisfying (2) is said to be a function of degree l (see Foster and Grimmer [4]). The set of all nonoscillatory solutions of degree l of (1^+) $((1^-))$ is denoted by $\mathcal{N}_l$. If we denote by $\mathcal{N}^+ (\mathcal{N}^-)$ the set of all nonoscillatory solutions of (1^+) $((1^-))$, then by Lemma 1

$$\mathcal{N}^+ = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \ldots \cup \mathcal{N}_{n-1} \quad \text{for } n \text{ odd},$$

$$\mathcal{N}^+ = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \ldots \cup \mathcal{N}_{n-1} \quad \text{for } n \text{ even}$$

and

$$\mathcal{N}^- = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \ldots \cup \mathcal{N}_n \quad \text{for } n \text{ even},$$

$$\mathcal{N}^- = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \ldots \cup \mathcal{N}_n \quad \text{for } n \text{ odd}.$$

In the sequel we will use the following notation, which is due to Willett [11].

Let $i_k \in \{1, 2, \ldots, n-1\}$, $1 \leqslant k \leqslant n-1$ and $t, s \in [t_0, \infty)$, we define

$$I_0 = 1,$$

$$I_k(t, s; r_{i_k}, \ldots, r_{i_1}) = \int_s^t r_{i_k}(x)\, I_{k-1}(x, s; r_{i_{k-1}}, \ldots, r_{i_1})\, dx.$$

It is easy to verify that for $1 \leqslant k \leqslant n-1$

$$\text{(3)} \qquad \begin{aligned} & I_k(t, s; r_{i_k}, \ldots, r_{i_1}) = (-1)^k I_k(s, t; r_{i_1}, \ldots, r_{i_k}), \\ & I_k(t, s; r_{i_k}, \ldots, r_{i_1}) = \int_s^t r_{i_1}(x)\, I_{k-1}(t, x; r_{i_k}, \ldots, r_{i_2})\, dx. \end{aligned}$$

We are interested in the two particular situations described in the following definitions.

Definition 1. *Equation* (1^+) *is said to have property* (A) *if for n even* (1^+) *is oscillatory (*i.e. $\mathcal{N}^+ = \emptyset$ *and for n odd* $\mathcal{N}^+ = \mathcal{N}_0$.

Definition 2. *Equation* (1^-) *is said to have property* (B) *if for n even* $\mathcal{N}^- = \mathcal{N}_0 \cup \mathcal{N}_n$ *and for n odd* $\mathcal{N}^- = \mathcal{N}_n$.

The purpose of this paper is establisch sufficient conditions for (1^+) and (1^-) to have property (A) (property (B)). For other related results regarding property (A) and property (B) of advanced linear or nonlinear differential equations the reader is referred e.g. to [2], [7], [8] and [10]. The main results of this paper are new different from analogous known ones.

Lemma 2. *Let* $L_i y(t)$ *exist for* $0 \leqslant i \leqslant n$. *Let* $0 \leqslant i \leqslant k \leqslant n-1$ *and* $t, s \in [T_y, \infty)$ *then*

$$L_i y(t) = \sum_{j=i}^{k} (-1)^{j-i} L_i y(s) I_{j-i}(s, t; r_j, \ldots, r_{i+1}) + \tag{4}$$
$$+(-1)^{k-i+1} \int_t^s I_{k-i}(x, t; r_k, \ldots, r_{i+1}) r_{k+1}(x) L_{k+1} y(x)\, dx .$$

This lemma is a generalization of Taylor's formula. The proof is immediate. Now, we use this lemma to derive some important relationships between *quasi*-derivatives of $y(t)$.

Lemma 3. *Let* $y(t)$ *be a positive function of degree* l, *with* $l \geqslant 2$. *Let* t_1 *be a number associated with* $y(t)$ *by Lemma* 1. *Then*

$$y(t) \geqslant \int_{t_1}^{t} I_{l-2}(t, x; r_1, \ldots, r_{l-2}) r_{l-1}(x) L_{l-1} y(x)\, dx, \quad t \geqslant t_1. \tag{5}$$

Proof. Applying Lemma 2 to $y(t)$ with $i = 0$, $k = l-2$, $s = t_1$, $t \geqslant s$ and using (3), we obtain

$$L_0 y(t) = \sum_{j=0}^{l-2} L_j y(t_1) I_j(t, t_1; r_1, \dots, r_j) +$$

$$+ \int_{t_1}^{t} I_{l-2}(t, x; r_1, \dots, r_{l-2}) r_{l-1}(x) L_{l-1} y(x)\, dx, \quad t \geqslant t_1,$$

which implies (5). □

Lemma 4. *Let $y(t)$ be a positive solution of degree l of* (1^+) *(of* (1^-)*), with* $l \leqslant n-1$. *Let* t_1 *be a number associated with* $y(t)$ *by Lemma* 1. *Then*

$$(6) \qquad L_l y(t) \geqslant \int_{t}^{\infty} I_{n-l-1}(x, t; r_{n-1}, \dots, r_{l+1}) p(x) f(y(\tau))\, dx, \quad t \geqslant t_1.$$

Proof. Let $y(t)$ be a positive function of degree l. Then from Lemma 2 with $i = l$, $k = n-1$, $s \geqslant t \geqslant t_1$, we have

$$(7) \qquad L_l y(t) \geqslant (-1)^{n-l} \int_{t}^{s} I_{n-l-1}(x, t; r_{n-1}, \dots, r_{l+1}) L_n y(x)\, dx, \quad t \geqslant t_1.$$

Letting $s \to \infty$ in (7) and taking into account (1^+) and the parity of $n+l$, the relation (7) leads to (6). On the other hand, applying (1^-) and Lemma 1 to (7), we obtain again (6). The proof is complete now. □

We introduce the notation

$$M_f = \max \left\{ \limsup_{y \to \infty} \frac{y}{f(y)}, \limsup_{y \to -\infty} \frac{y}{f(y)} \right\} \geqslant 0.$$

We have supposed that $\tau(t)$ is increasing function, therefore there exists its inverse function $\tau^{-1}(t)$.

Theorem 1. *Suppose that* $M_f < \infty$. *Define*

$$a_1(t) = \frac{1}{R_1(t)} \left\{ \int_{t}^{\tau(t)} r_1(x) \int_{x}^{\infty} I_{n-2}(s, x; r_{n-1}, \dots, r_2) p(s)\, ds\, dx + \right.$$

$$\left. + \int_{\tau^{-1}(t)}^{t} r_1(x) \int_{\tau(x)}^{\infty} I_{n-2}(s, x; r_{n-1}, \dots, r_2) p(s)\, ds\, dx \right\} +$$

$$a_l(t) = \int_{t}^{\infty} I_{n-l-1}(s, t; r_{n-1}, \dots, r_{l+1}) p(s) I_{l-1}(\tau(s), t; r_1, \dots, r_{l-1})\, ds$$

for $l \in \{2, 3, \dots, n-1\}$.

Further assume that for all $l \in \{1, 2, ..., n-1\}$ *with* $n+l$ *odd (even)*

$$\limsup_{t \to \infty} R_l(t)\, a_l(t) > M_f . \tag{8}$$

Then equation (1^+) *(equation* (1^-)*) has property (A) property (B)).*

Proof. Assume that (1^+) possesses a nonoscillatory solution $y(t)$, which is eventually positive. Then $y(t)$ satisfies (2) for all $t \geqslant t_1$ with integer $l \in \{1, 2, ..., n-1\}$ and moreover (6) holds.

Let $l \geqslant 2$. Then it is easy to see that $y(t) \to \infty$ as $t \to \infty$. Combining (6) with (5), one gets

$$L_l y(t) \geqslant \int_t^{\infty} I_{n-l-1}(s, t; r_{n-1}, ..., r_{l+1})\, p(s) \frac{f(y(\tau(s)))}{y(\tau(s))}\, y(\tau(s))\, ds \geqslant$$

$$\geqslant \inf_{x \geq t} \frac{f(y(\tau x)))}{y(\tau(x))} \int_t^{\infty} I_{n-l-1}(s, t; r_{n-1}, ..., r_{l+1})\, p(s) \times$$

$$\times \int_{t_1}^{\tau(s)} I_{l-2}(\tau(s), x; r_1, ..., r_{l-2})\, r_{l-1}(x)\, L_{l-1}\, y(x)\, dx\, ds \geqslant$$

$$\geqslant \inf_{z \geq y(r(t))} \frac{f(z)}{z} \int_t^{\infty} I_{n-l-1}(s, t; r_{n-1}, ..., r_{l+1})\, p(s) \times$$

$$\times \int_t^{\tau(s)} I_{l-2}(\tau(s), x; r_1, ..., r_{l-2})\, r_{l-1}(x)\, L_{l-1}\, y(x)\, dx\, ds, \quad t \geqslant t_1 .$$

Taking into account that $L_{l-1}\, y(t)$is increasing, the above inequalities yield

$$L_l\, y(t) \sup_{z \geq y(r(t))} \frac{z}{f(z)} \geqslant L_{l-1}\, y(t)\, a_l(t) . \tag{9}$$

Integration of the identity $L_l\, y(t) = L_l\, y(t)$ from t_1 to t provides

$$L_{l-1}\, y(t) \geqslant \int_{t_1}^{t} r_l(\sigma) L_l\, y(\sigma)\, d\sigma \geqslant L_l\, y(t)\, (R_l(t) - R_l(t_1)), \quad t \geqslant t_1,$$

which in view of (9) implies

$$\sup_{z \geq y(r(t))} \frac{z}{f(z)} \geqslant (R_l(t) - R_l(t_1))\, a_l(t),$$

$$M_f = \limsup_{z \to \infty} \frac{z}{f(z)} \geqslant \limsup_{t \to \infty} (R_l(t) - R_l(t_1))\, a_l(t) .$$

This contradicts (8).

Let $l = 1$. First observe that (8) together with general condition (*ii*) imposed on $f(x)$ imply

$$\int^{\infty} I_{n-1}(s, t_0; r_{n-1}, \dots, r_1)\, p(s) |f(c)|\, ds = \infty \quad \text{for all } c \neq 0, \tag{10}$$

which ty Theorem 1 of Kitamura and Kusano [6] ensures that (1^+) cannot have a nonoscillatory solution $y(t)$, such that

$$\lim_{t \to \infty} y(t) = c_0 \in I\!R - \{0\}. \tag{11}$$

Therefore, as $y(t)$ is increasing ($l = 1$),we see that $\lim_{t \to \infty} y(t) = \infty$. Hence from Lemma 4 with $l = 1$ it follows

$$L_1 y(t) \geqslant \inf_{x \geq t} \frac{f(y(\tau(x)))}{y(\tau(x))} \int_t^{\infty} I_{n-2}(s, t; r_{n-1}, \dots, r_2)\, p(s)\, y(\tau(s))\, ds, \quad t \geqslant t_1. \tag{12}$$

Integrating (12) from t to $\tau(t)$, we get

$$(y(\tau(t)) - y(t)) \sup_{z \geq y(\tau(t))} \frac{z}{f(z)} \geqslant$$

$$\geqslant \int_t^{\tau(t)} r_1(x) \int_x^{\infty} I_{n-2}(s, x; r_{n-1}, \dots, r_2)\, p(s)\, y(\tau(s))\, ds\, dx \geqslant \tag{13}$$

$$\geqslant y(\tau(t)) \int_t^{\tau(t)} r_1(x) \int_x^{\infty} I_{n-2}(s, x; r_{n-1}, \dots, r_2)\, p(s)\, ds\, dx, \quad t \geqslant t_1,$$

where we have used that $y(t)$ and $\tau(t)$ are increasing. Now integrating (12) from t_1 to t, we obtain

$$y(t) \geqslant \sup_{z \geq y(\tau(t))} \frac{z}{f(z)} \geqslant \int_{t_1}^{t} r_1(x) \int_x^{\infty} I_{n-2}(s, x; r_{n-1}, \dots, r_2)\, p(s)\, y(\tau(s))\, ds\, dx \geqslant$$

$$\geqslant \int_{\tau^{-1}(t)}^{t} r_1(x) \int_{\tau(x)}^{\infty} I_{n-2}(s, x; r_{n-1}, \dots, r_2)\, p(s)\, y(\tau(s))\, ds\, dx \geqslant$$

$$\geqslant y(\tau(t)) \int_{\tau^{-1}(t)}^{t} r_1(x) \int_{\tau(x)}^{\infty} I_{n-2}(s, x; r_{n-1}, \dots, r_2)\, p(s)\, ds\, dx, \quad t \geqslant t_1.$$

Combining the above inequalities with those in (13), one gets

$$y(\tau(t)) \sup_{z \geq y(r(t))} \frac{z}{f(z)} \geqslant y(r(t)) R_1(t) a_1(t)$$

$$M_f = \limsup_{z \to \infty} \frac{z}{f(z)} \geqslant \limsup_{t \to \infty} R_1(t) a_1(t).$$

We arrive at a contradictin to (8). Hence the first part of this theorem is proved. To prove this theorem for (1^-) we can use arguments analogous to those we have used for property (A). □

Theorem 1 generalizes Theorem 2 in [2], in which property (A) of the third order advanced equation is discussed. On the other hand, for the second order advanced equation

$$y''(t) + p(t) f(y(\tau(t))) = 0$$

Theorem 1 improves Theorem 1 and Corollary 2 in [8].

We remark that according to Theorem 1 in [6] condition (10) ensures that equation (1^+) has no nonoscillatory solution $y(t)$ satisfying (11). On the other hand, noting that if (1^+) with n odd has property (A), then every nonoscillatory solution $y(t)$ of (1^+) is decreasing. Therefore, the conclusions of Theorem 1 can be strengthened as follows

Theorem 2. *Let n be odd. Assume that* (10) *is satisfied. Let all assumption of Theorem* 1 *hold. Then every nonoscillatory solution $y(t)$ of (1^+) satisfies*

$$\lim_{t \to \infty} I_k(t, t_0; r_1, \ldots, r_k) L_k y(t) = 0 \quad 0 \leqslant k \leqslant n-1.$$

Proof. By Theorem 1, every nonoscillatory solution $y(t)$ of (1^+) belongs to the class $\mathcal{N}_0$, i.e. $y(t)$ is decreasing and according to Theorem 1 in [6] $\lim_{t \to \infty} y(t) = 0$ and (see [1])

$$\lim_{t \to \infty} I_k(t, t_0; r_1, \ldots, r_k) L_k y(t) = 0 \quad 0 \leqslant k \leqslant n-1.$$

□

Now we are prepared to compare our criterion for property (A) with those which are due to Oláh [8] and Werbowski [10].

Example 3. Let us consider the third order advanced equation

$$(14)\qquad y'''(t)+\frac{1}{t^3}y(1.55t)=0, \quad t\geqslant 1.$$

By Theorem 2 equation (14) has property (A). On the other hand, Oláh's and Werbowski's criteria for property (A) of the advanced equation

$$y'''(t)+p(t)\,y(\tau(t))=0,$$

takes form

$$(15)\qquad \limsup_{t\to\infty}\int_t^{\tau(t)}(s-t)^2p(s)\,ds>2, \quad \text{and}$$

$$(16)\qquad \liminf_{t\to\infty}\int_t^{\tau(t)}\int_s^{\infty}(\tau(x)-\tau(s))\,p(x)\,dx\,ds>\frac{1}{e},$$

respectively. It is easy to verify that those criteria cannot be applied to (14) since (15) and (16) fail for (14).

Now, let us consider the superlinear advance differential equation

$$(17^+)\qquad L_n(t)+p(t)|y(\tau(t))|^{\beta}\operatorname{sgn}y(\tau(t))=0,$$

$$(17^-)\qquad L_n(t)-p(t)|y(\tau(t))|^{\beta}\operatorname{sgn}y(\tau(t))=0,$$

for which (*i*) and (*ii*) are satisfied and $\beta>1$.

Theorem 1′. *Let all assumption of Theorem* 1 *hold with* $M_f=0$. *Then equation* (17^+) (*equation* (17^-)) *has property* (*A*) (*property* (*B*)).

Theorem 2′. *Let all assumption of Theorem* 2 *hold with* $M_f=0$. *Then every nonoscillatory solution* $y(t)$ *of* (17^+) *satisfies*

$$\lim_{t\to\infty}I_k(t,t_0;r_1,\dots,r_k)\,L_ky(t)=0, \quad 0\leqslant k\leqslant n-1.$$

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