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THE OSCILLATORY BEHAVIOR OF LINEAR RECURRENCE EQUATIONS

This paper gives sufficient conditions for the oscillation of linear recurrence equation.
Key words; recurrence equations, oscillatory solutions.

In this paper we consider the oscillatory behavior of solutions of recurrence equation

$$a_0(n)\,x(n)+a_1(n)\,x(n+1)+\,...\,+a_{m+1}(n)x(n+m+1)=0,\quad m\geq 1, \tag{L}$$

where $n\in N$, $a_k:N\to R$ for $k=0,1,...,m+1$.

A solution of equation (L) is a function $x:N\to R$ such that $\sup\{|x(n)|:n\geq n_0\}>0$ for any $n_0\in N$ and x satisfies (L) on N.

A solution x of equation (L) is called oscillatory if for every $n_1\in N$ there exists $n\geq n_1$ such that $x(n)x(n+1)\leq 0$. Otherwise it is called nonoscillatory.

Our aim is to establish sufficient conditions for the oscillation of all solutions of equation (L). It is worth to notice that existence of oscillatory solutions of equation (L) is connected with signs of coefficients a_k $(k=0,1,...,m+1)$. If all of them are positive (or negative) equation (L) possesses only oscillatory solutions. If not, equation (L) can have both oscillatory and nonoscillatory solutions. For example, the following recurrence equation

$$(4n^2+10n+3)\,x(n)+2(n+1)\,x(n+1)-(4n^2+6n+1)\,x(n+2)+2n\,x(n+3)=0$$

has oscillatory solution $x(n)=(-1)^n$ and nonoscillatory solution $x(n)=n$. So, we consider equation (L) with the following assumptions. Let for some $s\in\{1,2,...,m\}$ $a_s(n)<0$ and $a_k(n)\geq 0$ $(k=0,1,...,\,s-1,\,s+1,...,m+1)$, $n\in N$. Without loss of generality we may assume $a_s(n)=-1$. Then equation (L) takes the form

$$x(n+s)=a_0(n)\,x(n)+\,...\,+a_{s-1}(n)\,x(n+s-1)+ \\ +a_{s+1}(n)\,x(n+s+1)+\,...\,+a_{m+1}(n)\,x(n+m+1),\quad m\geq 1 \tag{E}$$

As usually we take $\prod_{j=k}^{t} q_j = 1$ and $\sum_{j=k}^{t} q_k = 0$ for $t < k$. Moreover, for convenience, we assume that inequalities about values of functions are satisfied for all large $n \in N$.

First, we prove two lemmas which are an improvement of the known results of Erbe and Zhang [1] and Györi and Ladas [2].

Lemma 1. *Consider the recurrence inequality*

$$z(n+1) - z(n) + P(n)\,z(n-k) \le 0, \tag{1}$$

where $n \in N$, *k is a positive integer,* $P: N \to R_+$. *If*

$$\limsup_{n\to\infty} \left\{ \sum_{i=0}^{k} P(n-i) + \sum_{i=1}^{k} P(n-i)\,P(n-k-i) \right\} > 1,$$

then (1) *has no positive solution.*

Lemma 2. *Consider the recurrence inequality*

$$y(n+1) - y(n) - Q(n)\,y(n+k+1) \ge 0, \tag{2}$$

where $n \in N$, *k is a positive integer,* $Q: N \to R_+$. *If*

$$\limsup_{n\to\infty} \left\{ \sum_{i=0}^{k} Q(n+i) + \sum_{i=1}^{k} Q(n+i)\,Q(n+k+i) \right\} > 1,$$

then (2) *has no positive solution.*

Proof of Lemmas 1 *and* 2. Suppose that the inequality (1) has a positive solutoion $z(n) > 0$ for $n \ge n_0$. Then for sufficiently large $n \ge n_1 \ge n_0$ we have

$$z(n-i) \ge z(n+1-i) + P(n-i)\,z(n-k-i), \quad i = 0, 1, \dots, k.$$

Summing now both sides of the above inequality from $i = 1$ to $i = k$ we obtain

$$z(n-k) \ge z(n) + \sum_{i=1}^{k} P(n-i)\,z(n-k-i). \tag{3}$$

From inequality (1) we get for $i = 1, 2, \dots, k$

$$z(n-k-i) \ge z(n+1-k-i) + P(n-k-i)\,z(n-2k-i)$$

and

(4) $$z(n-k-i) \geq z(n-k)+P(n-k-i)\,z(n-k).$$

Using now (1) and (4) in (3) we have

$$z(n-k) \geq z(n+1)+P(n)\,z(n-k) + \sum_{i=1}^{k} P(n-i)\,[1+P(n-k-i)]\;z(n-k) \geq$$

$$\geq \left\{ \sum_{i=0}^{k} P(n-i) + \sum_{i=1}^{k} P(n-i)\,P(n-k-i) \right\} z(n-k).$$

Dividing now both sides of the above inequality by $z(n-k)$ we get a contradiction. Thus Lemma 1 is proved. Similarly we can prove Lemma 2.

Now we study sufficient conditions for the oscillation of all solutions of equation (E).

Theorem 1. *If*

(5) $$\liminf_{n\to\infty} \sum_{i=0}^{m-s} B(n+i) \prod_{j=1}^{m-s+1} A(n+i+j) > \left(\frac{m-s+1}{m-s+2}\right)^{m-s+2},$$

where

$$A(n) = \sum_{k=0}^{s-2} a_k(n) \prod_{j=2}^{s-k} a_{s+1}(n-j) + a_{s-1}(n), \quad n > s,$$

and

$$B(n) = \sum_{k=s+1}^{m} a_k(n)\,a_{m+s+1-k}(n-s+k) + a_{m+1}(n),$$

then equation (E) *has only oscillatory solutions.*

Theorem 2. *Let*

(6) $$\limsup_{n\to\infty} \left\{ \sum_{i=0}^{m-s+1} B(n+i) \prod_{j=1}^{m-s+1} A(n+i+j) + \right.$$
$$\left. + \sum_{i=0}^{m-s+1} B(n+i)\,B(n+m-s+1+i) \prod_{j=1}^{m-s+1} A(n+i+j)\,A(n+m+i-s+1+j) \right\} > 1,$$

where A and B are as in Theorem 1. *Then every solution of equation* (E) *is oscillatory.*

Proof of Theorems 1 *and* 2. Suppose that x is a nonoscillatory solution of (E) and let $x(n) > 0$ for $n \geq n_0$. Then from equation (E) we have

$$x(n+s) \geq a_{s+i}(n)\,x(n+s+i), \quad i = 1,2,...,m-s,$$

which gives for $n \geq n_1 > n_0+m$

$$x(n+s-k) \geq x(n+s-1) \prod_{j=2}^{k} a_{s+1}(n-j), \quad k = 1,2,...,s \tag{7}$$

and

$$x(n+s+k) \geq a_{m+1-k}(n+k)\,x(n+m+1), \quad k = 1,2,...,m-s. \tag{8}$$

From inequalities (7) and (8) and equation (E) we obtain

$$x(n+s) \geq \left\{ \sum_{k=0}^{s-2} a_k(n) \sum_{j=2}^{s-k} a_{s+1}(n-j) + a_{s-1}(n) \right\} x(n+s-1) +$$

$$+ \left\{ \sum_{k=s+1}^{m} a_k(n)\, a_{m+s+1-k}(n-s+k) + a_{m+1}(n) \right\} x(n+m+1) =$$

$$= A(n)\,x(n+s-1) + B(n)\,x(n+m-1).$$

The last inequality gives

$$y(n+1) - y(n) - B(n) \sum_{j=1}^{m-s+1} A(n+j)\,y(n+m-s+2) \geq 0,$$

where $x(n+s) = y(n+1) \prod_{j=s+1}^{n} A(j)$. In view of Theorem 7.6. 1 of [2] and (5) the last inequality cannot possess positive solution, which contradicts the fact that y is positive. Thus Theorem 1 is proved. Similarly, in view of (6) and Lemma 2 we get that Theorem 2 is true.

We give now other sufficient conditions for the oscillation of all solutions of equation (E).

Theorem 3. *Let*

$$\liminf_{n\to\infty} \sum_{i=0}^{s-1} C(n-i) \prod_{j=1}^{s} D(n-i-j) > \left(\frac{s}{s+1}\right)^{s+1}, \tag{9}$$

where

$$C(n)=a_0(n)+\sum_{k=1}^{s-1} a_k(n)\,a_{s-k}(n-s+k), \quad n>s$$

and

$$D(n)=a_{s+1}(n)+\sum_{k=s+2}^{m+1} a_k(n)\prod_{j=2}^{k-s} a_{s-1}(n+j).$$

Then every solution of (E) *oscillates.*

Theorem 4. *If*

$$\limsup_{n\to\infty}\left\{\sum_{i=0}^{s} C(n-i)\prod_{j=1}^{s} D(n-i-j)+\right.$$

(10)

$$\left.+\sum_{i=1}^{s} C(n-i)\,C(n-s-i)\prod_{j=1}^{s} D(n-i-j)\,D(n-s-i-j)\right\}>1,$$

where C and D are the same as in Theorem 3, *then equation* (E) *possesses only oscillatory solutions.*

Proof of Theorems 3 *and* 4. Assume that x is a nonoscillatory solution of (E) and let $x(n)>0$ *for* $n\geq n_0>s$. Then equation (E) gives

$$x(n+s)\geq a_{s-i}(n)\,x(n+s-i) \quad \text{for} \quad i=1,2,\ldots,s.$$

Hence we have

(11) $$x(n+k)\geq a_{s-k}(n-s+k)\,x(n) \quad \text{for} \quad k=1,2,\ldots,s.$$

and

(12) $$x(n+s+k)\geq x(n+s+1)\prod_{j=2}^{k} a_{s-1}(n+j) \quad \text{for} \quad k=2,\ldots,m-s+1.$$

Applying now inequalities (11) and (12) in (E) we obtain

$$x(n+s)\geq\left\{a_0(n)+\sum_{k=1}^{s-1} a_k(n)\,a_{s-k}(n-s+k)\right\}x(n)+$$

$$+\left\{a_{s+1}(n)+\sum_{k=s+2}^{m+1} a_k(n)\sum_{j=2}^{k-s} a_{s-1}(n+j)\right\}x(n+s+1)=C(n)\,x(n)+D(n)\,x(n+s+1).$$

Hence we get

$$z(n+1)-z(n)+C(n)\prod_{j=1}^{s} D(n-j)z(n-s)\leq 0,$$

where $z(n)=x(n+s)\prod_{j=1}^{n-1} D(j)$. Applying now Theorem 2 of [3] to the above inequality, in view of (9) we obtain a contradiction with the fact that z is positive. Similarly we can prove Theorem 4, view of Lemma 1 and (10).

We give now two another oscillation critera for equation (E).

Theorem 5. *If*

$$\liminf_{n\to\infty} A(n)D(n-1)>\frac{1}{4}, \tag{13}$$

where A nad D are as in Theorems 1 and 3, *then equation* (E) *has only oscillatory solutions.*

Theorem 6. *Let*

$$\begin{aligned}\limsup_{n\to\infty}\{&A(n)D(n-1)+A(n-1)D(n-2)+\\ &+A(n-1)A(n-2)D(n-2)D(n-3)\}>1,\end{aligned} \tag{14}$$

where A nad D are as above. Then every solution of (E) *is oscillatory.*

Proof of Theorems 5 *and* 6. Let x be a nonoscillatory positive solution of (E). Thus, as in the proofs of the above theorems, inequalities (7) and (12) are true. Applying them in (E) we obtain

$$\begin{aligned}x(n+s)\geq&\left\{\sum_{k=0}^{s-2} a_k(n)\sum_{j=2}^{s-k} a_{s+1}(n-j)+a_{s-1}(n)\right\}x(n+s-1)+\\ &+\left\{a_{s+1}(n)+\sum_{k=s+2}^{m+1} a_k(n)\sum_{j=2}^{k-s} a_{s-1}(n+j)\right\}x(n+s+1)=\\ &=A(n)x(n+s-1)+D(n)x(n+s+1)\end{aligned}$$

which gives

$$z(n+1)-z(n)+A(n)D(n-1)z(n-1)\leq 0,$$

where $z(n)=x(n+s)\prod_{j=1}^{n-1}D(j)$. In view of conditions (13) and (14), similarly as in the proofs of Theorems 3 and 4, we see that last inequality cannot possess positive solutions. Thus the theorems are proved.

It is worth to mention that the conditions in the corresponding theorems are mutually independent. For exaple, let us check independence of theorems with lower limit. Consider the following recurrence equation

$$50x(n)+4n\,x(n+1)-50n^2x(n+2)+15n^3x(n+3)+50n^4x(n+4)=0,\quad n\in N.$$

It possesses only oscillatory solutions since condition (5) is satisfied. However conditions (9) and (13) are not fulfilled. Now consider the recurrence equation of the form

$$50n^4x(n)+15n^3x(n+1)-50n^2x(n+2)+4n\,x(n+3)+50x(n+4)=0,\quad n\in N.$$

The above equation has only oscillatory solutions, too, but in this case condition (9) is true and conditions (5) and (13) are not satisfied. At last consider the recurrence equation

$$4x(n)+5n\,x(n+1)-20n^2x(n+2)+12n^3x(n+3)+12n^4x(n+4)=0,\quad n\in N.$$

Every solution of this equation is oscillatory because the condition (13) is fulfilled however conditions (5) and (9) are not true.

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