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ON LIMIT PROPERTIES OF SOLUTION OF EQUATION $(\Delta - a^2)u(x,y) = 0$ IN THE HÖLDER METRICES

In this paper we present some theorems on the limit properties of the solution of equation $(\Delta - a^2)u(x,y) = 0$ in the Hölder spaces. These theorems extends the results given in [6].

Key words: degree of convergence, Hölder spaces, Dirichlet problem.

1. Notations

1.1. In the paper [5] the author considered the solution of the Dirichlet problem for the equation $(\Delta - a^2)u(x,y) = 0$, $a > 0$ in the halfplane $\mathbb{R}^2_+ = \{(x,y): |x| < \infty, \ y > 0\}$, where Δ is the Laplace operator.

This solution is given in the form:

$$u(x,y;f) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x-s)\,Q(s,y)\,ds \tag{1}$$

where

$$Q(s,y) := \sqrt{\frac{2}{\pi}}\, ay \frac{1}{\sqrt{y^2+s^2}} K_1\left(a\sqrt{y^2+s^2}\right), \tag{2}$$

K_ν is the Mac Donald function of the order ν ([7]), f is a given boundary function and $\mathbb{R} := (-\infty, +\infty)$.

In the paper [6] some theorems concerning the limit properties of solution (1) were given. In our paper we shall consider a similiar problem in generalized Hölder spaces H^∞ and $\tilde{H}^\omega$. We study the difference

$$L(x,y;f) := u(x,y;f) - f(x) \tag{3}$$

in the set $P := \left\{(x,y): |x| < \infty, \ 0 < y < \frac{1}{2}\right\}$ if $y \to 0+$, using the generalized Hölder's norms.

The similar problem concerning other singular integrals is considered in [1] and [2].

1.2. Let C be the space of real – valued functions uniformly continuous and bounded in $\mathbb{R}$ with the norm

$$\|f\|_C := \sup_{x\in\mathbb{R}} |f(x)|. \tag{4}$$

For given function $f\in C$ we denote by $\omega(t; f)$, $t > 0$, the modulus of continuity, i.e.

$$\omega(t; f) := \sup_{|h| \le t} \|\Delta_h f\|_C, \tag{5}$$

where

$$\Delta_h f(x) := f(x+h) - f(x) \quad x, h \in \mathbb{R}. \tag{6}$$

Let Ω be the set of functions of the modulus continuity types i.e. Ω is the set of all functions with properties:

(a) ω is continuous in $[0, \infty)$,

(b) ω is increasing for $t > 0$ and $\omega(0) = 0$,

(c) $\dfrac{\omega(t)}{t}$ is decreasing for $t > 0$.

Analogously as in [3] for given $\omega \in \Omega$ we define the space H^ω, i.e. the set of all functions f for which

$$\|f\|_\omega := \sup_{h>0} \frac{\|\Delta_h f\|_C}{\omega(h)} \tag{7}$$

is finite.

In the space H^ω we introduce the norm

$$\|f\|_{H^\omega} := \|f\|_C + \|f\|_\omega. \tag{8}$$

Similarly as in [3] we denote by $\tilde{H}^\omega$ the space of functions $f\in C$ such that

$$\lim_{h\to 0+} \frac{\|\Delta_h f\|_C}{\omega(h)} = 0. \tag{9}$$

In $\tilde{H}^\omega$ the norm is defined by (8).

H^ω and $\tilde{H}^\omega$ are called generalized Hölder's spaces. In the case $\omega(t) = t^\alpha$, $0 < \alpha \le 1$, H^ω is the classical Hölder space.

It is known ([3], [4]) that if $f \in H^\omega$, then

$$\omega(t; f) \le \omega(t)\|f\|_\omega, \quad t > 0, \tag{10}$$

and if $f \in \tilde{H}^{\omega}$, then

$$\omega(t;f) = o(\omega(t)), \quad \text{as } t \to 0^+. \tag{11}$$

Moreover, it is known [3]), that if ω, $\mu \in \Omega$ are such that the function

$$q(t) := \frac{\omega(t)}{\mu(t)}, \quad t > 0, \tag{12}$$

is nondecreasing, then $H^{\omega} \subseteq H^{\mu}$ and $\tilde{H}^{\omega} \subseteq \tilde{H}^{\mu}$.

In further sections we shall denote by $M_i(\cdot)$, $i = 1, 2, \ldots$, suitable positive constans depending on indicated parameters.

2. Auxiliary results

In this section we give some auxiliary inequalities, which we use in main theorems.

It is known ([6]) that

$$u(x, y; 1) = e^{-ay} \quad \text{for } y > 0, \tag{13}$$

$$Q(s, y) \leq \sqrt{\frac{2}{\pi}}\, y(y^2 + s^2)^{-1} \quad \text{for } (s, y) \in \mathbb{R}^2_+, \tag{14}$$

and there exists $s_0 \geq 1$ such that

$$Q(s, y) \leq M_1 y s^{-3/2} \exp(-as) \quad \text{for } s \geq s_0,\ y > 0. \tag{15}$$

Lemma 1. *If $f \in C$, then*

$$\|u(\cdot, y; f)\|_C \leq \|f\|_C \quad \text{for } y > 0$$

Proof. From (1) and (4) if follows that

$$\|u(\cdot, y; f)\|_C \leq \|f\|_C \int_{-\infty}^{+\infty} Q(s, y)\, ds = \|f\|_C\, u(\cdot, y; 1),$$

for $y > 0$. By (13) we have

$$u(x, y; 1) = e^{-ay} < 1, \quad \text{for } y > 0,$$

so the proof is complete.

Lemma 2. *If* $f \in H^{\omega}$, *then*

$$\|u(\cdot, y; f)\|_{\omega} \le \|f\|_{\omega} \quad \text{for } y > 0.$$

which proves that $u(\cdot, y; f) \in H^{\omega}$ *for every fixed* $y > 0$.

Proof. The formulae (1) and (6) imply

$$\Delta_h u(x, y; f) = u(x, y; \Delta_h f) \quad \text{for } |x| < \infty,\ y > 0,\ h \in \mathbb{R}.$$

Hence by Lemma 1 we have

$$\|\Delta_h u(\cdot, y; f)\|_C = \|u(\cdot, y; \Delta_h f)\|_C \le \|\Delta_h f\|_C, \quad \text{for } y > 0, \quad h > 0.$$

From the above and by (7) we get

$$\|u(\cdot, y; f)\|_{\omega} = \sup_{h>0} \frac{\|\Delta_h u(\cdot, y; f)\|_C}{\omega(h)} \le \sup_{h>0} \frac{\|\Delta_h f\|}{\omega(h)} = \|f\|_{\omega} \quad \text{for } y > 0.$$

Thus the proof is complete.

Similarly as Lemma 2 we can prove

Lemma 3. *If* $f \in \tilde{H}^{\omega}$, *then* $u(\cdot, y; f) \in \tilde{H}^{\omega}$ *for every fixed* $y > 0$.

3. Theorems

3.1. First we shall give an estimation of the norm of $L(\cdot, y; f)$, defined by (3), in the space C.

Theorem 1. *If* $f \in C$, *then*

$$\|L(\cdot, y; f)\|_C \le M_2(a)\, \omega(y|\ln y|; f) \tag{16}$$

for all $y \in (0, 1/2)$.

Proof. In view of (1), (2), (3) and (13) we have

$$L(x, y; f) = \int_{-\infty}^{+\infty} [f(x-s) - f(x)]\, Q(s, y)\, ds + f(x)(e^{-ay} - 1),$$

for $|x| < \infty$, $y > 0$. Hence

$$\|L(x,y;f)\|_C \le \int_{-\infty}^{+\infty} \omega(|s|;f)\,Q(s,y)\,ds + |e^{-ay}-1|\,\|f\|_C$$

for $y>0$. Using the inequality

$$\omega(\lambda t;f) \le (\lambda+1)\,\omega(t;f) \quad \text{for } \lambda>0,\ t\ge 0,$$

([4]), we get

$$\int_{-\infty}^{+\infty} \omega(|s|;f)\,Q(s,y)\,ds \le 2\omega(y|\ln y|;f)\int_0^{+\infty}\left(\frac{s}{|y\ln y|}+1\right)Q(s,y)\,ds$$

for $y\in(0,1/2]$. Reasoning analogously as in [6], we have

$$\int_0^{+\infty} sQ(s,y)\,ds = \left(\int_0^y + \int_y^{s_0} + \int_{s_0}^{+\infty}\right) sQ(s,y)\,ds =: I_1+I_2+I_3,$$

for $y\in(0,1/2)$, where s_0 is the number defined in (15). Applying (14) to I_1, we obtain

$$I_1 \le \sqrt{\frac{2}{\pi}}\int_0^y \frac{sy}{y^2+s^2}\,ds \le \frac{1}{2}y \le y|\ln y|,$$

and

$$I_2 \le \int_y^{s_0} \frac{sy}{y^2+s^2}\,ds \le M_3(s_0)\,y|\ln y|, \quad \text{for } y\in(0,1/2).$$

Using (15) to I_3, we get

$$I_3 \le M_1 \int_{s_0}^{+\infty} ys^{-1/2}e^{-as}ds \le M_1\,ys_0^{-1/2}\int_0^{+\infty} e^{-as}ds = M_4(s_0,a)\,y$$

for $y\in(0,1/2)$. By (13)

$$2\int_0^{+\infty} Q(s,y)\,ds = e^{-ay} < 1 \quad \text{for } y\in(0,1/2).$$

From the above it follows that

$$\int_{-\infty}^{+\infty} \omega(|s|;f)\,Q(s,y)\,ds \le M_5(s_0,a)\,\omega(y\,|\ln y|;f),$$

for $y\in(0,1/2)$. The Taylor formula implies

$$|e^{-ay}-1| = 1-e^{-ay} < ay, \quad \text{for } y>0,\ a>0.$$

Collecting these results we obtain (16) which completes the proof.

Theorem 1 and the inequalities (10), (11) imply the following corollaries.

Corollary 1. *If* $f \in H^{\omega}$, *then*

$$\|L(\cdot, y; f)\|_C \le M_6(a)\,\omega(y|\ln y|)\,\|f\|_{\omega} \quad \text{for } y \in (0, 1/2).$$

In particular, if $f \in H^{\omega}$, $\omega(t) = t^{\alpha}$, $0 < \alpha \le 1$, *then*

$$\|L(\cdot, y; f)\|_C \le M_7(a)\,(y|\ln y|)^{\alpha} \quad \text{for } y \in (0, 1/2).$$

Corollary 2. *If* $f \in \tilde{H}^{\omega}$, *then*

$$\|L(\cdot, y; f)\|_C = o(\omega(y|\ln y|)) \quad \text{as } y \to 0+.$$

In the case if $\omega(t) = t^{\alpha}$, $0 < \alpha \le 1$, *then*

$$\|L(\cdot, y; f)\|_C = o((y|\ln y|)^{\alpha}) \quad \text{as } y \to 0+.$$

The estimations presented in Corollaries 1 and 2 for the function $f \in H^{\omega}$ with $\omega(t) = t^{\alpha}$, i.e. $\omega(t; f) \le \|f\|_{\omega} t^{\alpha}$, can be improved. Namely, repeating exactly the proof of Theorem 1, it can be shown the following

Theorem 2. *If* $f \in H^{\omega}$, $\omega(t) = t^{\alpha}$, $0 < \alpha \le 1$ *for* $t > 0$, *then*

$$\|L(\cdot, y; f)\|_C \le M_8(\alpha) \begin{cases} y^{\alpha} & \text{if } 0 < \alpha < 1 \\ y|\ln y| & \text{if } \alpha = 1 \end{cases}$$

for all $y \in (0, 1/2)$.
If $f \in \tilde{H}^{\omega}$, $\omega(t) = t^{\alpha}$, $0 < \alpha \le 1$ *for* $t > 0$, *then*

$$\|L(\cdot, y; f)\|_C \begin{cases} o(y^{\alpha}) & \text{if } 0 < \alpha < 1 \\ o(y|\ln y|) & \text{if } \alpha = 1 \end{cases}$$

as $y \to 0+$.

3.2. Now we give the estimation of the norm of $L(\cdot, y; f)$, defined by (3), in the space H^{ω}.

Theorem 3. *If* $f \in H^{\omega}$ *and* $\mu \in \Omega$ *is the function such that* $q(h) = \dfrac{\omega(h)}{\mu(h)}$ *is nondecreasing for* $h > 0$, *then*

(17) $$\|L(\cdot, y; f)\|_{H^\mu} \leq M_9^* q(y|\ln y|)$$

for all $y \in (0, 1/2)$, *where* $M_9^* = M_9(a, \|f\|_C, \|f\|_\omega)$.

Proof. By the assumption on the function q we have $H^\omega \subseteq H^\mu$ and from (8) we can write

$$\|L(\cdot, y; f)\|_{H^\mu} = \|L(\cdot, y; f)\|_C + \|L(\cdot, y; f)\|_\mu,$$

where the estimation of $\|L(\cdot, y; f)\|_C$ is given in Theorem 1, i.e.

$$\|L(\cdot, y; f)\|_C \leq M_2(a, \mu)\|f\|_\omega q(y|\ln y|) \quad \text{for } y \in (0, 1/2).$$

By (7)

$$\|L(\cdot, y; f)\|_\mu = \sup_{h>0} \frac{\|\Delta_h L(\cdot, y; f)\|_C}{\mu(h)}.$$

Let us denote $y^* := y|\ln y|$, for every fixed $y \in (0,1/2)$. Then

$$\|L(\cdot, y; f)\|_\mu \leq T(y) + S(y) \quad \text{for } y \in (0, 1/2),$$

where

$$T(y) := \sup_{h>y^*} \frac{\|\Delta_h L(\cdot, y; f)\|_C}{\mu(h)}, \quad S(y) := \sup_{0<h\leq y^*} \frac{\|\Delta_h L(\cdot, y; f)\|_C}{\mu(h)},$$

By the monotonicity of μ it follows that

$$T(y) \leq \frac{1}{\mu(y^*)} \sup_{h>y^*} \|\Delta_h L(\cdot, y; f)\|_C.$$

For any $F \in C$ we have $\|\Delta_h F\|_C = \|F(x+h) - F(x)\|_C \leq 2\|F\|_C$. Hence using Corollary 1, we get

$$T(y) \leq \frac{1}{\mu(y^*)} 2\|L(\cdot, y; f)\|_C \leq 2M_6(a) \frac{1}{\mu(y^*)} \omega(y|\ln y|)\|f\|_\omega \leq$$

$$\leq 2M_6(a) q(y^*)\|f\|_\omega = 2M_6(a) q(y|\ln y|)\|f\|_\omega,$$

for every $y \in (0, 1/2)$. Further we have

$$\Delta_h L(x, y; f) = u(x, y; \Delta_h f) - \Delta_h f(x),$$

which by (4) and Lemma 1 yields

$$\|\Delta_h L(\cdot, y; f)\|_C \leq \|u(\cdot, y; \Delta_h f)\|_C + \|\Delta_h f\|_C \leq 2\|\Delta_h f\|_C.$$

Therefore

$$S(y) \le \sup_{0>h\le y^*} \frac{2\|\Delta_h f\|_C}{\mu(h)} \le 2q(y^*)\,\|f\|_\omega.$$

Combining the above estimations, we obtain (17), which completes the proof.
Analogously we can prove the following

Theorem 4. *Let* ω, μ *and* q *be as in Theorem* 3. *If* $f \in \tilde{H}^\omega$, *then* $\|L(\cdot, y; f)\|_{\tilde{H}^\mu} = o(q(y|\ln y|))$ *as* $y \to 0+$.

From Theorems 3, 4 we can deduce the following two corollaries.

Corollary 3. *Assume that* ω, μ *and* q *satisfy the conditions of Theorem* 3 *and morover,* $q(t) \le M_{10} t^\gamma$, $0 < \gamma < 1$, for $t > 0$.

If $f \in H^\omega$, *then*

$$\|L(\cdot, y; f)\|_{H^\mu} \le M_{11}^* (y|\ln y|)^\gamma \quad \textit{for}\ y \in (0, 1/2),$$

where $M_{11}^* = M_{11}(a, \|f\|_C, \|f\|_\omega, M_{10})$.

In addition, if $f \in \tilde{H}^\omega$, *then*

$$\|L(\cdot, y; f)\|_{\tilde{H}^\mu} = o((y|\ln y|)^\gamma) \quad \textit{as}\ y \to 0+.$$

Corollary 4. *Let* $\omega(t) = t^\alpha$, $\mu(t) = t^\beta$, $0 < \beta < \alpha \le 1$, *for* $t > 0$.

If $f \in H^\omega$, *then*

$$\|L(\cdot, y; f)\|_{H^\mu} = O((y|\ln y|)^{\alpha-\beta}) \quad \textit{for}\ y \in (0, 1/2)$$

Moreover, if $f \in \tilde{H}^\omega$, *then*

$$\|L(\cdot, y; f)\|_{\tilde{H}^\mu} = o((y|\ln y|)^{\alpha-\beta}) \quad \textit{as}\ y \to 0+.$$

The estimation given in Corollary 4 can be improved. Arguing analogously as in the proof of Theorem 3, we can obtain the following.

Theorem 5. *Let* $\omega(t) = t^\alpha$, $\mu(t) = t^\beta$, $0 < \beta < \alpha \le 1$, *for* $t > 0$.

If $f \in H^\omega$, *then*

$$\|L(\cdot, y; f)\|_{H^\mu} \begin{cases} O(y^{\alpha-\beta}) & \textit{if}\ 0 < \alpha < 1 \\ O(y^{1-\beta}|\ln y|) & \textit{if}\ \alpha = 1 \end{cases}$$

for $y \in (0, 1/2)$. *If* $f \in \tilde{H}^\omega$, *then*

$$\|L(\cdot, y; f)\|_{\tilde{H}^{\mu}} \begin{cases} o(y^{\alpha-\beta}) & \textit{if} \quad \alpha < 1 \\ o(y^{1-\beta}|\ln y|) & \textit{if} \quad \alpha = 1 \end{cases}$$

as $y \to 0+$.

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