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ON A SINGULAR INITIAL VALUE PROBLEM FOR A SYSTEM OF INTEGRO-DIFFERENTIAL EQUATIONS DEPENDING ON A PARAMETER

The purpose of the present paper is to study the existence and uniqueness of solutions of the singular system of integro-differential equations and continuous dependence of solutions on a parameter.

Key words: system of integro-differential equations, Banach space, Banach contraction principle.

In this paper we deal with the following problem:

$$y'(x) = f(x, y, \mu) + \int_{0+}^{x} g(x, s, y(s), \mu)\, ds, \quad y(0^+, \mu) = 0, \tag{1}$$

where $f(f_1, \dots, f_n)^T$, $g = (g_1, \dots, g_n)^T$, $y^{(i)} = (y_1^{(i)}, \dots, y_n^{(i)})^T$, $i = 0, 1$, $\mu \in R$. The functions f, g will be assumed to satisfy:

(*I*) $f \in C(D_1)$, $D_1 = \{(x, u, \mu) \in J \times R^n \times R : |u| \le \phi(x),\ J = (0, x_0]\}$,
$g \in C(D_2)$, $D_2 = \{(x, s, u, \mu) \in J \times J \times R^n \times R : |u| \le \phi(x)\}$,
where $x_0 > 0$, $0 < \phi(x) \in C^0(J)$, $\phi(0^+) = 0$.

(*II*) $|f(x, \bar{y}, \mu) - f(x, \bar{\bar{y}}, \mu)| \le M|\bar{y} - \bar{\bar{y}}|$ for every $(x, \bar{y}, \mu)$, $(x, \bar{\bar{y}}, \mu) \in D_1$,
$|g(x, s, \bar{y}, \mu) - g(x, s, \bar{\bar{y}}, \mu)| \le N|\bar{y} - \bar{\bar{y}}|$ for every $(x, s, \bar{y}, \mu)$, $(x, s, \bar{\bar{y}}, \mu) \in D_2$,
where $M \ge 0$, $N \ge 0$ *and* $|.|$ denotes the usual norm in R^n.

Similar problems for ordinary and integro-differential equations without a parameter were studied by many authors (see e.g. [1], [2], [3]).

The results of this paper extend those of [4]. We shall also consider the problem of continuous dependence on the parameter μ.

Theorem 1. *If the functions* $f(x, u, \mu)$, $g(x, s, u, \mu)$ *satisfy* (*I*), (*II*), *on* D_1

$$|f(x, u, \mu)| \le \Phi(x), \text{ where } 0 < \Phi(x) \in C(J), \int_{0+}^{x} \Phi(s)\, ds \le \alpha\phi(x), \ \alpha \in R^+$$

and on D_2

$|g(x,s,u,\mu)| \leq \psi(x,s)$, where $0 < \psi(x,s) \in C(J \times J)$, $\int_{0+}^{x}\int_{0+}^{s} \psi(s,t)\,dt\,ds \leq \beta\phi(x)$, $\beta \in R^{+}$, $\alpha+\beta = 1$ hold, then the initial problem (1) has for each $\mu \in R$ a unique solution $y(x,y)$ such that $(x, y(x,\mu), \mu) \in D_1$ for $x \in J$.

Proof. Denote H the Banach space of continuous vector-valued functins

$$h: J_0 \to R^n,\ J_0 = [0, x_0],\ |h(x)| \leq \phi(x) \text{ on } J$$

with norm

$$\|h\|_\lambda = \max_{x \in J_0} \{\exp(-\lambda x)|h(x)|\},$$

where λ is an arbitrary parameter. The identity

$$\left|\int_0^x \exp(\lambda t)\,dt\right| = \frac{1}{\lambda}\{\exp(\lambda x) - 1\}$$

is valid for every $x \in J_0$, $\lambda > 0$.

The initial value problem (1) is equivalent to the system of integral equations

$$y(x) = \int_{0+}^{x} f(s, y(s), \mu)\,ds + \int_{0+}^{x}\int_{0+}^{s} g(s,t,y(t),\mu)\,dt\,ds. \tag{2}$$

Define the operator T by right-hand side of (2):

$$T(h) = \int_{0+}^{x} f(s, h(s), \mu)\,ds + \int_{0+}^{x}\int_{0+}^{s} g(s,t,h(t),\mu)\,dt\,ds,$$

where $h \in H$. Let $\mu \in$ R be fixed. The transformation $\Psi = T(h)$ maps H continuously into itself because

$$|T(h)| \leq \int_{0+}^{x} |f(s,h(s),\mu)|\,ds + \int_{0+}^{x}\int_{0+}^{s} |g(s,t,h(t),\mu)|\,dt\,ds \leq$$

$$\leq \int_{0+}^{x} \Phi(s)\,ds + \int_{0+}^{x}\int_{0+}^{s} \psi(s,t)\,dt\,ds \leq \alpha\phi(x) + \beta\phi(x) = \phi(x)$$

for every $h \in H$. We shall prove that

$$\|T(h_2) - T(h_1)\|_\lambda \leq \left(\frac{M}{\lambda} + \frac{N}{\lambda^2}\right)\|h_2 - h_1\|_\lambda \tag{3}$$

for all h_1, $h_2 \in H$ and $\lambda > 0$. Using (II) and the definition $\|.\|_\lambda$ we have

$$|T(h_2)| - T(h_1)| \le \int_{0+}^{x} (f(s, h_2(s), \mu) - f(s, h_1(s), \mu))\, ds| + |\int_{0+}^{x}\int_{0+}^{s} (g(s, t, h_2(t), \mu) -$$

$$- g(s, t, h_1(t), \mu))\, dt\, ds| \le M \int_{0+}^{x} |h_2(s) - h_1(s)|\, ds + N \int_{0+}^{x}\int_{0+}^{s} |h_2(t) - h_1(t)|\, dt\, ds \le$$

$$\le M\|h_2(x) - h_1(x)\|_\lambda \int_{0+}^{x} \exp(\lambda s)\, ds + N\|h_2(x) - h_1(x)\|_\lambda \int_{0+}^{x}\int_{0+}^{s} \exp(\lambda t)\, dt\, ds \le$$

$$\le M\|h_2(x) - h_1(x)\|_\lambda \left(\frac{\exp(\lambda x)}{\lambda} - \frac{1}{\lambda}\right) + N\|h_2(x) - h_1(x)\|_\lambda \left(\frac{\exp(\lambda x)}{\lambda^2} - \frac{1}{\lambda^2} - \frac{x}{\lambda}\right) <$$

$$< \|h_2(x) - h_1(x)\|_\lambda \exp(\lambda x)\left(\frac{M}{\lambda} + \frac{N}{\lambda^2}\right).$$

Thus

$$\|T(h_2) - T(h_1)\|_\lambda = \max_{x \in J_0} \{\exp(-\lambda x)|T(h_2) - T(h_1)|\} \le \|h_2 - h_1\|_\lambda \left(\frac{M}{\lambda} + \frac{N}{\lambda}\right).$$

Now we choose $\lambda > 0$ so that $\left(\frac{M}{\lambda} + \frac{N}{\lambda^2}\right) < 1$ and apply the classical Banach contraction priniciple to T and the distance function $\|h_2 - h_1\|_\lambda$ to complete the proof.

Theorem 2. Let the assumptions of Theorem 1 be satisfied. If there exist constants L_1, L_2, and integrable functions $\gamma : J_0 \to J_0$, $\omega : J_0 \times J_0 \to J_0$ such that

$$|f(x, u, \mu_2) - f(x, u, \mu_1)| \le \gamma(x)|\mu_2 - \mu_1|, \text{ where } (x, u, \mu_1), (x, u, \mu_2) \in D_1,$$

$$|g(x, s, u, \mu_2) - g(x, s, u, \mu_1)| \le \omega(x, s)|\mu_2 - \mu_1|, \text{ where } (x, s, u, \mu_1), (x, s, u, \mu_2) \in D_2$$

and

$$\max_{x \in J_0}\left\{\exp(-\lambda x)\int_{0+}^{x} \gamma(s)\, ds\right\} \le L_1, \quad \max_{(x,s) \in J_0 \times J_0}\left\{\exp(-\lambda x)\int_{0+}^{x}\int_{0+}^{s} \omega(s, t)\, dt\, ds\right\} \le L_2$$

then the solution $y(x, \mu)$ of (1) fulfilling $y(0^+, \mu) = 0$ is continuous with respect to the variables $(x, \mu) \in J \times R$.

Proof. For $h \in H$ we define the transformation $T_\mu(h)$ by the rieght-hand side of (2). From (3) we obtain

$$\|T_\mu(h) - T_\mu(y)\|_\lambda \le \left(\frac{M}{\lambda} + \frac{N}{\lambda^2}\right)\|h - y\|_\lambda.$$

From assumptions of Theorem 2 we have

$$\exp(-\lambda x)|T_{\mu_2}(h)-T_{\mu_1}(h)| \leq \exp(-\lambda x)\left(\int_{0+}^{x}\gamma(s)\,ds+\int_{0+}^{x}\int_{0+}^{s}\omega(s,t)\,dt\,ds\right)|\mu_2-\mu_1|$$

and hence

$$\|T_{\mu_2}(h)-T_{\mu_1}(h)\|_\lambda \leq (L_1+L_2)\,|\mu_2-\mu_1|.$$

From Thoerem 1 there exists the unique function $h(x,\mu)\in H$ such that $T_\mu(x,\mu)) = h(x,\mu)$.

Therefore, we have

$$\|h(x,\mu_2)-h(x,\mu_1)\|_\lambda = \|T_{\mu_2}(h(x,\mu_2))-T_{\mu_2}(h(x,\mu_1))+T_{\mu_2}(h(x,\mu_1))-T_{\mu_1}(h(x,\mu_1))\| \leq$$

$$\leq \|T_{\mu_2}(h(x,\mu_2))-T_{\mu_2}(h(x,\mu_1))\|+\|T_{\mu_2}(h(x,\mu_1))-T_{\mu_1}(h(x,\mu_1))\| \leq$$

$$\leq \left(\frac{M}{\lambda}+\frac{N}{\lambda^2}\right)\|h(x,\mu_2)-h(x,\mu_1)\|_\lambda+(L_1+L_2)\,|\mu_2-\mu_1|.$$

Hence

$$\|h(x,\mu_2)-h(x,\mu_1)\|_\lambda\leq \left(1-\left(\frac{M}{\lambda}+\frac{N}{\lambda^2}\right)\right)^{-1}(L_1+L_2)\,|\mu_2-\mu_1|.$$

Consequently, the function $h(x,\mu)$ is uniformly continuous with respect to the variable $\mu\in R$; so $y(x,\mu)$ is also continuous with respect to two variables $(x,\mu)\in J\times R$, which completes the proof.

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