FASCICULI MATHEMATICI

Nr 25 1995

VINCENT ŠOLTÉS

OSCILLATORY PROPERTIES OF SOLUTIONS OF SECOND ORDER SUBLINEAR DIFFERENTIAL EQUATIONS

There are found oscillatory criteria for the second order sublinear differential equation $(r(t)y'(t))' + a(t)f(y(t)) = 0$, where $a(t)$ is not of a constant sign on $[t_0, \infty)$. There are also found sufficient conditions so that $\liminf_{t \to \infty} |y(t)| = 0$ for every solution $y(t)$ of this equation (but with nonzero right side).

Key words: second order sublinear differential equation, oscillation.

1. Introduction

We consider the second order nonlinear differential equation of the form

$$(r(t)y'(t))' + a(t)f(y(t)) = 0, \tag{1}$$

where $r(t) > 0$, $r(t) \in C[0, \infty)$, $a(t) \in C[0, \infty)$ and $f(y) \in C[-\infty, \infty)$ is non-decreasing with respect to y and such that $yf(y) > 0$ for $y \neq 0$.
We suppose that $f(y)$ satisfies a sublinearity condition

$$\int_0^{\pm\epsilon} \frac{dy}{f(y)} < \infty \quad \text{for every } \epsilon > 0. \tag{2}$$

We restrict our attention to those solutions of equation (1) which exist on $[t_0, \infty)$, $t_0 \geq 0$ and which are nontrivial in any neighbourhood of infinity.

A solution is called oscillatory if it has arbitrarily large zeros. The equation (1) is oscillatory, if all solutions of (1) are oscilatory.

The oscilatory properties of solutions of equation (1), and especially in its special case, when $r(t) = 1$ or $f(y(t)) = |y(t)|^\gamma \operatorname{sgn} y(t)$, have been studied by several authors. This paper is adjoined to paper [8] and extends results stated there.

We remark that we obtained different results from the one which you can obtain by direct transformation of the equation (1) to equation with $r(t) \equiv 1$.

Besides, there are given sufficient conditions for all solutions of equation

$$(3) \qquad (r(t)\,y'(t)) + a(t) f(y(t)) = b(t)$$

to satisfy the condition $\lim_{t\to\infty} \inf |y(t)| = 0$.

2. Main results

Let us denote:

$$R(t) = \int_0^t r(s)\, ds,$$

$$F(y) = \int_0^y \frac{dv}{f(v)}.$$

For each $s \ge 0$, define the function $A(s)$ by

$$A(s) = \lim_{T\to\infty} \frac{\int_s^T r(t) \int_s^t \frac{\delta^\tau(u)\, a(u)}{r(u)}\, du\, dt}{R(T)}$$

and

$$A_+(s) = \max(A(s), 0).$$

We suppose that

$$(4) \qquad \lim_{t\to\infty} R(t) = \infty,$$

$$(5) \qquad f'(y) F(y) \ge \frac{1}{c} \quad \text{for all } y \ne 0, \text{ where}$$

c is a positive constant and $\tau = \frac{1}{1+c} < 1$.

Theorem 1. *Let $f(y)$ satisfy* (2) *and* (5) *and let $\delta(t)$ be a positive concave function on $[0, \infty)$ such that*

$$(6) \qquad \left[\frac{\delta^\tau(t)\, r'(t)}{r(t)}\right] \le 0 \quad \textit{for all } t \ge 0.$$

If (4) *is satisfied and*

$$\limsup_{T\to\infty}\left[\int_0^T \frac{r(s)\,A_+^2(s)}{R(s)}\,ds\right]\left[\int_0^T \frac{R(s)\,(r(s)\,\delta(s))'^2}{r^3(s)\,\delta^2(s)}\,ds\right]^{-1} = \infty, \tag{7}$$

then the equation (1) *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1) and suppose for example $y(t)>0$ on $[t_0,\infty)$. We define the function $z(t)=\delta^\tau(t)\,F(y(t))$, where $\tau=\dfrac{1}{1+c}$. Differentiating $z(t)$ twice and using (1) we obtain

$$z''(t)=\tau(\tau-1)\,\delta^{\tau-2}(t)\,\delta^2(t)\,F(y(t))+\tau\delta^{\tau-1}(t)\,\delta''(t)F(y(t))+2\tau\delta^{\tau-1}(t)\,\delta'(t)\,\frac{y'(t)}{f(y(t))}-$$
$$-\frac{\delta^\tau(t)\,f'(y(t))\,y'^2(t)}{f^2(y(t))}-\frac{a(t)\,\delta^\tau(t)}{r(t)}-\frac{\delta^\tau(t)\,r'(t)\,y'(t)}{r(t)f(y(t))}, \tag{8}$$

from that by using the assumptions of the theorem we get

$$z''(t)+\frac{a(t)\,\delta^\tau(t)}{r(t)}\le-\frac{\delta^{\tau-2}(t)}{cF(y(t))}\left[(1-\tau)\,\delta'(t)\,F(y(t))-\frac{\delta(t)\,y'(t)}{f(y(t))}\right]^2-\frac{\delta^\tau(t)\,r'(t)\,y'(t)}{r(t)f(y(t))}. \tag{9}$$

Since $z''(t)=\delta^\tau(t)\,F(y(t))$, from the last relation we have

$$z''(t)+\frac{a(t)\,\delta^\tau(t)}{r(t)}\le-\frac{1}{c\,z(t))}\left[(1-\tau)\,\delta^{\tau-1}(t)\,\delta'(t)F(y(t))-\right.$$
$$\left.-\frac{\delta^\tau(t)\,y'(t)}{f(y(t))}\right]^2-\frac{\delta^\tau(t)\,r'(t)}{r(t)}\,\frac{d\,F(y(t))}{dt}. \tag{10}$$

Integrating (10) from s *to* T, $T>s>t_0$, and using the fact that $-\ z'(t)=$ $=-\tau\,\delta^{\tau-1}(t)\,\delta'(t)F(y(t))-\dfrac{\delta(t)\,y'(t)}{f(y(t))}$

we get

$$z'(T)-z'(s)+\int_s^T\frac{\delta^\tau(u)\,a(u)}{r(u)}\,du\le-\frac{1}{c}\int_s^T\frac{1}{z(u)}\cdot$$
$$\cdot[\delta^{\tau-1}(u)\,\delta'(u)\,F(y(u))-z'(u)]^2du-\frac{z(T)\,r'(T)}{r(T)}+ \tag{11}$$
$$+\frac{z(s)\,r'(s)}{r(s)}+\int_s^T\left[\frac{\delta^\tau(u)\,r'(u)}{r(u)}\right]'F(y(u))\,du.$$

Since $\frac{1}{z(u)}\left[\delta^{\tau-1}(u)\,\delta'(u)\,F(y(u))-z'(u)\right]^2=\frac{1}{z'(u)}\left[\frac{z(u)\,\delta'(u)}{\delta(u)}-z'(u)\right]^2=\frac{\delta^2(u)}{z(u)}\cdot$ $\cdot\left[\frac{z(u)}{\delta(u)}\right]'^2$, from the relation (11) we have

$$z(T)\,r(T)-z(s)\,r(s)-\frac{[z(t)\,r(t)]'_{t=s}}{r(s)}\int_s^T r(t)\,dt+\int_s^T r(t)\int_s^t\frac{\delta^\tau(u)\,a(u)}{r(u)}\,du\,dt\le-\frac{1}{c}\int_s^T r(t)\cdot \tag{12}$$
$$\cdot\int_s^t\frac{\delta^2(u)}{z(u)}\left[\frac{z(u)}{\delta(u)}\right]'^2du\,dt.$$

From (12) we have

$$0\le\lim_{T\to\infty}\frac{z(T)\,r(T)}{R(T)}=K_0<\infty, \tag{13}$$

$$0\le\lim_{T\to\infty}\frac{\int_s^T r(t)\int_s^t\frac{\delta^2(u)}{z(u)}\left[\frac{z(u)}{\delta(u)}\right]'^2du\,dt}{R(T)}=K_1(s)<\infty \tag{14}$$

and

$$A(s)\le\frac{[z(s)\,r(s)]'}{r(s)}=\frac{1}{r(s)}\left[\left(\frac{z(s)}{\delta(s)}\right)'r(s)\,\delta(s)+\frac{z(s)}{\delta(s)}(r(s)\,\delta(s))'\right],$$

then

$$\frac{1}{2}A_+^2(s)\le\left[\frac{z(s)}{\delta(s)}\right]'^2\delta^2(s)+\frac{z^2(s)}{r^2(s)\,\delta^2(s)}(r(s)\,\delta(s))'^2. \tag{15}$$

Finally from (13) it follows that

$$\frac{r(s)}{R(s)}\le\frac{\tilde N}{z(s)}$$

and then from the relation (15) we have

$$\frac{1}{2}\int_{t_0}^T\frac{r(s)\,A_+^2(s)}{R(s)}\,ds\le\tilde N\int_{t_0}^T\frac{1}{z(s)}\left[\frac{z(s)}{\delta(s)}\right]'^2\delta^2(s)\,ds+\tilde N^2\int_{t_0}^T\frac{R(s)\,(r(s)\,\delta(s))'^2}{r^3(s)\,\delta^2(s)}\,ds,$$

from that

$$\frac{1}{2}\left[\int_{t_0}^{T}\frac{R(s)A_+^2(s)}{R(s)}ds\right]\left[\int_{t_0}^{T}\frac{R(s)(r(s)\delta(s))'^2}{r^3(s)\delta^2(s)}ds\right]^{-1}\leq\tilde{N}^2+$$

$$+\tilde{N}K_1(t_0)\left[\int_{t_0}^{T}\frac{R(s)(r(s)\delta(s))'^2}{r^3(s)\delta^2(s)}ds\right]^{-1},$$

which is a contradiction with condition (7). So the proof of Theorem 1 is complete.

Theorem 2. *Let the conditions of Theorem* 1 *be satisfied, where instead of* (7) *let be*

$$\lim_{T\to\infty}\sup\left[\int_0^T\frac{r(s)A_+^2(s)ds}{R(s)}\right]\left[\int_0^T\frac{R(s)\delta'^2(s)}{r(s)\delta^2(s)}ds\right]^{-1}=\infty \tag{16}$$

If moreover, $r'(t)\leq 0$ *for all* $t\geq 0$, *then the equation* (1) *is oscillatory.*

Proof. From the relation (11) we have

$$A(s)\leq\frac{[z(s)r(s)]'}{r(s)}\leq z'(s)=\left[\frac{z(s)}{\delta(s)}\right]'\delta(s)+\frac{z(s)}{\delta(s)}\delta'(s),$$

from that

$$\frac{1}{2}A_+^2(s)\leq\left[\frac{z(s)}{\delta(s)}\right]'^2\delta^2(s)+\left[\frac{z(s)}{\delta(s)}\right]^2\delta'^2(s).$$

Proceeding as in the proof of Theorem 1 we obtain

$$\frac{1}{2}\left[\int_{t_0}^{T}\frac{r(s)A_+^2(s)}{R(s)}ds\right]\left[\int_{t_0}^{T}\frac{R(s)\delta'^2(s)}{r(s)\delta^2(s)}ds\right]^{-1}\leq\tilde{N}^2+\tilde{N}K_1(t_0)\left[\int_{t_0}^{T}\frac{R(s)\delta'^2(s)}{r(s)\delta^2(s)}ds\right]^{-1},$$

which is a contradiction with condition (16). This completes the proof of Theorem 2.

Remark. If $r(t)\equiv 1$, then from Theorem 1 resp. Theorem 2 follows the Theorem in paper [8].

Now let us consider the equation (3), where r, a, f are as in equation (1) and $b(t)\in C[0,\infty]$.

Theorem 3. *Let the conditions of Theorem* 1 *hold and furthermore let*

$$\int_0^\infty r(t) \int_0^t \frac{\delta^\tau(u)|b(u)|}{r(u)}\, du\, dt < \infty. \tag{17}$$

Then every solution $y(t)$ *of equation* (3) *satisfies the condition* $\liminf_{t\to\infty} = 0$.

Proof. Let $y(t)$ be a solution of equation (3) such that $\liminf_{t\to\infty} |y(t)| = d > 0$. It is easy to see that $y(t)$ is nonoscillatory.

Let, for example, $y(t) > 0$ on $[t_0, \infty)$. Then there exists a constant $d_1 > 0$ such that $y(t) > d_1$. Then $\frac{1}{f(y(t))} \leq \frac{1}{f(d_1)}$. We proof that as Theorem 1, but instead of (8) we get

$$z''(t) = \tau(\tau-1)\,\delta^{\tau-2}(t)\,\delta'^2(t)\,F(y(t)) + \tau\delta^{\tau-1}(t)\,\delta''(t)\,F(y(t)) + 2\tau\delta^{\tau-1}(t)\,\delta'(t)\,\frac{y'(t)}{f(y(t))} -$$

$$- \frac{\delta^\tau(t) f'(y(t))\, y'^2(t)}{f^2(y(t))} - \frac{a(t))\,\delta^\tau(t)}{r(t)} - \frac{\delta^\tau(t)\, r'(t)\, y'(t)}{r(t) f(y(t))} + \frac{\delta^\tau(t)\, b(t)}{r(t) f(y(t))}, \tag{18}$$

from that

$$z(T)r(T) - z(s)r(s) - \frac{[z(t)r(t)]'_{t=s}}{r(s)} \int_s^T r(t)\, dt + \int_s^T r(t) \int_s^t \frac{\delta^\tau(u)\, a(u)}{r(u)}\, du\, dt \leq$$

$$\leq -\frac{1}{c} \int_s^T r(t) \int_s^T \frac{\delta^2(u)}{z(u)} \left[\frac{z(u)}{\delta(u)}\right]'^2 du\, dt + \frac{1}{f(d_1)} \int_0^\infty r(t) \int_0^t \frac{\delta(u)|b(u)|}{r(u)}\, du\, dt \tag{19}$$

The next steps in this proof are the same as in the proof of Theorem 1. This completes the proof of Theorem 3. It is easy to see that Theorem 4 holds.

Theorem 4. *Let the conditions of Theorem* 2 *be satisfied and furthermore suppose* (17) *holds. Then every solution* $y(t)$ *of the equation* (3) *satisfies the condition* $\limsup_{t\to\infty} |y(t)| = 0$.

References

[1] Belohorec S., *Two Remarks on the Properties of Solutions of a Nonlinear Differential Equation*, Acta Fac. Rerum. Natur. Univ. Comen. Math., 22, 1969, pp. 19–26.

[2] Kamenev I. V., *Some Specially Nonlinear Oscillation Theorems*, Mat. Zametki, 10, 1971, pp. 129–134.

[3] Kwong M. K., Wong J. S. W., *On a Oscillation Theorem of Belohorec*, SIAM J. Math. Anal., 14, 1983, pp. 474–476.

[4] Ohriska J., *On the Oscillation of a Linear Differential Equation of Second Order*, Czech. Math. J., 39, 114, 1989, pp. 16–23.

[5] Philos Ch. G. *Oscillation of Second Order Linear Ordinary Differential Equations with Alternating Coefficients*, Bull. Austral. Math. Soc., 27, 1983, pp. 307–313.

[6] Seman J., *Oscillation Theorems for Second Order Nonlinear Delay Inequalities*, Math. Slovaca, 39, 1989, No. 3, pp. 313–322.

[7] Šoltés V., *Oscillation of Second Order Nonlinear Differential Equations with Alternating Coefficients*, Archivum Mathematicum (Brno) (in press).

[8] Wong J. S. W., *An Oscillation Theorem for Second Order Sublinear Differential Equations*, Proc. Amer. Math. Soc., 110, 1990, pp. 633–637.

[9] Wong J. S. W., *Oscillation of Sublinear Second Order Differential Equations with Integrable Coefficients*, J. Math. Anal. Appl., 162, 1991, pp. 476–481.

(Technical University, Department of Mathematics, Košice, Slovak Republic)
Received on 13.11.1992.