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OSCILLATION OF EVEN ORDER NONLINEAR FUNCTIONAL EQUATION WITH DEVIATING ARGUMENTS

There are found oscillatory criteria for the n-th order nonlinear functional differential equations with deviating arguments of the form $p_n(t)(p_{n-1}(t)(...(p_1(t)(p_0(t)y(t))')'...)')' + a(t)F(y(g(t))) = 0$.

Key words: functional differential equation, oscillatory solution.

1. Introduction

Consider the differential equation

$$L_n y(t) + a(t)F(y(g(t))) = 0, \tag{1}$$

where n is even,

$$L_n y(t) = p_n(t)(p_{n-1}(t)(...(p_1(t)(p_0(t)y(t))')'...)')',$$

p_i, $i = 0, 1, ..., n$ are positive and continuous functions on $\langle t_0, \infty)$ such that $\int^{\infty} p_i^{-1}(t)\,dt = \infty$ for $i = 1, 2, ..., n-1$, $a(t) \geq 0$ is continuous on $\langle t_0, \infty)$, $F: R \to R$ is continuous, g is continuous on $\langle t_0, \infty)$ and $g(t) \to \infty$ as $t \to \infty$ and $x \cdot F(x) > 0$ for $x \neq 0$.

The expressions

$$L_0 y(t) = p_0(t)y(t), \quad L_i y(t) = p_i(t)[L_{i-1} y(t)]', \qquad i = 1, 2, ..., n \tag{2}$$

are called the quasi-derivatives of function y at the point $t \in \langle t_0, \infty)$. We will consider the solutions of equation (1) which exist on $\langle T_y, \infty)$ and satisfy the condition

$$\sup\{|y(t)| : t_1 \leq t < \infty\} > 0 \text{ for every } t_1 \in \langle T_y, \infty). \tag{3}$$

Such a solution will be called *oscillatory* if its set of zeros is unbounded and will be called *nonoscillatory* otherwise. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

Many authors discuss the oscillatory properties of the equation (1). S.R. Grace gave in the paper [1] some oscillation criteria for the special case of the equation (1) that is to say if $p_i(t) \equiv 1$ for $i = 0, 1, 2, \dots, n$. This paper extends some results established there for the equation with the quasi-derivatives.

2. Definitions and basic lemmas

The following notation will be used throughout this paper:

$$R_\alpha = (-\infty, -\alpha> \cup <\alpha, \infty) \quad \text{if } \alpha > 0$$

$$= (-\infty, 0) \cup (0, \infty) \quad \text{if } \alpha = 0$$

$C(R) = \{F : R \to R | F \text{ is continuous and } x \cdot F(x) > 0 \text{ if } x \neq 0\}$
$C^1(R_\alpha) = \{F \in C(R) | F \text{ is continuously differentiable in } R_\alpha\}$
$C_p(R_\alpha) = \{F \in C(R) | F \text{ is of bounded variation on every interval } \langle a, b\rangle \subset R_\alpha\}$.
We will use the known notation which is used for example in [5].

(4) $$I_0 \equiv 1,\ I_k(t, a; p_{i_1}, p_{i_2}, \dots, p_{i_k}) =$$

$$= \int_a^t p_{i_1}^{-1}(t_{i_1}) \int_a^{t_{i_1}} p_{i_2}^{-1}(t_{i_2}) \int_a^{t_{i_2}} \dots \int_a^{t_{i_{k-1}}} p_{i_k}^{-1}(t_{i_k})\, dt_{i_k}\, dt_{i_{k-1}} \dots dt_{i_1},$$

$$1 \le k \le n-1, \quad t_0 \le a \le t < \infty;$$

(5) $$x_j(t, a) = p_0^{-1}(t)\, I_{j-1}(t, a; p_1, p_2, \dots, p_{j-1}), \quad j = 1, 2, \dots, n;$$

(6) $$P_0(t, a) \equiv 1, \ P_j(t, a) = I_j(t, a; p_1, \dots, p_j), \quad \text{for } j = 1, 2, \dots, n-1;$$

(7) $$Q_n(t, a) \equiv 1, \ Q_j(t, a) = I_{n-j}(t, a; p_{n-1}, p_{n-2}, \dots, p_j),$$

for $j = 1, 2, \dots, n-1$.

Remark 1. In the case all $p_i(t) \equiv 1$

$$P_j(t, a) = \frac{(t-a)^j}{j!}, \qquad j = 0, 1, 2, \dots, n-1 \text{ and}$$

$$Q_j(t, a) = \frac{(t-a)^{n-j}}{(n-j)!}, \qquad j = 0, 1, \dots, n.$$

In the following we will use the well known generalizated Kiguradze lemmas published for example in [5].

Lemma 1. *Let* $\int^{\infty} p_i(t)^{-1}dt = \infty$ *for* $i = 1, 2, ..., n-1$ *and let* y *be positive function on the interval* $\langle t_1, \infty)$, $t_1 \geq t_0$ *such that* $L_n y$ *exists on* $\langle t_1, \infty)$, *is of constant sign and is not identically zero on any interval of the form* $\langle t_2, \infty)$, $t_2 \geq t_1$.

Then there exists an integer l, $0 \leq l \leq n$, *such that* $n+l$ *is odd for* $L_n y \leq 0$ *and* $n+l$ *is even for* $L_n y \geq 0$ *such that*

$$l \leq n-1 \text{ implies } (-1)^{l+j} L_j y(t) > 0 \text{ for every } t \geq t_1,$$
$$j = l, l+1, ..., n-1,$$

$$l > 1 \text{ implies } L_i y(t) > 0 \quad \text{for all large } t, \quad i = 1, 2, ..., l-1.$$

Lemma 2. *Let* l *be an integer,* $1 \leq l \leq n-1$, $a \in \langle t_0, \infty)$. *If the function* y *satisfies*

$$L_0 y(a), ..., L_{l-1} y(a) \geq 0, \; L_{l+1} y(t) \leq 0 \quad \text{for } a \leq t < \infty$$

then

(a) $$\left(\frac{L_i y(t)}{L_i x_{l+1}(t, a)}\right)' \leq 0, \qquad i = 0, 1, ..., l, \quad a < t < \infty,$$

(b) $$L_i y(t) \geq L_{i+1} y(t) \frac{L_i x_{l+1}(t, a)}{L_{i+1} x_{l+1}(t, a)}, \quad i = 0, 1, ..., l, \quad a < t < \infty,$$

(c) $$\frac{L_0 y(t)}{P_l(t, a)} \quad \text{is a nonincreasing function in } (a, \infty),$$

(d) $$L_0 y(t) \geq L_i y(t) \frac{P_l(t, a)}{L_i x_{l+1}(t, a)}, \qquad i = 1, 2, ..., l, \quad a < t < \infty$$

and

(e) $$L_0 y(t) \geq L_l y(t) P_l(t, a), \quad a \leq t < \infty.$$

In the paper [1] is presented this lemma.

Lemma 3. *Suppose* $\alpha \geq 0$ *and* $F \in C(R)$. *Then* $F \in C_p(R_\alpha)$ *if and only if* $F(y) = G(y)H(y)$ *for all* $y \in R_\alpha$, *where* $G: R_\alpha \to (0, \infty)$, *nondecreasing on* $(-\infty, -\alpha)$ *and nonincreasing on* (α, ∞) *and* $H: R_\alpha \to \mathrm{R}$ *and nondecreasing in* R_α.

Definition. We call G in Lemma 3 *a positive component of F, H and nondecreasing component of F and the ordered pair* (G, H) *a pair of components of F.*

Suppose that there exists a continuous function $\sigma: \langle t_0, \infty) \to \langle t_0, \infty)$ such that

$$\sigma(t) \leq \min\{t, g(t)\} \text{ and } \sigma(t) \to \infty \text{ as } t \to \infty.$$

Define functions

$$R_i(u,t) = a(u)\frac{Q_i(u,t)}{p_n(u)} G\left(\frac{k^*}{p_0(g(u))} P_{n-1}(g(u), t_2)\right) \cdot H\left(\frac{x_{l+1}(g(u), t_2) L_{l-1} x_{l+1}(\sigma(u), t_2)}{L_{l-1} x_{l+1}(g(u), t_2) L_{l-1} x_{l+1}(u, t_2)}\right) \text{ for } \tag{8}$$

$i \in \{1, 3, 5, \ldots, n-1\}$, where $t_2 \geq t_0$, $|k^*| \geq 1$.

3. Main results

Theorem 1. *Let* $\alpha \geq 1$ *and* (G, H) *is a pair of components of F. Suppose moreover that*

$$-H(-xy) \geq H(xy) \geq KH(x)H(y), \quad xy > 0, \tag{9}$$

where K is a positive constant and

$$\int^{\infty} \frac{du}{H(u)} < \infty, \quad \int^{-\infty} \frac{du}{H(u)} < \infty \tag{10}$$

If for every integer $l \in \{1, 3, \ldots, n-1\}$

$$\int^{\infty} R_l(u,t)\,du = \infty \text{ for every } k^* \geq 1 \tag{11}$$

or

$$\lim_{t \to \infty} \int_t^{\infty} R_l(u,t)\,du = \infty, \tag{12}$$

then equation (1) *is oscillatory.*

Proof. Let $y(t)$ is nonoscillatory solution of equation (1). Suppose $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_0$. Lemma 1 implies that there exists $l \in \{1, 3, ..., n-1\}$ such that

$$L_k y(t) > 0 (k = 0, 1, ..., l-1),$$

$$(-1^{k+l} L_k y(t) > 0 \ (k = l, l+1, ..., n-1) \text{ for every } t \geq t_1 \geq t_0.$$

In particular $L_n y(t) < 0$, $L_{n-1} y(t) > 0$ so there exist a constant $k_1 > 0$ and $t_2 \geq t_1$ such that $L_{n-1} y(t) \leq k_1$ for every $t \geq t_2$.

If we multiply this relation by function $\frac{1}{p_{n-1}}$, integrate from t_2 to t and use the fact that

$$\int^{\infty} p_{n-1}(t)\, dt = \infty$$

we get

$$L_{n-2} y(t) \leq \bar{k}_1 \int_{t_2}^{t} \frac{1}{p_{n-1}(s_1)} ds_1 \text{ for every } t \geq t_2.$$

Repeating this consideration we have

$$y(t) \leq k_2 \frac{1}{p_0(t)} I_{n-1}(t, t_2; p_1, p_2, ..., p_{n-1}),$$

where $k_2 \geq 1$.

Since $g(t) \to \infty$ as $t \to \infty$ there exists $t_3 \geq t_2$ such that (using (6))

$$y(g(t)) \leq k_2 \frac{1}{p_0(g(t))} P_{n-1}(g(t), t_2) \text{ for every } t \geq t_3.$$

By Lemma 3 we have

$$F(y(g(t)) = G(y(g(t)))\, H(y(g(t))) \geq G\left(k_2 \frac{1}{p_0(g(t))} P_{n-1}(g(t), t_2)\right) H(y(g(t)))$$

and so the inequality is satisfied.

$$L_n\, y(t) + a(t)\, G\left(k_2 \frac{1}{p_0(g(t))} P_{n-1}(g(t), t_2)\right) H(y(g(t))) \leq 0 \tag{14}$$

for every $t \geq t_3$.

It is easy to show that

$$L_l y(t) = \sum_{j=l}^{n-1} (-1)^{j-l} I_{j-l}(s,t;p_j,\dots,p_{l+1}) L_j y(s) +$$

$$+(-1)^{n-l} \int_t^s \frac{I_{n-1-l}(u,t;p_{n-1},\dots,p_{l+1}) L_n y(u)}{p_n(u)} du,$$

hence from Lemma 1 and (7) we have

$$L_l y(t) \geq -\int_t^s \frac{1}{p_n(u)} Q_{l+1}(u,t) L_n y(u) \, du.$$

Using (14) we obtain

$$L_l y(t) \geq \int_t^s \frac{1}{p_n(u)} Q_{l+1}(u,t) a(u) G\left(\frac{k_2}{p_0(g(u))} P_{n-1}(g(u),t_2)\right) H(y(g(u))) \, du \tag{15}$$

Lemma 2 case (a) implies that

$$\frac{L_{l-1} y(t)}{L_{l-1} x_{l+1}(t,t_2)}$$

is a nonincreasing function and hence

$$\frac{L_{l-1} y(\sigma(t))}{L_{l-1} x_{l+1}(\sigma(t),t_2)} \geq \frac{L_{l-1} y(t)}{L_{l-1} x_{l+1}(t,t_2)}. \tag{16}$$

From Lemma 2 case (d) for $i = l-1$ we have

$$L_0 y(t) \geq L_{l-1} y(t) \frac{P_l(t,t_2)}{L_{l-1} x_{l+1}(t,t_2)},$$

hence

$$y(t) \geq \frac{x_{l+1}(t,t_2)}{L_{l-1} x_{l+1}(t,t_2)} L_{l-1} y(t).$$

Since $g(t) \to \infty$ holds too

$$y(g(t)) \geq \frac{x_{l+1}(g(t),t_2)}{L_{l-1} x_{l+1}(g(t),t_2)} L_{l-1} y(g(t)). \tag{17}$$

From (16) we have

$$L_{l-1} y(\sigma(t)) \geq \frac{L_{l-1} x_{l+1}(\sigma(t),t_2)}{L_{l-1} x_{l+1}(t,t_2)} L_{l-1} y(t)$$

and because $L_{l-1}\, y(t) > 0$ and $L_{l-1}\, y(t)$ is increasing holds

$$L_{l-1}\, y(g(t)) > L_{l-1}\, y(\sigma(t)).$$

Then from (13) we have

$$y(g(t)) \geq \frac{x_{l+1}(g(t), t_2)\, L_{l-1}\, x_{l+1}(\sigma(t), t_2)}{L_{l-1}\, x_{l+1}(g(t), t_2)\, L_{l-1}\, x_{l+1}(t, t_2)}\, L_{l-1}\, y(t). \tag{18}$$

Multiply (15) by function $\frac{1}{p_l(t)}$ using (18) and (9). We obtain

$$\frac{[L_{l-1}\, y(t)]'}{H(L_{l-1} y(t))} \geq \frac{K}{p_l(t)} \int_t^s \frac{1}{p_n(u)}\, Q_{l+1}(u, t)\, a(u)\, G\left(\frac{k_2}{p_0(g(u))} P_{n-1}(g(u), t_2)\right) \cdot$$

$$\cdot H\left(\frac{x_{l+1}(g(u), t_2)\, L_{l-1}\, x_{l+1}(\sigma(u), t_2)}{L_{l-1}\, x_{l+1}(g(u), t_2)\, L_{l-1}\, x_{l+1}(u, t_2)}\right) du.$$

Integrating the last relation from t_3 to s, $s \geq t_3$, we get

$$\int_{t_3}^{s} \frac{[L_{l-1}\, y(t)]'}{H(L_{l-1}\, y(t))}\, dt \geq K \int_t^s R_l(u, t)\, du$$

When $s \to \infty$ we have

$$K \int_t^\infty R_l(u, t)\, du \leq \int_{L_{l-1} y(t_3)}^{\infty} \frac{dv}{H(v)} < \infty,$$

what is a contradiction to the assumption (11) or (12). The proof when $y(t) < 0$ is similar.

The proof is complete.

Theorem 2. *Let* $\alpha \geq 1$, $\sigma(t) \geq 0$ *for* $t \geq t_0$ *and* (G, H) *is a pair of components of* F. *Suppose that*

$$\frac{H(u)}{u} \geq c > 0 \quad \textit{for } u \neq 0. \tag{19}$$

If for every $l \in \{1, 3, \ldots, n-1\}$ *is*

$$(20)\qquad \limsup_{t\to\infty}\left(\int_{\sigma(t)}^{t}\frac{1}{p_n(u)}Q_{l+1}(u,\sigma,(t))\,a(u)\,G\left(\frac{k^*}{p_0(g(u))}P_{n-1}(g(u),t_2)\right)\cdot\right.$$

$$\cdot x_{l+1}(\sigma(u),t_2)\,du+L_{l-1}x_{l+1}(\sigma(t),t_2)\cdot$$

$$\left.\cdot\int_{t}^{\infty}\frac{1}{p_n(u)}Q_{l+1}(u,\sigma,(t))\,a(u)\,G\left(\frac{k^*}{p_0(g(u))}P_{n-1}(g(u),t_2)\right)\frac{x_{l+1}(\sigma(u),t_2)}{L_{l-1}x_{l+1}(\sigma(u),t_2)}\,du\right)>$$

$$>\frac{1}{c}\quad \textit{for every } |k^*|\geq 1,$$

then is equation (1) *osillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1). We can assume that for example $y(t)>0$, $y(g(t))>0$ for every $t\geq t_0\geq\alpha$ because $y(t)$ is a nondecreasing function (l is at least 1). Using (19) and (17) for $\sigma(t)$, from relation (15) we have

$$L_l\,y(\sigma(t))\geq c\int_{\sigma(t)}^{t}\frac{1}{p_n(u)}Q_{l+1}(u,\sigma(t))\,a(u)\,G\left(k_2\frac{1}{p_0(g(u))}P_{n-1}(g(u),t_2)\right)\cdot$$

$$\cdot\frac{x_{l+1}(\sigma(u),t_2)}{L_{l-1}x_{l+1}(\sigma(u),t_2)}L_{l-1}\,y(\sigma(u))\,du+$$

$$+c\int_{t}^{\infty}\frac{1}{p_n(u)}Q_{l+1}(u,\sigma(t))\,a(u)\,G\left(k_2\frac{1}{p_0(g(u))}P_{n-1}(g(u),t_2)\right)\cdot$$

$$\cdot G\left(\frac{k_2}{p_0(g(u))}P_{n-1}(g(u),t_2)\right)\frac{x_{l+1}(\sigma(u),t_2)}{L_{l-1}x_{l+1}(\sigma(u),t_2)}L_{l-1}\,y(\sigma(u))\,du.$$

In the first integral we use the fact that the function $\dfrac{L_{l-1}\,y(t)}{L_{l-1}x_{l+1}(t,a)}$ is nonincreasing (Lemma 2 case (a) for $i=l-1$) and in the second integral that the functions $L_{l-1}\,y(t)$ and $\sigma(t)$ are nondecreasing and then we get

$$\frac{L_{l-1}x_{l+1}(\sigma(t),t_2)\,L_l\,y(\sigma(t))}{c\,L_{l-1}\,y(\sigma(t))}\geq$$

$$\geq\int_{\sigma(t)}^{t}\frac{1}{p_n(u)}Q_{l+1}(u,\sigma(t))\,a(u)\,G\left(\frac{k_2}{p_0(g(u))}P_{n-1}(g(u),t_2)\right)x_{l+1}(\sigma(u),t_2)\,du+$$

$$+L_{l-1}\,x_{l+1}(\sigma(t),t_2)\int_{t}^{\infty}\frac{1}{p_n(u)}Q_{l+1}(u,\sigma(t))\,a(u)\,G\left(\frac{k_2}{p_0(g(u))}P_{n-1}(g(u),t_2\right)\cdot$$

$$\cdot\frac{x_{l+1}(\sigma(u),t_2)}{L_{l-1}x_{l+1}(\sigma(u),t_2)}\,du.$$

Lemma 2 case (b) for $i = l-1$ implies that

$$\frac{L_{l-1}\, x_{l+1}(\sigma(t), t_2)\, L_l y(\sigma(t))}{L_{l-1}\, y(\sigma(t))} \le 1, \tag{21}$$

consequently

$$\limsup_{t\to\infty}\Bigg(\int_{\sigma(t)}^{t} \frac{1}{p_n(u)}\, Q_{l+1}(u, \sigma(t))\, a(u)\, G\Big(\frac{k_2}{p_0(g(u))}\, P_{n-1}(g(u), t_2)\cdot$$
$$\cdot\, x_{l+1}(\sigma(u), t_2)\, du + L_{l-1}\, x_{l+1}(\sigma(t), t_2)\cdot$$
$$\cdot\int_{t}^{\infty} \frac{1}{p_n(u)}\, Q_{l+1}\,(u, \sigma,(t))\, a(u)\, G\Big(\frac{k_2}{p_0(g(u))}\, P_{n-1}(g(u), t_2)\Big) \frac{x_{l+1}(\sigma(u), t_2)}{L_{l-1}\, x_{l+1}(\sigma(u), t_2)}\, du\Bigg) \le \frac{1}{c},$$

and this is a contradiction with the assumption (20). The proof when $y(t) < 0$ is similar. This completes the proof of Theorem 2.

Theorem 3. *Let* $\alpha \ge 1$ *and* (G, H) *be a pair of components of* F. *Let hold* (9) *and*

$$\int_{0}^{\pm\mathscr{E}} \frac{du}{H(u)} < \infty \quad \text{for every } \mathscr{E} > 0. \tag{22}$$

If for every $l \in \{1, 3, \ldots, n-3\}$

$$\int^{\infty} \frac{H(L_{l-1}\, x_{l+1}(s, t_2))}{p_{l+1}(s)} \int_{s}^{\infty} R_{l+2}(u, s)\, du\, ds = \infty \tag{23}$$

and for $l = n-1$

$$\int^{\infty} \frac{1}{p_n(u)}\, a(u)\, G\Big(\frac{k^*}{p_0(g(u))}\, P_{n-1}(g(u), t_2)\Big)\cdot \tag{24}$$
$$\cdot H\Big(\frac{x_n(g(u), t_2)\, L_{l-2}\, x_n(\sigma(u), t_2)}{L_{l-2}\, x_n(g(u), t_2)}\Big)\, du = \infty$$

for every $|k^*| \ge 1$, *then the equation* (1) *is oscillatory.*

Proof. Let $y(t)$ be nonoscillatory solution of equation(1). Suppose $y(t) > 0$ and $y(g(t)) > 0$ for every $t \ge t_0 \ge \alpha$. Consider two cases:

Case 1. $l<n-1$

It is evident that holds

$$-L_{l+1}y(t)=\sum_{j=l+1}^{n-1}(-1)^{j-l}I_{j-l-1}(s,t;p_j,\dots,p_{l+2})L_jy(s)+$$

$$+(-1)^{n-l}\int_t^s\frac{I_{n-l-2}(u,t;p_{n-1},\dots,p_{l+2})L_ny(u)}{p_n(u)}\,du.$$

Using Lemma 1 and (14) we have

$$-L_{l+1}y(t)\geq\int_t^s\frac{1}{p_n(u)}Q_{l+2}(u,t)a(u)G\left(\frac{k_2}{p_0(g(u))}P_{n-1}(g(u),t)\right)\cdot H(y(g(u)))\,du,$$

hence by using relations (18), (9) and (21) we obtain

$$\frac{-L_{l+1}y(t)}{H(L_ly(t))}\geq K^2\cdot H(L_{l-1}x_{l+1}(t,t_2))\int_t^s R_{l+2}(u,t)\,du.$$

From the last relation after the multiplying by function $\frac{1}{p_{l+1}(t)}$ and integrating from t_2 to t we have

$$\int_{t_2}^t\frac{H(L_{l-1}x_{l+1}(v,t_2))}{p_{l+1}(v)}\int_v^\infty R_{l+2}(u,v)\,du\,dv\leq\frac{1}{K^2}\int_{L_ly(t)}^{L_ly(t_2)}\frac{dv}{H(v)},$$

hence for $t\to\infty$ we obtain the contradiction with (22), (23).

Case 2. $l=n-1$

From the realtions (14) and (18) we have

$$-L_ny(t)\geq a(t)G\left(k_2\frac{1}{p_0(g(t))}P_{n-1}(g(t),t_2)\right)H\left(\frac{x_n(g(t),t_2)L_{n-2}x_n(\sigma(t),t_2)}{L_{n-2}x_n(g(t),t_2)L_{n-2}x_n(t,t_2)}L_{n-2}y(t)\right).$$

Since from Lemma 2 case (b) for $i=n-2$ we have

$$L_{n-2}y(t)\geq L_{n-1}y(t)L_{n-2}x_n(t,a),$$

from the last relation we get

$$\int_{t_2}^t\frac{1}{p_n(s)}a(s)G\left(\frac{k_2}{p_0(g(s))}P_{n-1}(g(s),t_2)\right)\cdot$$

$$\cdot H\left(\frac{x_n(g(s),t_2)L_{n-2}x_n(\sigma(s),t_2)}{L_{n-2}x_n(g(s),t_2)}\right)ds\leq\frac{1}{K}\int_{L_{n-1}y(t)}^{L_{n-1}y(t_2)}\frac{dv}{H(v)},$$

hence for $t \to \infty$ we have again the contradiction to (22) and (24). The proof when $y(t) < 0$ is similar. The proof is complete.

Remarak 2. If $p_i(t) \equiv 1$ for $i = 0, 1, \ldots, n$, then we obtain from Theorems 1,2,3, the Theorems 1,2,3 in paper [1].

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