FASCICULI MATHEMATICI

Nr 25 1995

STANĚK SVATOSLAV

ON A CERTAIN FIVE-POINT BOUNDARY VALUE PROBLEM FOR SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS DEPENDING OF THE PARAMETER

We obtain sufficient conditions under which the boundary value problem $x'' = f(t, x, x', \lambda)$, $x(t_0) - x(0) = A$, $x(1) = B$, $x(2) - x(t_1) = C$, depending on the parameter λ, has unique solution for each A, B, $C \in R$.

Key words: one-parameter second-order differential equation, boundary value problem, multipoint boundary conditions, surjective mapping in $\mathbf{R}^n$.

1991 *Mathematics Subject Classification.* 34B10, 34B15.

1. Introduction

Consider the differential equation

$$x'' = f(t, x, x', \lambda), \tag{1}$$

depending on the parameter λ, together with the boundary conditions

$$x(t_0) - x(0) = A, \ x(1) = B, \ x(2) - x(t_1) = C. \tag{2}$$

Here $f \in C^0(\langle 0, 2\rangle \times \mathbf{R}^3)$, $0 < t_0 < 1 < t_1 < 2$, $A, B, C \in \mathbf{R}$.

We say that $x \in C^2(\langle 0, 2\rangle)$ is a solution of boundary value problem (BVP for short) (1), (2) if a $\lambda_0 \in \mathbf{R}$ exists such that x is a solution of (1) for $\lambda = \lambda_0$ and x satisfies (2).

In this paper we will give sufficient conditions under which there exists a unique solution of BVP (1), (2) for each A, B, $C \in \mathbf{R}$. To find sufficient conditions we will use a result in [7] based on a surjective mapping in $\mathbf{R}^n$ (see [2], p. 23, and [8]).

We observe that BVP (1), $x'(0) = A$, $x(1) = B$, $x(2) = C$ has been studied in [7] and sufficient conditions for the two-parameter differential equation $y'' + (g(t, \lambda, \mu) + r(t))\, y = 0$ having a nontrivial solution y satisfying $y(t_1) = x(t_2) = y(t_3) = 0$ $(-\infty < t_1 < t_2 < t_3 < \infty)$ are stated in [1] and [3].

Some BVPs for differential and functional differential equations depending on the parameter have been considered for example in [4]–[6] using the Schauder linearization technique the Schauder fixed point theorem.

2. Lemmas, results

In what follows we shall assume that f satisfies some of the following assumptions:

(H_1) The Cauchy problem $x(0)=x_0$, $x'(0)=x_1$ for equation (1) has a unique solution $x(t,x_0,x_1,\lambda)$ on $\langle 0,2\rangle$ for each $(x_0,x_1,\lambda)\in\mathbf{R}^3$;

(H_2) $f(t,.,y,\lambda)$ is increasing on $\mathbf{R}$ for each fixed $(t,y,\lambda)\in\langle 0,2\rangle\times\mathbf{R}^2$;

(H_3) $f(t,x,y,.)$ is increasing on $\mathbf{R}$ for each fixed $(t,x,y)\in\langle 0,2\rangle\times\mathbf{R}^2$;

(H_4) For each numbers $S>0$, $M>0$ and $L>0$ there exists a number $K>0$ such that

$$f(t,x,y,\lambda)\geq S \text{ for all } t\in\langle 0,2\rangle,\ x\geq -M,\ |y|\leq L,\ \lambda\geq K,$$

$$f(t,x,y,\lambda)\leq -S \text{ for all } t\in\langle 0,2\rangle,\ x\leq M,\ |y|\leq L,\ \lambda\leq -K;$$

(H_5) For each numbers $L_1>0$ and $K_1>0$ there exists a number $M_1>0$ such that

$$f(t,x,y,\lambda)>0 \text{ for all } t\in\langle 0,1\rangle,\ x\geq M_1,\ |y|\leq L_1,\ |\lambda|\leq K_1,$$

$$f(t,x,y,\lambda)<0 \text{ for all } t\in\langle 0,1\rangle,\ x\leq -M_1,\ |y|\leq L_1,\ |\lambda|\leq K_1,$$

(H_6) For any bounded set $\mathscr{D}\subset\mathbf{R}^2$ there exists a positive increasing function $w(z;\mathscr{D})$, $z\in\langle 0,\infty)$, such that

$$\int_0^\infty \frac{z\,dz}{w(z;\mathscr{D})}=\infty$$

and

$$|f(t,x,y,\lambda)|\leq w(|y|;\mathscr{D}) \text{ for all } y\in\mathbf{R} \text{ and } (t,x,\lambda)\in\langle 0,t_0\rangle\times\mathscr{D}.$$

The existence and uniqueness of a solution of BVP (1), (2) is given in the following proposition.

Propostion 1. *Let assumption* (H_1) *and the following two assumptions be satisfied:*

(H_7) *BVP* (1), (2) *has at most one solution for each* $(A,B,C)\in\mathbf{R}^3$,

(H_8) *If* $\{x(t,x_n,y_n,\lambda_n)\}$ *is an arbitrary sequence of solutions of* (1) *such that the sequences* $\{x(t_0,x_n,y_n,\lambda_n)-x(0,x_n,y_n,\lambda_n)\}$, $\{x(1,x_n,y_n,\lambda_n)\}$ and $\{x(2,x_n,y_n,\lambda_n)\}-x(t_1,x_n,y_n,\lambda_n)\}$ *are bounded, then the sequences* $\{x_n\}$, $\{y_n\}$ and $\{\lambda_n\}$ *are bounded, too.*

Then there exists a unique solution of BVP (1), (2) *for each* $(A, B, C)\in\mathbf{R}^3$ *and this solution as well as its derivative are continuous functions of the variables* (t,A,B,C) *in* $\langle 0,2\rangle\times\mathbf{R}^3$.

Proof. The assertion of our proposition follows from Theorem 1 and Corollary 1 in [7]. ■

The following two lemmas give some sufficient conditions which ensure the fulfilment of assumptions (H_7) and (H_8).

Lemma 1. *Suppose asumptions* (H_2)–(H_3) *are fulfilled. Then BVP* (1), (2) *has at most one solution for each* $(A,B,C)\in\mathbf{R}^3$.

Proof. Let x_j be a solution of (1) for $\lambda=\lambda_j$ satisfying (2) with $x=x_j$ $(j=1,2)$. Setting $w=x_2-x_1$, we have $w(t_0)-w(0)=w(1)=w(2)-w(t_1)=0$. Assume $\lambda_2\geq\lambda_1$ and $a:=\max\{w(t);\ 0\leq t\leq 2\}>0$. Then $a=w(\tau)>0$, $w'(\tau)=0$, $w''(\tau)\leq 0$ for a $\tau\in(0,2)$ which contradicts $w''(\tau)=f(\tau,x_2(\tau),\ x_2'(\tau),\ \lambda_2)-f(\tau,x_1(\tau),\ x_2'(\tau),\lambda_1)>0$. Therefore $a=0$ and then $w(1)=w'(1)=0$, $w''(1)\leq 0$. Since $w''(1)=f(1,x_1(1),x_1'(1),\lambda_2)-f(1,x_1(1),\ x_1'(1),\ \lambda_1)\geq 0$ we have (cf. (H_3)) $\lambda_1=\lambda_2$. Assume $b:=\min\{w(t);\ 0\leq t\leq 2\}=w(\varepsilon)<0$ for an $\varepsilon\in(0,2)$. Then $w'(\varepsilon)=0$, $w''(\varepsilon)\geq 0$ which contradicts $w''(\varepsilon)=f(\varepsilon,x_2(\varepsilon),x_2'(\varepsilon),\lambda_1)-f(\varepsilon,x_1(\varepsilon),x_2'(\varepsilon),\lambda_1)<0$. Hence $b=0$ and therefore $w=0$, that is $x_1=x_2$. ■

Lemma 2. *Suppose assumption* (H_4) *is fulfilled. Let* $x(t)$ *be a solution of* (1) *for* $\lambda=\lambda_0$ *on* $\langle 0,2\rangle$ *such that*

$$|x(t_0)-x(0)|\leq Q,\ |x(1)|\leq Q,\ |x(2)-x(t_1)|\leq Q \tag{3}$$

which a positive constant Q.

Then $|\lambda_0|<K$, *where* K *corresponds in assumption* (H_4) *to*

$$S=2Q\max\left\{\frac{1}{t_0(1-t_0)^2},\ \frac{1}{t_0t_1(2-t_1)},\ \frac{1}{(2-t_1)^2}\left(1+\frac{1}{t_0}\right)\right\}, \tag{4}$$

$$M=Q\max\{2,1/t_0\},\ L=2S.$$

Proof. Suppose x is a solution of (1) for $\lambda = \lambda_0$ satisfying (3) with a positive constant Q and K corresponds in (H_4) to S, M and L defined by (4). Assume $\lambda_0 \geq K$. We see that $Q \geq |x(t_0) - x(0)| = |x'(\xi)|\, t_0$ for a $\xi \in (0, t_0)$, hence

(5) $$|x'(\xi)| \leq Q/t_0 \;(<L).$$

The next part of the proof we will divide into four cases.

1) Let $x(\xi) \geq M$. Then $x''(\xi) \geq S$, $x'(t) \geq \min\left\{x'(\xi) + \dfrac{2Q(t-\xi)}{t_0(1-t_0)^2}, L\right\} \geq$

$\geq -Q/t_0 + 2Q(t-\xi)/(t_0(1-t_0)^2)$ for $t \in \langle \xi, 1 \rangle$ and we have

$$x(1) = x(\xi) + \int_\xi^1 x'(s)\, ds \geq Q/t_0 - Q(1-\xi)/t_0 + Q(1-\xi)^2/(t_0(1-t_0)^2 > Q,$$

a contradiction.

2) Let $x(\xi) \leq -M$. Then $x(\eta) = -M$, $x'(\eta) \geq 0$ for an $\eta \in \langle \xi, 1)$ and $x'(t) \geq \min\{x'(\eta) + 2Q(t-\eta)/(t_0 t_1 (2-t_1)), L\} > 2Q(t-1)/(t_0 t_1 (2-t_1))$ for $t \in \langle \eta, 2)$. Therefore

$$x(2) - x(t_1) = \int_{t_1}^2 x'(s)\, ds > \{2Q/(t_0 t_1 (2-t_1)\} \int_{t_1}^2 (s-1)\, ds = Q/t_0 > Q$$

and we have a contradiction.

3) Let $|x(t)| \leq M$ for $t \in \langle \xi, 2 \rangle$. Then $x'(t) \geq \min\{x'(\xi) + \dfrac{2Q(1+(1/t_0))(t-\xi)}{(2-t_1)^2}, L\}$ for $t \in \langle \xi, 2 \rangle$, hence $x'(t) > -\dfrac{Q}{t_0} + \dfrac{2Q}{(2-t_1)^2} \cdot \left(1 + \dfrac{1}{t_0}\right)(t - t_1)$ on $\langle t_1, 2 \rangle$

and

$$x(2) - x(t_1) = \int_{t_2}^2 x'(s)\, ds > -Q(2-t_1)/t_0 + \; + Q(1 + (1/t_0)) =$$
$$= Q(t_0 + t_1 - 1)/t_0 > Q$$ whis is impossible.

4) Let $|x(\xi)| < M$ and $|x(\varepsilon)| > M$ for an $\varepsilon \in (\xi, 2)$. If $x(\varepsilon) > M$, then $x(v) = M$, $x'(v) \geq 0$ for a $v \in (\xi, \varepsilon)$, therefore $x'(t) > 0$ on $(v, 2\rangle$, $x(2) > M$, $x(t_1) > M - Q$. Since

$$M - 2Q < x(t_1) - x(1) = x'(\rho)(t_1 - 1)$$

for a $\rho \in (1, t_1)$, we have $x'(\rho) > (M - 2Q)/(t_1 - 1)\,(\geq 0)$ and then

$$x'(t) \geq \min\left\{x'(\rho) + \frac{2Q}{(2-t_1)^2}\left(1 + \frac{1}{t_0}\right)(t-\rho), L\right\} > \frac{2Q}{(2-t_1)^2}\left(1 + \frac{1}{t_0}\right)(t - t_1)$$

for $t \in \langle t_1, 2 \rangle$ hence

$$x(2) - x(t_1) = \int_{t_1}^2 x'(s)\, ds > \frac{2Q}{(2-t_1)^2}\left(1 + \frac{1}{t_0}\right)\int_{t_1}^2 (s - t_1)\, ds = Q\left(1 + \frac{1}{t_0}\right) > Q,$$

a contradiction.

If $x(\varepsilon) < -M$ and $\xi < \varepsilon < 1$, then $x(\eta) = -M$, $x'(\eta) \geq 0$ for an $\eta \in (\varepsilon, 1)$, hence

$$x'(t) \geq \min\left\{x'(\eta) + \frac{2Q(t-\eta)}{t_1 t_0 (2-t_1)}, L\right\} \text{ for } t \in \langle \eta, 2\rangle .$$

The last inequality yields $x'(t) > 2Q(t-1)/(t_1 t_0(2-t_1))$ on $\langle t_1, 2)$ and we have

$$x(2) - x(t_1) = \int_{t_1}^{2} x'(s)\,ds > \frac{Q(1-(t_1-1)^2)}{t_1 t_0 (2-t_1)} = \frac{Q}{t_0} > Q,$$

a contradiction. This proves $x(t) \geq -M$ on $\langle \xi, 1\rangle$ and then $x'(t) \geq \min\left\{x'(\xi) + \frac{2Q(t-\xi)}{t_0(1-t_0)^2}, L\right\}$ on $\langle \xi, 1\rangle$, consequently $x'(1) \geq -\frac{Q}{t_0} + \frac{2Q(t-\xi)}{t_0(1-t_0)^2} > 0$. Therefore $x'(t) \geq \min\left\{x'(1) + \frac{2Q(t-1)}{(2-t_1)^2}\left(1+\frac{1}{t_0}\right), L\right\} > \frac{2Q(t-t_1)}{(2-t_1)^2}\left(1+\frac{1}{t_0}\right)$ for $t \in \langle 1, 2\rangle$ and $x(2) - x(t_1) = \int_{t_1}^{2} x'(s)\,ds > \frac{2Q}{(2-t_1)^2}\left(1+\frac{1}{t_0}\right)\int_{t_1}^{2}(s-t_1)\,ds = Q\left(1+\frac{1}{t_0}\right) > Q$, a contradiction.

For $\lambda_0 \leq -K$ the proof is similar and therefore is omitted. ■

Lemma 3. *Let assumptions* (H_5) *and* (H_6) *be satisfied. Let* $x(t)$ *be a solution f* (1) *for* $\lambda = \lambda_0$ *on* $\langle 0, 2\rangle$ satisfying (3) *whith a positive constant* Q. *Then*

$$|x(t)| \leq M_1 + \frac{Q}{t_0}(1+t_0)(=:P) \textit{ for } t \in \langle 0, 2\rangle,$$

where $M_1 (\geq Q)$ *corresponds in assumption* (H_5) *to* $L_1 = Q/t_0$, $K_1 \geq |\lambda_0|$, and

$$|x'(0)| \leq \max\{Q/t_0, \ \Psi^{-1}(2P)\},$$

where Ψ^{-1} *denotes the inverse function to* Ψ, $\Psi(t) = \int_{Q/t_0}^{t} \frac{v\,dv}{w(v; \mathscr{D}_0)}$, $t \in \langle Q/t_0, \infty)$ *with* $w(v, \mathscr{D}_0)$ *corrersponds in assumption* (H_6) *to* $\mathscr{D}_0 = \langle -P, P\rangle \times \langle -K_1, K_1\rangle$.

Proof. Let x be a solution of (1) for $\lambda = \lambda_0$ on $\langle 0, 2\rangle$ satisfying (3) with a positive constant Q. Let $M_1(\geq Q)$ correspond in condition (H_5) to $L_1 = Q/t_0$ and $K_1 \geq |\lambda_0|$ and set $\mathscr{D}_0 = \langle -P, P\rangle \times \langle -K_1, K_1\rangle$. Assume $x(0) > M_1 + Q(1+t_0)/t_0$. By virtue of $x(1) \leq Q$ we have $x(\tau) = M_1$ and

$x(t) > M_1$ on $\langle 0, \tau)$ for a $\tau \in (0,1)$. Assume $a := \max\{x(t);\ \tau \le t \le \le 1\} = x(\alpha) > M_1$, $\alpha \in (\tau, 1)$. Then $x(\alpha) > M_1, x'(\alpha) = 0$, $x''(\alpha) \le 0$ which contradicts $x''(\alpha) = f(\alpha, x(\alpha), 0, \lambda_0) > 0$. Therefore $a = M_1$ and since $x(t_0) \ge \ge x(0) - Q$ we have $\tau \in (t_0, 1)$. Next we have $Q \ge x(0) - x(t_0) = = -x'(\eta)\, t_0$, $Q/t_0 < x(t_0) - x(\tau) = -x'(\varepsilon)(\tau - t_0)$ for an $\eta \in (0, t_0)$ and an $\varepsilon \in (t_0, \tau)$, consequently

$$x'(\eta) \ge -Q/t_0,\ x'(\varepsilon) < -Q/(t_0(\tau - t_0)) < -Q/t_0 .$$

Then $-Q/t_0 = x'(\rho)$, $x''(\rho) \le 0$ for a $\rho \in (\eta, \varepsilon)$ which contradicts $x''(\rho) = = f(\rho, x(p), x'(p), \lambda_0) > 0$. This proves $x(0) \le \mathrm{P}$. If x has a local maximum at a point $t = \beta \in (0, 2)$ and $x(\beta) > P$, then $x'(\beta) = 0$, $x''(\beta) \le 0$ which contradicts $x''(\beta) = f(\beta, x(\beta), 0, \lambda_0) > 0$. Therfore $x(t) \le P$ on $\langle 0, 2 \rangle$.

Simillary we can prove $x(t) \ge -P$ on $\langle 0, 2 \rangle$.

Suppose $|x'(0)| > \max\{Q/t_0, \Psi^{-1}(2P)\}$, say $x'(0) < -\max\{Q/t_0, \Psi^{-1}(2P)\}$. Since $Q \ge |x(0) - x(t_0)| = |x'(\xi)|\, t_0$ for a $\xi \in (0, t_0)$, we have $|x'(\xi)| \le Q/t_0$, thus $x'(t) < -Q/t_0$ on $\langle 0, \eta)(\subset <0, \xi))$ while $x'(\eta) = -Q/t_0$. Set $T = \min\{x'(t); 0 \le t \le \eta\} = x'(v)$ for a $v \in \langle 0, \eta)$. With regard to (H_6), $|x''(t)| = |f(t, x(t), x'(t), \lambda_0)| \le w(|x'(t)|; \mathcal{D}_0)$ for $t \in \langle 0, t_0 \rangle$, hence

$$\frac{x'(t)|x''(t)|}{w(-x'(t); \mathcal{D}_0)} \ge x'(t) \text{ on } \langle 0, \eta \rangle$$

and

$$\Psi(-T) = \int_{Q/t_0}^{-T} \frac{s\,ds}{w(s; \mathcal{D}_0)} \le \int_{\eta}^{v} \frac{x'(t)|x''(t)}{w(-x'(t); \mathcal{D}_0)}\, dt \le \int_{\eta}^{v} x'(t)\, dt = x(v) - x(\eta) \le 2P .$$

Then $-x'(0) \le -x'(v) = -T \le \Psi^{-1}(2P)$, a contradiction. ■

Lemma 4. *Suppose assumptions* $(\mathrm{H}_4)-(\mathrm{H}_6)$ *are satisfied. Let* $x_k(t)$, $k = 1, 2, \ldots$ *be a sequence of solutions of* (1) *for* $\lambda = \lambda_k$ *on* $\langle 0, 2 \rangle$ such that

$$|x_k(t_0) - x_k(0)| \le Q,\ |x_k(1)| \le Q,\ |x_k(2) - x(t_1)| \le Q,\quad k \in \mathrm{N},$$

where Q *is a positive constant. Then the sequences*

$$\{x_k(0)\},\ \{x_k'(0)\} \quad \text{and } \{\lambda_k\}$$

are bounded.

Proof. With respect to Lemma 2 $|\lambda_k| \le K$ for $k \in \mathrm{N}$, where K corresponds in condition (H_4) to S, M and L defined by (4). Using Lemma 3, $|x_k(0)| \le M_1 + Q(1 + t_0)/t_0 (=: P)$ for $k \in \mathrm{N}$, where $M_1 (\ge Q)$ corresponds in condition (H_5) to $L_1 = Q/t_0$ and $K_1 = K$, and $|x_k'(0)| \le \max\{Q/t_0, \Psi^{-1}(2P)\}$ for $k \in \mathbf{N}$, where Ψ^{-1} denotes the inverse function to Ψ, $\Psi : \langle Q/t_0, \infty) \to \langle 0, \infty)$, $\Psi(t) = \int_{Q/t_0}^{t} \frac{v\,dv}{(v; \mathscr{D})}$ with $w(z; \mathscr{D}_0)$ corresponds in condition (H_6) to $\mathscr{D}_0 = \langle -P, P\rangle \times \langle -K_1, K_1\rangle$. ■

3. Existence and uniqueness theorem

Theorem 1. *Let assumptions* $(H_1)-(H_6)$ *be satisfied. Then BVP* (1), (2) *has a unique solution for each* $(A, B, C) \in \mathbf{R}^3$ *and this solution as well as its derivative are continuous functions of the variables* $(t, A, B, C) \in \langle 0, 2\rangle \times \mathbf{R}^3$.

Proof. With regard to Proposition 1 it is sufficient to show that assumptions (H_7) and (H_8) are fulfilled. Assumptions $(H_2)-(H_3)$ imply (cf. Lemma 1) that assumption (H_7) is satisfied and assumptions $(H_4)-(H_6)$ give conditions guaranteeing the validity of assumption (H_8) (cf. Lemma 4). ■

Example 1. Consider the differential equation

$$x'' = t^2 + x + t|x'| + (1 + |e^t + x'|)\,\lambda. \tag{6}$$

Equation (6) satisfies all assumptions of Theorem 1 and therefore BVP (6), (2) has a unique solution $x(t; A, B, C)$ for each $(A, B, C) \in \mathbf{R}^3$. Furthermore, $x(t; A, B, C)$ and $x'(t; A, B, C)$ are continuous functions of the variables (t, A, B, C) on $\langle 0, 2\rangle \times \mathbf{R}^3$.

References

[1] Arscott F. M., *Two-parameter Eigenvalue Problems in Differential Equations*, Proc. London Math. Soc., (3), 14, 1964, pp. 459–470.

[2] Deimling K., *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, Heidelberg 1985.

[3] Greguš M., Neuman F., Arscott F. M., *There-point Boundary Value Problems in Differential Equations*, J. London Math. Soc., (2), 3, 1971, pp. 429–436.

[4] Staněk S., *Three Point Boundary Value Problem for Nonlinear Second-order Differential Equations with Parameter*, Czech. Math. J., 42(117), 1992, pp. 241–256.

[5] Staněk S., *Multi-point Boundary Value Problem for a Class of Functional Differentional Equations with Parameter*, Math. Slovaca, No. 1, 42, 1992, pp. 85–96.

[6] Staněk S., *On a Class of Five-point Boundary Value Problems in Second-order Functional Differentional Equations with Parameter*, Acta Math. Hungar. (to appear).

[7] Staněk S., *On Certain Three-point Regular Boundray Value Problems for Nonlinear Second-order Differentional Equations Depending on the Parameter*, (1992) Submitted for Publication.
[8] Šeda V., *A Correct Problem at a Resonance, Differential and Integral Eqs*, (2), 4, 1989, pp. 389–396.

(Department of Mathematical Analysis, Faculty of Science, Palacký University, 772 00 Olomouc, Czech Republic)
Received on 9.9.1992 and, in revised form, on 15.7.1993.