FASCICULI MATHEMATICI

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ON A PROPERTY OF BEZIER POLYNOMIALS

In this paper there are discussed effective methods of calculating the Bezier coefficients $b_{i_0 i_1 i_2}$ of Bezier polynomial Q_n, defined with respect to a certain basis $\mathscr{B}$, where the coefficients of Bezier polynomial P_n with respect to another basis $\mathscr{A}$ are given.

Key words: orthogonal polynomials and functions, special properties of functions of several variables.

MS Classification: 33C25, 26B35.

1. Preliminary results

Let $\mathscr{A} = \{A_0, A_1, A_2\}$, be the set of three nonlinear points in the real plane R^2. Further on we will call $\mathscr{A}$ a point basis of plane R^2 or more simply a basis of R^2. Then every point $x \in R^2$ can be expressed as:

$$x = t_0 A_0 + t_1 A_1 + t_2 A_2 \tag{1}$$

where $t_0 + t_1 + t_2 = 1$.

Numbers t_0, t_1, t_2 are called the barycentric coordinates of the point x with respect to the basis $\mathscr{A}$.

Let $P_n(t_0, t_1, t_2)$ be a Bezier polynomial of degree n:

$$P_n(t_0, t_1, t_2) = \sum_{i_0+i_1+i_2=n} a_{i_0 i_1 i_2} \Phi^n_{i_0 i_1 i_2}(t_0, t_1, t_2) \tag{2}$$

where $0 \le i_0, i_1, i_2 \le n$, and

$$\Phi^n_{i_0 i_1 i_2}(t_0, t_1, t_2) = \frac{n!}{i_0!\, i_1!\, i_2!}\, t_0^{i_0}\, t_1^{i_1}\, t_2^{i_2} \tag{3}$$

The coefficients $a_{i_0 i_1 i_2}$ are called Bezier points, end in their natural ordering they form the Bezier net of the triangular region $T(A_0, A_1, A_2)$. (see Fig. 2)

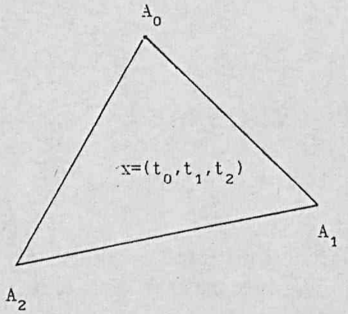

Fig. 1

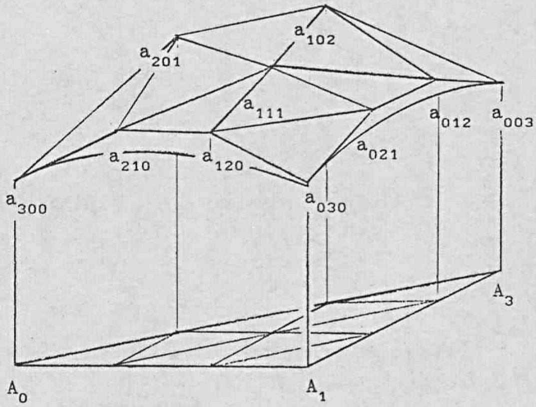

Fig. 2

Now, we provide the following notation. For the Bezier polynomial $P_n(t_0, t_1, t_2)$, let $P_{i_0 i_1 i_2}(t_0, t_1, t_2)$ be the Bezier polynomial of degree i_0 given by formula (4):

$$(4) \qquad P_{i_0 i_1 i_2}(t_0, t_1, t_2) = \sum_{j_0+j_1+j_2=i_2} a_{i_0+j_0, i_1+j_1, j_2} \Phi^{i_2}_{j_0 j_1 j_2}(t_0, t_1, t_2)$$

where $i_0+i_1+i_2 = n$, and $P_{n,0}(t_0, t_1, t_2)$, $P_{n,1}(t_0, t_1, t_2)$, $P_{n,2}(t_0, t_1, t_2)$ are Bezier polynomials of degree $n-1$ given by (5), (6), (7), respectively:

$$(5) \qquad P_{n,0}(t_0, t_1, t_2) = \sum_{i_0+i_1+i_2=n-1} a_{i_0+1, i_1, i_2} \Phi^{n-1}_{i_0 i_1 i_2}(t_0, t_1, t_2)$$

$$(6) \qquad P_{n,1}(t_0, t_1, t_2) = \sum_{i_0+i_1+i_2=n-1} a_{i_0, i_1+1, i_2} \Phi^{n-1}_{i_0 i_1 i_2}(t_0, t_1, t_2)$$

$$(7) \qquad P_{n,2}(t_0, t_1, t_2) = \sum_{i_0+i_1+i_2=n-1} a_{i_0, i_1, i_2+1} \Phi^{n-1}_{i_0 i_1 i_2}(t_0, t_1, t_2)$$

Using the equalities (5), (6), (7) we can write the Bezier polynomial $P_n(t_0, t_1, t_2)$ as:

$$(8) \qquad P_n = t_0 P_{n,0} + t_1 P_{n,0} + t_2 P_{n,0}$$

It is a well-know recursion formula of Bezier polynomial (see [1]).

Remark 1. It is easy to show that:

$$P_{n,0}(t_0,t_1,t_2) = P_{1,0,n-1}(t_0,t_1,t_2) \tag{9}$$

$$P_{n,0}(t_0,t_1,t_2) = P_{0,1,n-1}(t_0,t_1,t_2) \tag{10}$$

Hence, the equality (8) can be writen as:

$$P_n = t_0 P_{1,0,n-1} + t_1 P_{0,1,n-1} + t_2 P_{n,0} \tag{11}$$

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Remark 2. (de Casteljau algorithm). The recursion formula (8) provides the following algorithm for the evaluation of $P_n(t_0,t_1,t_2)$ (so called de Casteljau algorithm):

$$P_n(t_0,t_1,t_2) = a^n_{0,0,0} \tag{12}$$

where

$$a^r_{i_0,i_1,i_2} = t_0\, a^{r-1}_{i_0+1,i_1,i_2} + t_1\, a^{r-1}_{i_0,i_1+1,i_2} + t_2\, a^{r-1}_{i_0,i_1,i_2+1}$$

and $i_0+i_1+i_2 = n-r$, and $a^0_{i_0,i_1,i_2} = a_{i_0,i_1,i_2}$.

■

Remark 3. It is clear that

$$P_{1,0,n-1}(t_0,t_1,t_2) = a^{r-1}_{1,0,0} \tag{13}$$

$$P_{0,1,n-1}(t_0,t_1,t_2) = a^{r-1}_{0,1,0} \tag{14}$$

and in general

$$P^i_{i_0 i_1 i_2}(t_0,t_1,t_2) = a^2_{i_0 i_1 0} \tag{15}$$

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2. Main Theorem

Now, let $\mathscr{A} = \{A_0, A_1, A_2\}$ and $\mathscr{B} = \{B_0, B_1, B_2\}$ be two different collections of points (point bases) in the real plane R^2. Then every point $x \in R^2$ can be expressed as

(16) $$x = t_0 A_0 + t_1 A_1 + t_2 A_2$$

where t_0, t_1, t_2 are the barycentric coordinates of x with respect to $\mathscr{A}$, or as:

(17) $$x = u_0 B_0 + u_1 B_1 + u_2 B_2$$

where u_0, u_1, u_2 are the barycentric coordinates of x with respect to $\mathscr{B}$. Let P_n and Q_n be two Bezier polynomials given by the formulas (18) and (19), respectively

(18) $$P_n(t_0, t_1, t_2) = \sum_{i_0+i_1+i_2=n} a_{i_0 i_1 i_2} \Phi^n_{i_0 i_1 i_2}(t_0, t_1, t_2)$$

(19) $$Q_n(u_0, u_1, u_2) = \sum_{i_0+i_1+i_2=n} b_{i_0 i_1 i_2} \Phi^n_{i_0 i_1 i_2}(u_0, u_1, u_2)$$

where numbers t_0, t_1, t_2 are barycentric coordinates of x with respect to $\mathscr{A}$ and u_0, u_1, u_2 are barycentric coordinates of x with respect to $\mathscr{B}$.

We suppose that we have the coefficiens $a_{i_0 i_1 i_2}$ of $P_n(t_0, t_1, t_2)$ for any i_0, i_1, i_2 $(0 \leq i_0, i_1, i_2 \leq n;\ i_0 + i_1 + i_2 = n)$. We want to evaluate the coefficiens $b_{i_0 i_1 i_2}$ of $Q_n(t_0, t_1, t_2)$ so that the equality (20) will by satisfied for every $x \in R^2$.

(20) $$Q_n(x) = P_n(x)$$

The above problem can be solved in two steps: the first step considers a particular case, when $\mathscr{A} \cap \mathscr{B} = \{A_1, A_j\}$, for $i \neq j$, where $i, j = 0, 1, 2$, the second one generalities the case.

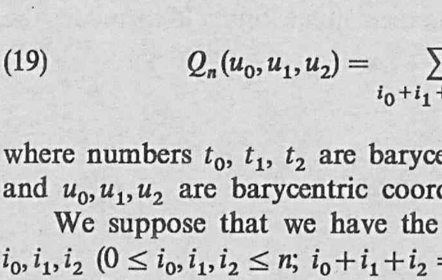

Fig. 3

First, we consider the case, when the sets $\mathscr{A}$ and $\mathscr{B}$ have two common poinst, i.e.: $\mathscr{A} \cap \mathscr{B} = \{A_i, A_j\}$, where $i \neq j$ and $i, j = 0, 1, 2$. We may assume, without loss of generality, that $\mathscr{A} \cap \mathscr{B} = \{A_0, A_1\}$ (see Fig. 3).

Let coefficiens of polynomial P_n and Q_n be set in tow arrays as shown in Fig. 4.

Let numbers v_0, v_1, v_2 be barycentric coordinates of B_2 with respect to $\mathscr{A}$, ie:

(21) $$B_2 = v_0 A_0 + v_1 A_1 + v_2 A_2$$

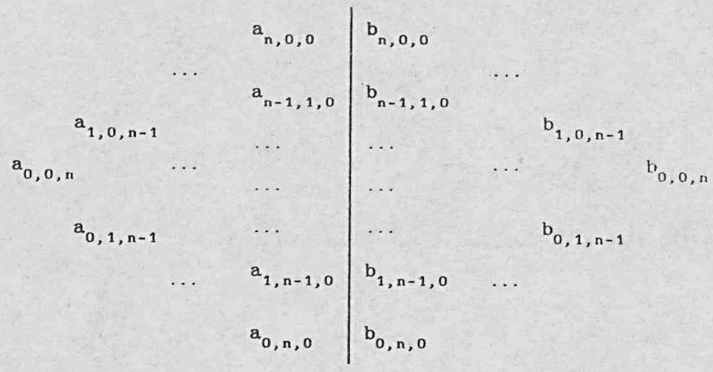

Fig. 4

Theorem 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two different bases in the real plane R^2 such that $\mathcal{A} \cap \mathcal{B} = \{A_0, A_1\}$ (see Fig. 4) and Bezier polynomials $P_n(t_0, t_1, t_2)$ and $Q_n(u_0, u_1, u_2)$ where t_0, t_1, t_2 and u_0, u_1, u_2 are barycentric coordinates of x with respect to $\mathcal{A}$ and $\mathcal{B}$ respectively. Then $P_n(x) = Q_n(x)$ for every $x \in R^2$ if and only if the equality* (22) *holds for $i_0, i_1, i_2 \geq 0$ and $i_0 + i_1 + i_2 = 0$,*

$$b_{i_0 i_1 i_2} = P_{i_0 i_1 i_2}(v_0, v_1, v_2) \tag{22}$$

where v_0, v_1, v_2 are the barycentric coordinates of B_2 with respect to $\mathcal{A}$.

Proof. Let us calculate the value of the polynomial $Q_n(x)$ at a given point $x \in R^2$. Let u_0, u_1, u_2 be barycentric coordinates of x with respect to $\mathcal{B}$. Then

$$Q_n(x) = Q_n(u_0, u_1, u_2) = \sum_{i_0+i_1+i_2=n} b_{i_0 i_1 i_2} \Phi^n_{i_0 i_1 i_2}(u_0, u_1, u_2) \tag{23}$$

Replacing $b_{i_0 i_1 i_2}$ on the right side of (23) by the right side of (22) we obtain:

$$Q_n(x) = \sum_{i_0+i_1+i_2=n} \Phi^n_{i_0 i_1 i_2}(u_0, u_1, u_2) P_{i_0 i_1 i_2}(v_0, v_1, v_2) =$$

$$= \sum_{i_0+i_1+i_2=n} \Phi^n_{i_0 i_1 i_2}(u_0, u_1, u_2) \sum_{j_0+j_1+j_2=i_2} a_{i_0+j_0, i_1+j_1, j_2} \Phi^{i_2}_{j_0 j_1 j_2}(t_0, t_1, t_2) =$$

$$= \sum_{i_0+i_1+j_0+j_1+j_2=n} a_{i_0+j_0, i_1+j_1, j_2} \Phi^n_{i_0 i_1 i_2}(u_0, u_1, u_2) \Phi^{j_2}_{j_0 j_1 j_2}(t_0, t_1, t_2) \tag{24}$$

It is easy to show that

$$\Phi^{n}_{i_0 i_1 i_2}(u_0, u_1, u_2)\, \Phi^{i_2}_{j_0 j_1 j_2}(t_0, t_1, t_2) = \tag{25}$$
$$= \frac{n!}{i_0!\, i_1!\, j_0!\, j_1!\, j_2!}\, u_0^{i_0} u_1^{i_1} (u_2 v_0)^{j_0} (u_2 v_1)^{j_1} (u_2 v_2)^{j_2}.$$

We can write the sum on the right side of (24) as:

$$\sum_{i_0+i_1+j_0+j_1+j_2=n} = \sum_{k_0+k_1+k_2=n} \sum_{i_0+j_0=k_0} \sum_{i_1+j_1=k_1} \tag{26}$$

Applying the equalities (25) and (26) to expression (24) it is easy to show that:

$$Q_n(x) = \sum_{k_0+k_1+j_2=n} a_{k_0 k_1 j_2} \frac{n!}{k_0!\, k_1!\, j_2!} \left(\sum_{i_0+j_0=k_0} \frac{k_0!}{i_0!\, j_0!} (u_2 v_0)^{j_0} u^{i_0} \right) \cdot \tag{27}$$
$$\cdot \left(\sum_{i_1+j_1=k_1} \frac{k_1!}{i_1!\, j_1!} (u_2 v_1)^{j_1} u_1^{i_1} \right) \cdot (u_2 v_2)^{j_2}$$

We remark that

$$\sum_{i_0+j_0=k_0} \frac{k_0!}{i_0!\, j_0!} (u_2 v_0)^{j_0} u_0^{i_0} = (u_0 + u_2 v_0)^{k_0} \tag{28}$$

and

$$\sum_{i_1+j_1=k_1} \frac{k_1!}{i_1!\, j_1!} (u_2 v_1)^{j_1} u_1^{i_1} = (u_1 + u_2 v_1)^{k_1}. \tag{29}$$

Hence, we can write the equality (27) as

$$Q_n(x) = \sum_{k_0+k_1+j_2=n} a_{k_0 k_1 j_2} \frac{n!}{k_0!\, k_1!\, j_2!} (u_0 + u_2 v_0)^{k_0} (u_1 + u_2 v_1)^{k_1} (u_2 v_2)^{j_2} \tag{30}$$

Since u_0, u_1, u_2 are barycentric coordinates of x with respect to $\mathscr{B}$, we can write x as:

$$x = u_0 B_0 + u_1 B_1 + u_2 B_2 \tag{31}$$

Since $B_0 = A_0$, $B_1 = A_1$ and $B_2 = v_0 A_0 + v_1 A_1 + v_2 A_2$ (see equality (21)), x can be expressed by:

$$x = u_0 A_0 + u_1 A_1 + u_2 (v_0 A_0 + v_1 A_1 + v_2 A_2) = \tag{32}$$
$$= (u_0 + u_2 v_0) A_0 + (u_1 + u_2 v_1) A_1 + u_2 v_2 A_2$$

On the other hand $x = t_0 A_0 + t_1 A_1 + t_2 A_2$ because t_0, t_1, t_2 are barycentric coordinates of x with respect to $\mathscr{A}$. Hence we obtain following equalities:

$$t_0 = u_0 + u_2 v_0$$

$$t_1 = u_1 + u_2 v_1$$

$$t_2 = u_2 v_2$$

Hence, we can write equality (30) as (see (18)):

$$Q_n(x) = \sum_{k_0+k_1+j_2=n} a_{k_0 k_1 j_2} \frac{n!}{k_0!\, k_1!\, j_2!} t_0^{k_0} t_1^{k_1} t_2^{j_2} = P_n(x)$$

Finally, we obtain $Q_n(x) = P_n(x)$ for any $x \in R^2$.

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Remark 4. Since $P_{i_0 i_1 i_2}(t_0, t_1, t_2) = a^2_{i_0 i_1 0}$ (see Remark 3) we can calculate the coefficients $b_{i_0 i_1 i_2}$ of the polynomial $Q_n(x)$ which is equal to $P_n(x)$ (see Theorem 1), considering indirect points $a^{i_2}_{i_0 i_1 0}$ in the Casteljau algoritm of the calculation of the value of polynomial P_n at the point B_2.

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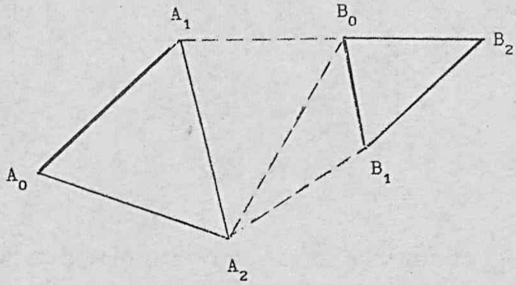

Fig. 5

Now, let $\mathscr{A}$ and $\mathscr{B}$ be two point bases. After replacing one point of $\mathscr{A}$ by a point of $\mathscr{B}$ (in Fig. 5 point A_0 was replaced by B_0) we obtain a certain basis $\mathscr{A}^1$. Applying Theorem 1 to bases A and A^1 we can calculate all coefficients of certain polynomial $P_n^1(x)$ with respect to $\mathscr{A}^1$. If $\mathscr{A}^1 = \mathscr{B}$ then $P_n^1(x) = Q_n(x)$, otherwise we use the above process to bases $\mathscr{A}^1$ and $\mathscr{B}$ and polynomials $P_n^1(x)$ and $Q_n(x)$, respectively, etc.

Using the above procedure we obtain the coefficients $b_{i_0 i_1 i_2}$ of $Q_n(x)$ in at most tree steps.

Conclusion. The coefficients $b_{i_0 i_1 i_2}$ of polynomial $Q_n(x)$ with respect to $\mathscr{B}$ can be calculated from the coefficients of polynomial $P_n(x)$ by at most three calculations of the value of certain polynomials at the points of $\mathscr{B}$. Each point will be used at most once.

3. Application

Now, we consider the area T which is composed by two triangles T_1 and T_2, as shown in Fig. 6, and function $F: T \to R$, $F|_{T_1} = P_n$ and $F|_{T_2} = Q_n$, where P_n and Q_n are Bezier polynomials of degree n. In general, F doesn't have to be continuous.

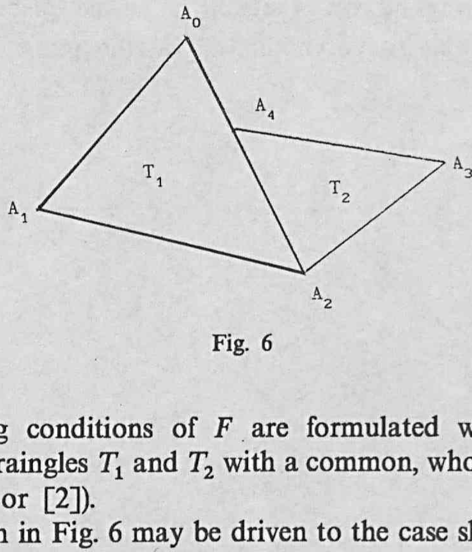

Fig. 6

The smoothing conditions of F are formulated when the area T is composed by two traingles T_1 and T_2 with a common, whole edge, as shown in Fig. 7 (see in [1] or [2]).

The case shown in Fig. 6 may be driven to the case shown in Fig. 7. This can be done by dividing the triangle T_1 (as shown in Fig. 8) and applying Theorem 1 to triangles T_1' and T_1'' and polynomial $P_n(x)$. On this way we obtain coefficients of a certain polynomial $P_n'(x)$ with respect to the basis $\{A_1, A_2, A_4\}$, which is equal to $P_n(x)$ for any point $x \in R^2$. Now, we can use the smoothing conditions to calculate coefficients of the polynomial $Q_n(x)$.

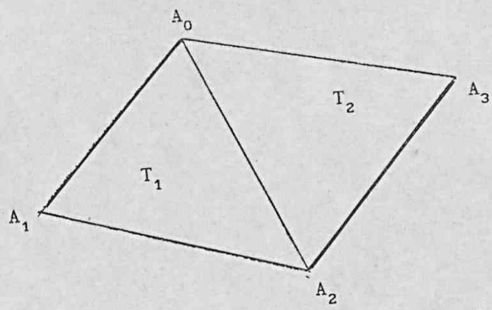

Fig. 7

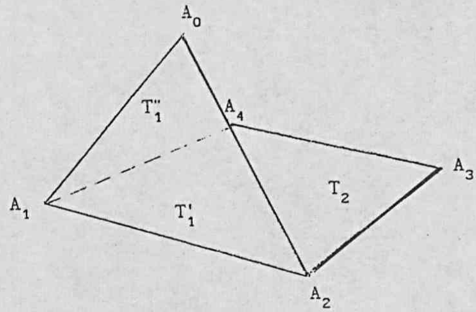

Fig. 8

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