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# OSCILLATIONS OF SOME LINEAR DIFFERENCE EQUATIONS

The purpose of this paper is to establish sufficient conditions for the oscillation of solutions of some linear difference equations.

Key words: oscillations, linear difference equations.

## 1. Introduction

In the past several years the oscillation and nonoscillation of solutions of difference equations have been extensively investigated. See, for example [1, 3–4, 6–7, 10–15] and the references cited therein. In particular, the oscillations of the solutions of the neutral difference equations have been studied in [2, 5, 8, 9]. For the general theory of difference equations the reader is referred to the recent books [1, 4, 7]. Our main purpose in this paper is to establish sufficient conditions for the oscillation of solutions of some linear difference equations.

Consider the difference equations of the form

$$\Delta\left(y_n+\sum_{i=1}^{m} p_i y_{n-k_i}\right)+\sum_{j=1}^{\nu} q_j y_{n-r_j}=0, \quad n\in N \tag{1}$$

where $N=\{0,1,2,...,\}$, $\Delta$ is the forward difference operator defined by $\Delta x_n = x_{n+1}-x_n$ for any sequence $(x_n)$ of real numbers and

$$p_i\in R \text{ and } k_i\in N \quad \text{for } i=1,...,m;$$

$$q_j\in(0,\infty) \text{ and } r_j\in N \quad \text{for } j=1,...,\nu.$$

Let

$$p=\max\{k_1,...,k_m,r_1,...,r_\nu\}.$$

By a solution of (1) we mean a sequence $(y_n)$ which is defined for $n\geqslant -p$ and which satisfieds (1) for all $n\in N$. A nontrivial solution $(y_n)$ of Eq. (1) is called oscillatory if for every $k\in N$ there exists $n\geqslant k$ such that $x_n x_{n+1}\leqslant 0$. Otherwise it is called nonoscillatory.

## 2. Some basic lemmas

In this section we group some lemmas which will be useful in our study of Eq. (1).

*Lemma* 1. *Let* $(y_n)$ *be a solution of* Eq. (1). *Set*

$$z_n = y_n + \sum_{i=1}^{m} p_i y_{n-k_i}. \tag{2}$$

*Then* $(z_n)$ *is a solution of* Eq. (1), *i.e.*

$$\Delta z_n + \sum_{i=1}^{m} p_i \Delta z_{n-k_i} + \sum_{j=1}^{v} q_j z_{n-r_j} = 0.$$

*Proof.* Denote the left hand side of Eq. (1) by $L(y)$. We show that $L(z) = 0$, where $z = (z_n)$ is defined by (2). In fact, since Eq. (1) is linear and autonomous we have

$$\begin{aligned}
L(z) &= \Delta z_n + \sum_{i=1}^{m} p_i \Delta z_{n-k_i} + \sum_{j=1}^{v} q_j z_{n-r_j} = \\
&= \Delta y_n + \sum_{i=1}^{m} p_i \Delta y_{n-k_i} + \sum_{i=1}^{m} p_j \left( \Delta y_{n-k_i} + \sum_{s=1}^{m} p_s \Delta y_{n-k_i-k_s} \right) + \\
&\quad + \sum_{j=1}^{v} q_j \left[ y_{n-r_j} + \sum_{s=1}^{m} p_s y_{n-r_j-k_s} \right] = \\
&= \sum_{i=1}^{m} p_i \left[ \Delta y_{n-k_i} + \sum_{s=1}^{m} p_s \Delta y_{n-k_i-k_s} \right] + \sum_{j=1}^{v} q_j \sum_{s=1}^{m} p_s y_{n-r_j-k_s} = \\
&= \sum_{i=1}^{m} p_i \left[ \Delta \left( y_{n-k_i} + \sum_{s=1}^{m} p_s y_{n-k_i-k_s} \right) \right] + \sum_{i=1}^{m} p_i \sum_{j=1}^{v} q_j y_{n-k_i-r_j} = \\
&= \sum_{i=1}^{m} p_i \left\{ \Delta \left[ y_{n-k_i} + \sum_{s=1}^{m} p_s y_{n-k_i-k_s} \right] + \sum_{j=1}^{v} q_j y_{n-k_i-r_j} \right\} = 0
\end{aligned}$$

that is $(z_n)$ is a solution of (1).

The next lemma shows that if Eq. (1) has nonoscillatory solution then it also has a nonoscillatory solution with "nice" properties wich are useful in the study of Eq. (1).

*Lemma* 2. *Let* $(y_n)$ *be an eventually positive solution of* Eq. (1) *and define* $(z_n)$ *by* (2) *and* $(w_n)$ *by*

(3) $$w_n = z_n + \sum_{i=1}^{m} p_i z_{n-k_i}.$$

*Then*

a) $(w_n)$ *is a solution of* Eq. (1),

b) *either*

(4) $$z_n > 0, \ \Delta z_n < 0 \quad \textit{and} \quad \lim_{n\to\infty} z_n = 0$$

*or*

(5) $$z_n < 0, \ \Delta z_n < 0 \quad \textit{and} \quad \lim_{n\to\infty} z_n = -\infty,$$

c) *either*

(6) $$w_n > 0, \quad \Delta w_n < 0, \ \Delta^2 w_n > 0 \quad \textit{and} \quad \lim_{n\to\infty} w_n = 0,$$

*or*

(7) $$w_n > 0, \quad \Delta w_n > 0, \ \Delta^2 w_n > 0 \quad \textit{and} \quad \lim_{n\to\infty} w_n = \infty.$$

*Furthermore*

(4) *holds if and only if* (6) *holds*

*and*

(5) *holds if and only if* (7) *holds.*

*Remark.* Evidently, all the above inequalities hold eventually.

*Proof*

a) This follows from Lemma 1.

b) From (1) and (2) we have

(8) $$\Delta z_n = -\sum_{j=1}^{v} q_j y_{n-r_j} < 0 \quad \text{eventually.}$$

Hence $(z_n)$ is eventually decreasing. Therefore either $\lim_{n\to\infty} z_n = -\infty$ or

$$\lim_{n\to\infty} z_n = L$$

that is either (5) holds or (9).

If (9) holds, then from (8) we get

$$0 = \lim_{n\to\infty} \sum_{j=1}^{v} q_j y_{n-r_j}$$

and so

$$\lim_{n\to\infty} y_n = 0.$$

Hence, by (2), we have $\lim_{n\to\infty} z_n = 0$ that is (4) holds.

c) From (b) we see that either $z_n > 0$ or $z_n < 0$ eventually. First assume $z_n > 0$. Then

$$\Delta w_n = -\sum_{j=1}^{v} q_j y_{n-r_j} < 0 \quad \text{eventually}$$

and so $(w_n)$ eventually decreases. Since $\lim_{n\to\infty} z_n = 0$ we have $\lim_{n\to\infty} w_n = 0$, and so $w_n > 0$. Further we get

$$\Delta^2 w_n = -\sum_{j=1}^{v} q_j \Delta z_{n-r_j} > 0.$$

On the other hand, if $z_n < 0$, then

$$\Delta w_n = -\sum_{j=1}^{v} q_j z_{n-r_j} > 0 \quad \text{eventually.}$$

Now since $\Delta w_n > 0$ and $\Delta^2 w_n > 0$ we have $\lim_{n\to\infty} w_n = \infty$ which implies $w_n > 0$ eventually. Therefore (4) implies (6) and (5) implies (7). It follows easily that (6) implies (4) and (7) implies (5). The proof is complete.

The next lemma provides sufficient conditions for (6) or (7) to hold.

*Lemma* 3. *Let* $(y_n)$ *be an eventually positive solution of* Eq. (1) *and let the sequence* $(z_n)$ *and* $(w_n)$ *be defined by* (2) *and* (3). *Then*

a) *If* $p_i \geqslant 0$ *for* $i = 1, \ldots, m$, *then* (6) *is true.*

b) *If* $p_i \leqslant 0$ *for* $i = 1, \ldots, m$, *then* $\sum_{i=1}^{m} p_i \neq -1$.

*Furthermore*

$$\sum_{i=1}^{m} p_i > -1 \quad \textit{implies (6)},$$

*while*

$$\sum_{i=1}^{m} p_i < -1 \quad \textit{implies (7)}.$$

c) *Suppose* $p_i > 0$ *for* $i = 1, \ldots, m^*$ *and* $p_i < 0$ *for* $i = m^* + 1, \ldots, m$. *Denote*

$$k^+ = \max_{1 \leqslant i \leqslant m^*} k_i, \quad k^- = \min_{m^*+1 \leqslant i \leqslant m} k_i \tag{10}$$

*and assume* $k^+ \leqslant k^-$. *Then* $\sum_{i=1}^{m} p_i \neq -1$.

*Moreover,*

$$\sum_{i=1}^{m} p_i > -1 \quad \textit{if and only if (6) holds},$$

*and*

$$\sum_{i=1}^{m} p_i < -1 \quad \textit{if and only if (7) holds}.$$

*Proof*

a) Since $p_i \geqslant 0$ $(i = 1, \ldots, m)$ and $(y_n)$ is eventually positive we see from (2) that $z_n > 0$ eventually and by Lemma 2 (6) holds.

b) Now, since $(z_n)$ decreases, we have

$$0 < w_n = z_n + \sum_{i=1}^{m} p_i z_{n-k_i} \leqslant z_n\left(1 + \sum_{i=1}^{m} p_i\right).$$

Hence

$$\sum_{i=1}^{m} p_i \neq -1 \quad \text{and so}$$

$$1+\sum_{i=1}^{m} p_i > 0 \quad \text{implies } z_n > 0$$

and

$$1+\sum_{i=1}^{m} p_i < 0 \quad \text{implies } z_n < 0.$$

Therefore the result follows from Lemma 2.

c) Since

$$n-k^+ \leqslant n-k_i \qquad i = 1,\dots,m^*,$$

$$n-k^- \geqslant n-k_i \qquad i = m^*+1,\dots,m,$$

and $(z_n)$ is a decreasing sequence, we get

$$0 < w_n = z_n + \sum_{i=1}^{m^*} p_i z_{n-k_i} + \sum_{i=m^*+1}^{m} p_i z_{n-k_i} \leqslant$$

$$\leqslant z_{n-k^+}\left(1+\sum_{i=1}^{m^*} p_i\right) + z_{n-k^-}\left(\sum_{i=m^*+1}^{m} p_i\right) \leqslant z_{n-k^+}\left(1+\sum_{i=1}^{m} p_i\right).$$

Thus our assertion follows analogously as in the case b).

## 3. Main results

The first result provides sufficient conditions for the nonoscillatory solutions of Eq. (1) to tend to $\infty$ as $n \to \infty$.

*Theorem* 1. *Let* $(y_n)$ *be an eventually positive solution of* Eq. (1). *If* $p_i \geqslant 0$ *for* $i = 1,\dots,m-1$, $p_m < 0$, $\max_{1 \leqslant i \leqslant m-1} k_i \leqslant k_m$ *and*

$$\sum_{i=1}^{m} p_i < -1$$

*then*

$$\lim_{n \to \infty} y_n = \infty.$$

*Proof.* By Lemma 3 c) with $m^* = m-1$, it follows that

$$\lim_{n\to\infty} z_n = -\infty .$$

Note that

$$p_m y_{n-k_m} < y_n + \sum_{i=1}^{m} p_i y_{n-k_i} = z_n \to -\infty \text{ as } n \to \infty,$$

which implies

$$\lim_{n\to\infty} y_n = \infty .$$

*Theorem* 2. *Let* $(y_n)$ *be an eventually positive solution of* Eq. (1). *If* $p_i \geqslant 0$ $(i = 1, \ldots, m^*)$, $p_i < 0$ $(i = m^*+1, \ldots, m)$, $k^+ \leqslant k^-$ *where* $k^+, k^-$ *are defined by* (10) and

$$\sum_{i=1}^{m} p_i < -1,$$

*then*

$$\lim_{n\to\infty} y_n = \infty.$$

*Proof.* By Lemma 3 c) we have $\lim_{n\to\infty} z_n = -\infty$. Also

$$\sum_{i=m^*+1}^{m} p_i y_{n-k_i} < y_n + \sum_{i=1}^{m^*} p_i y_{n-k_i} + \sum_{i=m^*+1}^{m} p_i y_{n-k_i} = z_n \to -\infty$$

as $n \to \infty$, which implies that $y_n \to \infty$ as $n \to \infty$. Now we present sufficient conditions for the oscillations of solutions of Eq. (1).

*Theorem* 3. *Each of the following two conditions is sufficient to imply that all solutions of* Eq. (1) *are oscillatory:*

a) $p_i \leqslant 0$ *for* $i = 1, \ldots, m$ *and* $\sum_{i=1}^{m} p_i = -1$,

b) $p_i \geqslant 0$ *for* $i = 1, \ldots, m^*$, $p_i < 0$ *for* $i = m^*+1, \ldots, m$,
$k^+ \leqslant k^-$(*where* $k^+$ *and* $k^-$ *are defined in* (10) *and* $\sum_{i=1}^{m} p_i = -1$.

*Proof*

a) Suppose the contrary. Without loss of generality let $(y_n)$ be an eventually positive solution of (1). By Lemma 3b), we have $\sum_{i=1}^{m} p_i \neq -1$, which is a contradiction. The proof is complete.

b) The proof of b) follows from Lemma 3c similarly.

*Remark.* In particular, for $m = v = 1$ from Theorem 3a we get a known result (Theorem 1 (i) in [5]).

*Theorem* 4. *Suppose* $p_i \geqslant 0$ *for* $i = 1,\dots,m$, $r_j > k = \max_{1 \leqslant i \leqslant m} k_i$ *for* $j = 1,\dots,v$ *and that the inequality*

$$\Delta d_n + \sum_{j=1}^{v} \frac{q_j}{1+\sum_{i=1}^{m} p_i} d_{n-(r_j-k)} \leqslant 0 \tag{11}$$

*has no eventually positive solution. Then every solution of* Eq. (1) *is oscillatory.*

*Proof.* If not, we assume that Eq. (1) has an eventually positive solution $(y_n)$. Then, by Lemma 3a and Lemma 2 for the sequence $(w_n)$ defined in (3) we have $w_n > 0$, $\Delta w_n < 0$, $\Delta^2 w_n > 0$ eventually and

$$\Delta w_n + \sum_{i=1}^{m} p_i \Delta w_{n-k_i} + \sum_{j=1}^{v} q_j w_{n-r_j} = 0.$$

In view of the monotonicity of $(\Delta w_n)$ we may write

$$\Delta w_{n-k} + \sum_{i=1}^{m} p_i \Delta w_{n-k} + \sum_{j=1}^{v} q_j w_{n-r_j} \leqslant 0,$$

and so

$$\Delta w_n + \sum_{j=1}^{v} \frac{q_j}{1+\sum_{i=1}^{m} p_i} w_{n-(r_j-k)} \leqslant 0.$$

This contradics the assumption that (11) has no eventually positive solution, and the proof is complete.

*Theorem* 5. *Assume that* $p_i \leqslant 0$ *for* $i = 1, \dots, m$,

$$\sum_{i=1}^{m} p_i > -1, \; r_j > k = \min_{1 \leqslant i \leqslant m} k_i \quad \textit{for } j = 1, \dots, \nu,$$

*and the inequality* (11) *has no eventually positive solution. Then every solution of* Eq. (1) *is oscillatory.*

The proof of Theorem 5 follows by argument similar to the proof of Theorem 4, and so will be omitted.

*Theorem* 6. *Assume that* $p_i \leqslant 0$ *for* $i = 1, \dots, m$,

$$\sum_{i=1}^{m} p_i < -1, \quad r_j < k = \min_{1 \leqslant i \leqslant m} k_i \quad \textit{for } j = 1, \dots, \nu,$$

*and the inequality*

$$\Delta d_n + \sum_{j=1}^{\nu} \frac{q_j}{1 + \sum_{i=1}^{m} p_j} d_{n+k-r_j} \geqslant 0 \tag{12}$$

*has no eventually positive solution. Then every solution of* Eq. (1) *is oscillatory.*

*Proof.* If not, let $(y_n)$ be an eventually positive solution of (1). Then, by Lemma 3b) and Lemma 2a) we know that $w_n > 0$, $\Delta w_n > 0$, $\Delta^2 w_n > 0$ eventually and

$$\Delta w_n + \sum_{i=1}^{m} p_i \Delta w_{n-k_i} + \sum_{j=1}^{\nu} q_j w_{n-r_j} = 0.$$

By the assumptions and monotonicity of $(\Delta w_n)$ we get

$$\Delta w_{n-k} + \sum_{i=1}^{m} p_i \Delta w_{n-k} + \sum_{j=1}^{\nu} q_j w_{n-r_j} \leqslant 0.$$

which yields

$$\Delta w_n + \Delta w_n \sum_{i=1}^{m} p_i + \sum_{j=1}^{\nu} q_j w_{n+k-r_j} \leqslant 0$$

or

$$\Delta w_n + \sum_{j=1}^{v} \frac{q_j}{1+\sum_{i=1}^{m} p_i} w_{n+k-r_j} \geqslant 0$$

Thus $(w_n)$ is an eventually positive solution of (12) and this contradiction completes the proof of the theorem.

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