F A S C I C U L I   M A T H E M A T I C I

Nr 25 1995

ALEKSANDER WASZAK

ON SOME MODULAR SPACES OF DOUBLE SEQUENCES III

There are considered and investigated modular and countably modular spaces of double sequences, which are generated by translation operator and sequential φ-modulus.

Keywords: sequence spaces, modular spaces.

MS Classification: 46A45, 46E30.

1. Terminology and notation

Throughout this note X denotes the space of all real, bounded double sequences with convergence in the sense of Pringsheim. Sequences belonging to X will be denoted by

$$x = (t_{\mu\nu}) = (t_{\mu\nu})_{\mu,\nu=1}^{\infty} = ((x)_{\mu\nu}) = ((x)_{\mu\nu})_{\mu,\nu=1}^{\infty},\ |x| = (|t_{\mu\nu}|),\ x_p = (t_{\mu\nu}^p).$$

The (m,n) – translation of the sequence x we define as a sequence $((\tau_{mn}x)_{\mu\nu})$, where

$$(\tau_{mn}x)_{\mu\nu} = \begin{cases} t_{\mu,\nu} & \text{for } \mu < m \text{ and } \nu < n, \\ t_{\mu+m,\nu+n} & \text{for } \mu \geqslant m \text{ and } \nu \geqslant n, \\ t_{\mu,\nu+n} & \text{for } \mu < m \text{ and } \nu \geqslant n, \\ t_{\mu+m,\nu} & \text{for } \mu \geqslant m \text{ and } \nu < n. \end{cases}$$

If $M_{\mu\nu}(x) \equiv M_{\mu\nu}^{mn}(x)$ denotes $|(\tau_{00}x)_{\mu\nu} - (\tau_{m0}x)_{\mu\nu} - (\tau_{0n}x)_{\mu\nu} + (\tau_{mn}x)_{\mu\nu}|$ for $m,n = 0,1,2,\ldots$ and $\mu,\nu = 1,2,\ldots$, then

$$M_{\mu\nu}(x) = |t_{\mu,\nu} - t_{\mu,\nu+n} - t_{\mu+m,\nu} + t_{\mu+m,\nu+n}|$$

for $\mu \geqslant m$, $\nu \geqslant n$ and $M_{\mu\nu}(x) = 0$ for elsewhere μ and ν.

Let (a_{rs}) be a sequence of positive numbers with $a = \inf\limits_{r,s} a_{rs} > 0$.

Let $(\varphi_j)_{j=1}^{\infty}$ be a sequence of φ-function (see for instance [3, 4 or 5]) and let Ψ be a nonnegative, nondecreasing function of $u \geqslant 0$ such that $\Psi(u) \to 0$ as $u \to 0+$, $\Psi(u)$ not vanishing identically.

The sequential φ_j-modulus of the sequence $x \in X$ is defined by the formula

$$(1) \qquad \omega_{\varphi_j}(x; r, s) = \sup_{m \geqslant r} \sup_{n \geqslant s} \sum_{\mu,\nu=1}^{\infty} \varphi_j(M_{\mu\nu}(x)),$$

where r and s are nonnegative integers and φ_j is a given φ-function.

For every sequence of convex φ-functions (φ_j) and a given function Ψ we may define the sets

$$X_{\varphi_j}(\Psi) = X_j(\Psi) \equiv \{x \in X : a_{rs} \Psi(\omega_{\varphi_j}(\lambda x; r, s)) \to 0 \text{ as } r, s \to \infty \text{ for a } \lambda > 0\}$$

and pseudomodulars

$$(2) \qquad \rho_{\varphi_j}(x) \equiv \rho_j(x) = \sup_{r,s} a_{rs} \Psi(\omega_{\varphi_j}(x; r, s)).$$

Moreover, we may introduce an F-norm

$$\|x\|_{\rho_{\varphi_j}} \equiv \|x\|_j = \inf\left\{u > 0 : \rho_j\left(\frac{x}{u}\right) \leqslant u\right\}$$

and $\bar{s}$-homogeneous norm

$$\|x\|_{\rho_{\varphi_j}}^{\bar{s}} \equiv \|x\|_j^{\bar{s}} = \inf\left\{u > 0 : \rho_j\left(\frac{x}{u^{1/\bar{s}}}\right) \leqslant 1\right\}$$

2. Countably modulared spaces

By means of the sequence of pseudomodulars (2) we shall define the sequence of modular spaces (x_{ρ_j}), where

$$X_{\rho_{\varphi_j}} \equiv X_{\rho_j} = \{x \in X : \rho_j(\lambda x) \to 0 \text{ as } \lambda \to 0+\}$$

and extended real-valued functionals (which are modulars on X) $\rho_0(x) = \sup_j \rho_j(x)$, $\rho_\sigma(x) = \sup_k \frac{1}{k} \sum_{j=1}^{k} \rho_j(x)$, $\rho_s(x) = \sum_{j=1}^{\infty} \rho_j(x)$, $\rho_w(x) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\rho_j(x)}{1+\rho_j(x)}$.

In consequence, we may obtain the following countably modulared spaces X_{ρ_0}, X_{ρ_σ}, X_{ρ_s} and X_{ρ_w} (for definitions compare [6] and [7]).

Theorem 1. *Let Ψ be a function which satisfies the condition (Δ_2) for small u, and let (φ_j) be a given sequence of φ-functions which satisfy the condition*
(a) *there exist positive constants K, c, u_0 and an index j_0 such that $\varphi_j(cu) \leqslant K\varphi_{j_0}(u)$ for all $j \geqslant j_0$ and $0 \leqslant u \leqslant u_0$.*
The countably modulared spaces X_{ρ_0}, X_{ρ_σ} and X_{ρ_w} are identical.

Proof. It is obviously that $X_{\rho_0} \subset X_{\rho_\sigma} \subset X_{\rho_w}$. Let us suppose that $x \in X_{\rho_w} = \bigcap_{j=1}^{\infty} X_{\rho_j}$. Then we have $a_{rs}\Psi(\omega_{\varphi_j}(\lambda x; r, s)) \to 0$ as $\lambda \to 0+$ for each j separately and for all r and s. By assumptions and the formula (2) we obtain

$$\sup_{m \geqslant r} \sup_{n \geqslant s} \sum_{\mu,\nu=1}^{\infty} \varphi_j(\lambda M_{\mu\nu}) \to 0 \text{ as } \lambda \to 0+$$

for each j seperately and all r and s. In consequence

$$\varphi_j(\lambda M_{\mu\nu}(x)) \to 0 \text{ as } \lambda \to 0+$$

for $\mu \geqslant m \geqslant r$, $\nu \geqslant n \geqslant s$, where r and s are arbitrary and for each j seperately. In the following $\lambda M_{\mu\nu}(x) \leqslant u_0$ and by (a)

$$\varphi_j \lambda M_{\mu\nu}(x) \leqslant K\varphi_{j_0}\left(\frac{\lambda}{c} M_{\mu\nu}(x)\right)$$

for m, n, μ, ν as previously and for all $j \geqslant j_0$ and for sufficiently small $\lambda > 0$. Thus, for $j \geqslant j_0$ we have the following inequalities

$$\omega_{\varphi_j}(\lambda x: r, s) \leqslant K\omega_{\varphi_{j_0}}\left(\frac{\lambda}{c} x; r, s\right),$$

$$a_{rs}\Psi(\omega_{\varphi_j}(\lambda x: r, s)) \leqslant \bar{K} a_{rs}\Psi\left(\omega_{\varphi_{j_0}}\left(\frac{\lambda}{c} x; r, s\right)\right)$$

where $\bar{K}$ denotes a certain constant defined by the condition (Δ_2). In consequence

$$\rho_j(\lambda x) \leqslant \bar{K}\rho_{j_0}\left(\frac{\lambda}{c} x\right)$$

for all $j \geqslant j_0$ and for sufficiently small $\lambda > 0$. Thus, $x \in X_{\rho_0}$.

Under the same assumptions as in Theorem 1 we have the following theorem

Theorem 2. *Let* $x_p \in X_{\rho_w}$. *The condition* $x_p \xrightarrow{\rho_w} 0$ *implies* $x_p \xrightarrow{\rho_0} 0$.

Proof. Condition $x_p \xrightarrow{\rho_w} 0$ implies that there exists $\lambda_0 > 0$ (dependent on x_p) such that $\rho_w(\lambda_0 x_p) \to 0$ as $p \to \infty$. In consequence $\rho_j(\lambda_0 x_p) \to 0$ as $p \to \infty$ for each j separately, and in particular $\rho_{j_0}(\lambda_0 x_p) \to 0$ as $p \to \infty$. Thus, for $\lambda_0 = \dfrac{\lambda}{c}$ we find N such that

$$\text{(i)} \qquad \rho_{j_0}\left(\frac{\lambda}{c} x_p\right) < \frac{\varepsilon}{k}$$

for $p > N$, where ε, k are some positive numbers.

By assumptions we have that φ-functions φ_j are equicontinuous at 0, and that $\varphi_j(\lambda x_p) \to 0$ as $\lambda \to 0+$ for each j separately and for all p. Condition (a) implies that there exist $c > 0$ and an index j_0 such that for every $u' > 0$ there is a k' such that $\varphi_j(u) \leqslant k'\varphi_{j_0}\left(\dfrac{u}{c}\right)$ for all $0 \leqslant u \leqslant u'$ and for all $j \geqslant j_0$. Then $\lambda M_{\mu\nu}(x_p) \leqslant u'$ and $\varphi_j(\lambda M_{\mu\nu}(x_p)) \leqslant k'\varphi_{j_0}\left(\dfrac{\lambda}{c} M_{\mu\nu}(x_p)\right)$ for sufficiently small $\lambda > 0$ and for $j \geqslant j_0$, $\mu \geqslant m \geqslant r$, $\nu \geqslant n \geqslant s$, where r and s are some positive integers. Thus,

$$\text{(ii)} \qquad \rho_j(\lambda x_p) \leqslant \bar{K}\rho_{j_0}\left(\frac{\lambda}{c} x_p\right)$$

for all $j \geqslant j_0$, for all p and for sufficiently small $\lambda > 0$, where $\bar{K}$ is a constant defined by the condition (Δ_2) and (a). The inequelities (i) and (ii) lead to the condition

$$\rho_j(\lambda x_p) \leqslant \varepsilon \quad \text{for } p \geqslant N \text{ and } j \geqslant j_0.$$

If we choose N_1 in such a manner that $\rho_j(\lambda_0 x_p) < \varepsilon$ for $p \geqslant N_1$ and $j = 1, 2, \ldots, j_0 - 1$, then finally $\rho_j(\lambda_1 x_p) < \varepsilon$ for all j, $p \geqslant \max\{N, N_1\}$ and $\lambda_1 = \min\{\lambda, \lambda_0\}$. Finally, $x_p \xrightarrow{\rho_0} 0$.

3. The space $X_j(\Psi)$

Theorem 3. *Let one of the following two conditions hold:*

1° φ_j *satisfy the condition* (b)

(b) *there exists an* $\bar{\alpha} > 0$ *such that for every* $u > 0$ *and each* α *satisfying the inequality* $0 < \alpha \leqslant \bar{\alpha}$ *there holds the inequality* $\varphi_j(\alpha u) \leqslant \frac{1}{2}\varphi_j(u)$,

2° Ψ *satisfies* (Δ_2) *for small u.*

Then $X_j(\Psi)$ *is a vector space.*

Proof. For some λ, $\alpha > 0$ and for x, $y \in X$ we have

$$\omega_{\varphi_j}(\lambda\alpha x; r, s) = \sup_{m \geqslant r} \sup_{n \geqslant s} \sum_{\mu,\nu}^{\infty} \varphi_j(\lambda\alpha M_{\mu\nu}(x)) \leqslant \frac{1}{2} \sup_{m \geqslant r} \sup_{n \geqslant s} \sum_{\mu,\nu}^{\infty} \varphi_j(\lambda M_{\mu\nu}(x)) =$$
$$= \frac{1}{2}\omega_{\varphi_j}(\lambda x; r, s) \text{ and } \omega_{\varphi_j}(\lambda\alpha y; r, s) \leqslant \frac{1}{2}\omega_{\varphi_j}(\lambda y; r, s).$$

In consequence

$$a_{rs}\Psi\left(\omega_{\varphi_j}\left(\frac{1}{2}\alpha\lambda(x+y); r, s\right)\right) \leqslant a_{rs}\Psi(2\omega_{\varphi_j}(\alpha\lambda x; r, s)) + a_{rs}\Psi(2\omega_{\varphi_j}(\alpha\lambda y; r, s)) \leqslant$$

$$\leqslant a_{rs}\Psi(\omega_{\varphi_j}(\lambda x; r, s)) + a_{rs}\Psi(\omega_{\varphi_j}(\lambda y; r, s)) \to 0 \text{ as } r, s \to \infty \text{ for some } \lambda > 0.$$

Thus, $\lambda x \in X_j(\Psi)$ and $x + y \in X_j(\Psi)$.

By inequality $\varphi_j(u+v) \leqslant \varphi_j(2u) + \varphi_j(2v)$, we have
$\omega_{\varphi_j}(x+y; r, s) \leqslant \omega_{\varphi_j}(2x; r, s) + \omega_{\varphi_j}(2y; r, s)$
for every r and s. This condition and monotonicity of the function Ψ implies

$$a_{rs}\Psi\left(\omega_{\varphi_j}\left(\frac{1}{2}\lambda(x+y); r, s\right)\right) \leqslant a_{rs}\Psi(2\omega_{\varphi_j}(\lambda x; r, s)) + a_{rs}\Psi(2\omega_{\varphi_j}(\lambda y; r, s)).$$

Since x, $y \in X_j(\Psi)$ then $a_{rs}\Psi(\omega_{\varphi_j}(\lambda x; r, s)) \to 0$ and $a_{rs}\Psi(\omega_{\varphi_j}(\lambda y; r, s)) \to 0$ ar $r, s \to \infty$, where λ is some positive number. These two conditions and assumption on (a_{rs}) give that $\Psi(\omega_{\varphi_j}(\lambda x; r, s)) \to 0$ and $\Psi(\omega_{\varphi_j}(\lambda y; r, s)) \to 0$ as $r, s \to \infty$, for some $\lambda > 0$.

Now, by properties of Ψ we obtain that there exist r_0 and s_0 such that $\Psi(\omega_{\varphi_j}(\lambda x; r, s)) < \delta$ and $\Psi(\omega_{\varphi_j}(\lambda x; r, s) < \delta$ for all $r \geqslant r_0$ and $s \geqslant s_0$, where λ is some positive number. Consequently,
$\omega_{\varphi_j}(\lambda x; r, s) \leqslant M$ and $\omega_{\varphi_j}(\lambda y; r, s)) \leqslant M$
for $r \geqslant r_0$ and $s \geqslant s_0$. In the following $\Psi(2\omega_{\varphi_j}(\lambda x; r, s)) \leqslant K_1 \Psi(\omega_{\varphi_j}(\lambda x; r, s))$, $\Psi(2\omega_{\varphi_j}(\lambda y; r, s)) \leqslant K_1 \Psi(\omega_{\varphi_j}(\lambda y; r, s))$ for $r \geqslant r_0$ and $s \geqslant s_0$.

Finally

$a_{rs}\Psi\left(\omega_{\varphi_j}\left(\frac{1}{2}\lambda(x+y);r,s\right)\right) \leqslant K_1(a_{rs}\Psi(\omega_{\varphi_j}(\lambda x;r,s))+a_{rs}\Psi(\omega_{\varphi_j}(\lambda y;r,s))) \to 0$ as $r,\ s \to \infty$.

It is easy to chock that:

Theorem 4. *Let the function Ψ has property $\Psi(0) = 0$ and satisfies condition (Δ_2) for fmall u and let (φ_j) be a sequence of φ-functions which satisfy the condition (Δ_2) for all u. Then $x \in X_j(\Psi)$ if and only if*

$$\lim a_{rs}\Psi(\omega_{\varphi_j}(\lambda x;r,s)) = 0 \quad \textit{as } r,\ s \to \infty,$$

for every $\lambda > 0$.

4. Properties of some quotient spaces

In this part let us suppose that $\varphi_j = \varphi$ for all j and that one of the two conditions hold: 1° or 2°.

Let $\bar{c}$ denotes the space of sequences $(t_{\mu\nu})$ such that
$t_{\mu,1} = t_{1,\nu} = u_1$ for $\mu,\ \nu = 2, 3, \ldots$,
$t_{\mu,\nu} = u_2$ for $\mu \geqslant 2$ and $\nu \geqslant 2$,
$t_{1,1} = 2u_1 - u_2$, where u_1 and u_2 are arbitrary numbers.

It is obvious that

Theorem 5.

1) $\bar{c} = \{x \in X: \rho(x) = 0\}$,

2) *$\bar{c}$ is subspace of the space of all convergent double sequences,*

3) *if Ψ is concave and φ is $\bar{s}$-convex with some $0 < \bar{s} \leqslant 1$, then $x \in \bar{c}$ if and only if $\|x\|_\rho = 0$,*

4) *if φ is convex, then $x \in \bar{c}$ if and only if $\|x\|_\rho = 0$.*

In the following we shall consider quotient spaces

$$\tilde{X}_\rho = \tilde{X}_\rho/\bar{c} \quad \text{and } \tilde{X}(\Psi) = X(\Psi)/\bar{c}$$

with elements $\tilde{x},\ \tilde{y}, \ldots$

Moreover, let

$$\tilde{\rho}(\tilde{x}) = \inf\{\rho(y): y \in \tilde{x}\},\ \|\tilde{x}\|_\rho = \|x\|_\rho,\ \|\tilde{x}\|_\rho^{\bar{s}} = \|x\|_\rho^{\bar{s}},$$

(compare [1]).

Theorem 6. *Let φ satisfy conditions*:

(c) *for every* $\varepsilon > 0$ there exist $A > 0$ *and* $\bar{\alpha} > 0$ *such that for every* α *and* u *satisfying the inequalities* $0 < \alpha \leqslant \bar{\alpha}$, $0 < u \leqslant A$ *there holds the inequality* $\varphi(\alpha u) \leqslant \varepsilon\varphi(u)$.

(d) *for every* $\eta > 0$ *there exists an* $\varepsilon > 0$ *such that for all* $u > 0$ *the inequality* $\varphi(u) < \varepsilon$ *implies* $u < \eta$.

Let Ψ *be increasing, continuous,* $\Psi(0) = 0$ *and satisfying the condition*:

(e) *for arbitrary* $v_1 > 0$ *and* $\delta_1 > 0$ *there exists an* $\eta_1 > 0$ *such that there holds the inequality* $\Psi(\eta u) \leqslant \delta_1 \Psi(u)$ *for all* $0 \leqslant u \leqslant v_1$ *and* $0 \leqslant \eta \leqslant \eta_1$.

Moreover, let one of the following conditions hold:
Ψ *is concave or* φ *is convex.*

Then $\tilde{X}_\rho$ *is a Fréchet space with respect to the F norm* $\|\cdot\|_\rho$.

Proof. Let $x_p \in \tilde{x}_p$, $x_p = (t^p_{\mu,\nu})^{\infty}_{\mu,\nu=0}$ be such that

(3) $$t^p_{1,\nu} = t^p_{\mu,1} = 0 \quad \text{for all } \mu, \nu \text{ and } p,$$

and let $(\tilde{x}_p)$ be a Cauchy sequence in $\tilde{X}_\rho$. For every $\varepsilon > 0$ one can find an N such that $|x_p - x_q|_\rho < a\Psi(\varepsilon)$ for $p, q > N$, where a is defined by (a_{rs}). Thus, there exists u_ε such that $0 < u_\varepsilon < a\Psi(\varepsilon)$ and $a_{rs}\Psi\left(\omega_\varphi\left(\frac{x_p - x_q}{u_\varepsilon}; r, s\right)\right) \leqslant u_\varepsilon$ for $p, q > N$ and all $r, s \geqslant 1$. Hence,

$$\omega_\varphi\left(\frac{x_p - x_q}{u_\varepsilon}; r, s\right) \leqslant \Psi_{-1}\left(\frac{u_\varepsilon}{a_{rs}}\right) \leqslant \Psi_{-1}\left(\frac{u_\varepsilon}{a}\right) < \varepsilon$$

for $p, q > N$, $r, s \geqslant 1$, wher Ψ_{-1} denotes the inverse function to Ψ. Applying definition of ω_φ, we have

(4) $$\sum_{\mu=m,\nu=n}^{\mu=\bar{m},\nu=\bar{n}} \varphi\left(\frac{1}{u_\varepsilon} \cdot M_{\mu\nu}(x_p - x_q)\right) < \Psi_{-1}\left(\frac{u_\varepsilon}{a}\right) < \varepsilon$$

for $p, q > N$, $\bar{m} \geqslant \mu \geqslant m \geqslant r \geqslant 1$ and $\bar{n} \geqslant \nu \geqslant n \geqslant s \geqslant 1$.

By (d), for every $\eta > 0$ one can find an $\varepsilon > 0$ such that

(5) $$\frac{1}{u_\varepsilon} \cdot M_{\mu\nu}(x_p - x_q) < \eta$$

for $p, q > N$, $\mu \geqslant m \geqslant 1$, $\nu \geqslant n \geqslant 1$. In the following we have $|t^p_{\mu+m,\nu+n} - t^q_{\mu+m,\nu+n}| < A_1 + A_2 + A_3 + M_{\mu\nu}(x_p - x_q)$, where $A_1 = |t^p_{\mu,\nu} - t^q_{\mu,\nu}|$, $A_2 = |t^p_{\mu+m,\nu} - t^q_{\mu+m,\nu}|$, $A_3 = |t^p_{\mu,\nu+n} - t^q_{\mu,\nu+n}|$. First, let us remark that by (3),

$A_1 = A_2 = A_3 = 0$ for $r = s = 1$ and $\mu = \nu = 1$ and we see that $(t_{22}^p)_{p=1}^{\infty}$ is a Cauchy sequence. Next, by induction we obtain that $(t_{\mu\nu}^p)_{p=1}^{\infty}$ are Cauchy sequences for all μ, ν. Hence these sequences are convergent. We write $x = (t_{\mu,\nu})_{\mu,\nu=0}^{\infty}$, where $t_{\mu\nu} = 0$ for $\mu = 0$ or $\nu = 0$ and $t_{\mu\nu} = \lim_{p\to\infty} t_{\mu\nu}^p$ for $\mu, \nu = 1, 2, \ldots$. Taking $q \to \infty$ in (5) we have

$$\sum_{\mu=m,\nu=n}^{\mu=\bar{m},\nu=\bar{n}} \varphi\left(\frac{1}{u_\varepsilon} M_{\mu\nu}(x_p - x)\right) \leqslant \Psi_{-1}\left(\frac{u_\varepsilon}{a_{rs}}\right)$$

for $p > N$, $\bar{m} \geqslant m \geqslant r \geqslant 1$, $\bar{n} \geqslant n \geqslant s \geqslant 1$; and for $\bar{m}, \bar{n} \to \infty$ we obtain

$$\sum_{\mu=m,\nu=n}^{\infty} \varphi\left(\frac{1}{u_\varepsilon} M_{\mu\nu}(x_p - x)\right) \leqslant \Psi_{-1}\left(\frac{u_\varepsilon}{a_{rs}}\right)$$

for $p > N$, $m \geqslant r \geqslant 1$ and $n \geqslant s \geqslant 1$.

Consequently,

$$\omega_\varphi\left(\frac{x_p - x}{u_\varepsilon}; r, s\right) \leqslant \Psi_{-1}\left(\frac{u_\varepsilon}{a_{rs}}\right)$$

for $p > N$ and $r, s \geqslant 1$, so

$$a_{rs} \Psi\left(\omega_\varphi\left(\frac{1}{u_\varepsilon}(x_p - x); r, s\right)\right) \leqslant u_\varepsilon, \quad \text{for } p > N,\ r, s \geqslant 1. \tag{6}$$

We are going to prove that $\rho(\lambda(x_p - x)) \to 0$ as $\lambda \to 0+$ for large p. Let N be chosen as above. For $\varepsilon, \lambda > 0$ and $p > N$ we have

$$\omega_\varphi(\lambda(x_p - x); r, s) = \omega_\varphi\left(\lambda u_\varepsilon \cdot \frac{x_p - x}{u_\varepsilon}; r, s\right) = \sup_{m \geqslant r} \sup_{n \geqslant s} \sum_{\mu,\nu}^{\infty} \varphi\left(\lambda u_\varepsilon M_{\mu\nu}\left(\frac{x_p - x}{u_\varepsilon}\right)\right).$$

If we take $p \to \infty$ in (7) then

$$M_{\mu\nu}\left(\frac{x_p - x}{u_\varepsilon}\right) \leqslant \eta.$$

By (c) with $\bar{\varepsilon} = \varepsilon$, $\eta = A$, $\alpha = \lambda u_\varepsilon \leqslant \bar{\alpha}$ for $u = \frac{1}{u_\varepsilon} M_{\mu\nu}(x_p - x)$ we have

$$\varphi\left(\lambda u_\varepsilon M_{\mu\nu}\left(\frac{x_p - x}{u_\varepsilon}\right)\right) \leqslant \bar{\varepsilon}\varphi\left(\frac{1}{u_\varepsilon} M_{\mu\nu}(x_p - x)\right)$$

for $p > N$ and $\mu \geqslant m \geqslant 1$, $\nu \geqslant n \geqslant 1$. Hence

$$\omega_\varepsilon((\lambda(x_p-x);r,s)\leqslant\bar{\varepsilon}\omega_\varphi\left(\frac{x_p-x}{u_\varepsilon};r,s\right)\leqslant\bar{\varepsilon}\Psi_{-1}\left(\frac{u_\varepsilon}{a_{rs}}\right)\leqslant\bar{\varepsilon}\cdot\varepsilon.$$

Finally, for $0<\lambda<\bar{\alpha}/u_\varepsilon$ we have

$$\rho(\lambda(x_p-x))\leqslant\sup_{r,s}a_{rs}\Psi\left(\bar{\varepsilon}\Psi_{-1}\left(\frac{u_\varepsilon}{a_{rs}}\right)\right).$$

In the following, we apply the condition (e) with $v_1=\Psi_{-1}\left(\frac{u_\varepsilon}{a}\right)$, $u=\Psi_{-1}\left(\frac{u_\varepsilon}{a_{rs}}\right)$.
For $\delta_1>0$ and $\bar{\varepsilon}=\eta_1$ we have

$$\Psi\left(\bar{\varepsilon}\Psi_{-1}\left(\frac{u_\varepsilon}{a_{rs}}\right)\right)\leqslant\delta_1\Psi\left(\Psi_{-1}\left(\frac{u_\varepsilon}{a_{rs}}\right)\right)=\delta_1\frac{u_\varepsilon}{a_{rs}}.$$

Thus

$$\rho(\lambda(x_p-x)\leqslant\sup_{r,s}a_{rs}\delta_1\frac{u_\varepsilon}{a_{rs}}=\delta_1u_\varepsilon$$

for $0<\lambda u_\varepsilon\leqslant\bar{\alpha}$. Since u_ε is fixed, this implies $\rho(\lambda(x_p-x))\to0$ as $\lambda\to0+$, for $p>N$, i.e. $x_p-x\in X_\rho$ for sufficiently large p. Since X_ρ is a vector space, then $x\in X_\rho$. By (6), $\rho\left(\frac{1}{u_\varepsilon}(x_p-x)\right)\leqslant u_\varepsilon$ for $p>N$. Thus $\|x_p-x\|_\rho<u_\varepsilon<a\Psi(\varepsilon)$ for $p>N$. Finally, $\|x_p-x\|_\rho\to0$ as $p\to\infty$, and this proves the completeness of the space X_ρ.

Theorem 7. *Let functions* Ψ *and* φ *sastisfy the same assumptions as in Theorem* 2 *and* 6. *Then* $\tilde{X}(\Psi)\cap\tilde{X}_\rho$ *is a Fréchet space with respect to the F-norm* $\|\cdot\|_\rho$.

Proof. It is sufficient to remark that $\tilde{X}(\Psi)\cap\tilde{X}_\rho$ is a closed subspace of $\tilde{X}_\rho$ with respect to F-norm $\|\cdot\|_\rho$. Let $\tilde{x}_p\to\tilde{x}$ in $\tilde{X}_\rho$, $\tilde{x}_p\in\tilde{X}(\Psi)\cap\tilde{X}_\rho$, $x_p\in\tilde{x}_p$, $x\in\tilde{x}$. Then for every $\lambda>0$

$$a_{rs}\Psi(\omega_\varphi(\lambda(x_p-x);r,s))\to0\text{ as }p\to\infty$$

uniformly with respect to r and s.
Applying property of ω_φ and the condition (Δ_2) for φ with a constant $K>0$, we obtain

$$\omega_\varphi(\lambda x;r,s)\leqslant\omega_\varphi(2\lambda(x_p-x);r,s)+\omega_\varphi(2\lambda x_p;r,s)\leqslant$$
$$\leqslant K(\omega_\varphi(\lambda(x_p-x);r,s)+\omega_\varphi(\lambda x_p;r,s)).$$

Taking $\lambda > 0$ fixed, by properties of Ψ, we may find a $\bar{p}$ such that $\Psi(\omega_\varphi(\lambda(x_p - x); r, s)) < \delta$ for $p \geqslant \bar{p}$ and for all r and s, where δ is some positive constant. Hence, there exists $M > 0$ such that $\omega_\varphi(\lambda(x_p - x); r, s) \leqslant M$ for $p \geqslant \bar{p}$ and all r, s. If k is chosen such that $K \leqslant 2^k$, then by the inequality $\Psi(u+v) \leqslant \Psi(2u) + \Psi(2v)$ and condition (Δ_2) for Ψ, for small u with a constant $K_1 > 0$, we obtain $a_{rs}\Psi(\omega_\varphi(\lambda x; r, s)) \leqslant a_{rs}\Psi(2K\omega_\varphi(\lambda(x_p - x); r, s))) +$ $+ a_{rs}\Psi(2K\,\omega_\varphi(\lambda x_p; r, s)) \leqslant K_1^{k+1} a_{rs}(\Psi(\omega_\varphi(\lambda(x_p - x); r, s)) + \Psi(\omega_\varphi(\lambda x_p; r, s)))$ for $p \geqslant \bar{p}$ and all r, s. Let us fix $\varepsilon > 0$. There is an index $p_0 > \bar{p}$ such that $a_{rs}\Psi(\omega_\varphi(\lambda(x_{p_0} - x); r, s)) < \frac{1}{2}\varepsilon K_1^{-(k+1)}$.

But $x_{p_0} \in X(\Psi)$ and so by Theorem 1, we obtain

$$a_{rs}\Psi(\omega_\varphi(\lambda x_{p_0}; r, s)) \to 0 \text{ as } r, s \to \infty.$$

Thus, there exist r_0 and s_0 such that

$$a_{rs}\Psi(\omega_\varphi(\lambda x_{p_0}; r, s)) < \frac{1}{2}\varepsilon K_1^{-(k+1)}$$

for all $r \geqslant r_0$ and $s \geqslant s_0$. Finally

$$a_{rs}\Psi(\omega_\varphi(\lambda x; r, s)) \leqslant K_1^{k+1}\left(\frac{1}{2}\varepsilon K_1^{-(k+1)} + \frac{1}{2}\varepsilon K_1^{-(k+1)}\right) = \varepsilon$$

for all $r \geqslant r_0$ and $s \geqslant s_0$, and this shows that $x \in X(\Psi)$. Since, by Theorem 6, $x \in X_\rho$, then $x \in X(\Psi) \cap X_\rho$ and so $x \in \tilde{X}(\Psi) \cap \tilde{X}_\rho$.

We may also considered Theorem 6 and 7 with modular convergence (with respect to the modular $\tilde{\rho}(\tilde{x})$) in place of F-norm convergence.

The space $\tilde{X}_\rho$ is called $\tilde{\rho}$-complete, if any $\tilde{\rho}$-Cauchy sequence is $\tilde{\rho}$-convergent to an element $\tilde{x} \in \tilde{X}_\rho$. The sequence $(\tilde{x})_p$, $x_p \in \tilde{X}_\rho$ is said to be $\tilde{\rho}$-Cauchy, if there exists a $k > 0$ such that for every $\varepsilon > 0$ there is an N such that $\tilde{\rho}(k(\tilde{x}_p - \tilde{x}_q)) < \varepsilon$ for all p, $q > N$.

Theorem 8. *Under the assumptions of Theorem* 6 *or Theorem* 7, *the space* $\tilde{X}_\rho$ *or* $\tilde{X}(\Psi) \cap \tilde{X}_\rho$ *are* $\tilde{\rho}$*-complete, respectively.*

The proofs of these theorems are analogous to proofs of Theorem 6 and Theorem 7.

References

[1] Jędryka T., Musielak J. *Some Remarks on F-modular Spaces, Functiones et Approximatio*, 2, 1976, pp. 83–100.

[2] Krasnosel'skii M.A., Rutickii Ya.B., *Convex Functions and Orlicz Spaces*, Groningen 1961.

[3] Kufner A., John O., Fučik S., *Function Spaces*, Prague 1977.
[4] Musielak J., *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math., 1034, Springer Verlag, Berlin–Heidelberg–New York–Tokyo 1983.
[5] Musielak J., Orlicz W., *On Modular Spaces*, Studia Math., 18, 1959, pp. 49–65.
[6] Musielak J., Waszak A., *On Some Countably Modulared Spaces*, Studia Math., 38, 1970, pp. 51–57.
[7] Musielak J., Waszak A., *Some New Countably Modulared Spaces*, Commentationes Math., 15, 1971, pp. 209–215.
[8] Musielak J., Waszak A., *Sequence Spaces Generated by Moduli of Smoothnes* (in print).
[9] Nakano H., *Generalized Modular Spaces*, Studia Math., 31, 1968, pp. 439–449.
[10] Waszak A., *On Some Modular Spaces of Double Sequences I*, Commentationes Math., 30, 1991, pp. 473–482.
[11] Waszak A., *On Some Modular Spaces of Double Sequences II*, Discussiones Math., 12, 1992, pp. 109–116.

(University of Adam Mickiewicz, Faculty of Math. and Inf. Poznań, Poland)
Received on 6.5.1993.