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OSCILLATION CRITERIA AND COMPARISON THEOREMS FOR DELAY DIFFERENCE EQUATIONS

This paper is concerned with difference equations which can be considered as discrete analogues of neutral delay differential equations. We establish several oscillation criteria for these equations. The techniques that we have used include Riccati type transformations, comparison theorems, and successive approximations.

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Section 1

Discrete analogues of differential equations with deviating arguments have been investigated in a few recent studies (see for example [1, 6, 7, 8, 9, 11]). In this paper, we are concerned with a class of difference equations of the form

$$\Delta(r_n \Delta(x_n + p_n x_{n-k})) + q_n f(x_{n-\tau(n)}) = 0, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

where k is a positive integer, $0 \le p_n \le 1$ for $n \ge 0$, $\{r_n\}_0^\infty$ is a positive sequence, $\{q_n\}_0^\infty$ is a nonnegative sequence with infinitely many nonzero terms, $\tau(n) \ge -1$ is an integer, and $\{n-\tau(n)\}_0^\infty$ is an increasing sequence such that $\lim_{n\to\infty}(n-\tau(n)) = \infty$, and finally the everywhere continuous function f satisfies the sign condition $xf(x) > 0$ for $x \ne 0$. Equation (1.1) can be considered as a discrete analog of a second order neutral differential equation with delay.

By a solution of equation (1.1), we mean a sequence $\{x_n\}$ which is defined for

$$n \ge \min\{-k, \min_{i\ge 0}\{i-\tau(i)\}\}$$

and satisfies equation (1.1) for $n \geq 0$. A solution $\{x_n\}$ is said to be oscillatory if its terms are not eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory.

We shall establish several oscillation criteria for equation (1.1) when the coefficient sequences $\{r_n\}$, $\{p_n\}$, $\{q_n\}$ and the function f satisfy some appropriate conditions to be specified later. Here we mention that the sequence $\{r_n\}$ will be assumed to satisfy

$$\sum_{n=0}^{\infty} r_n^{-1} = \infty \tag{1.2}$$

except in Theorem 4.5.

Section 2

We first consider a class of linear neutral type difference equations of the form

$$\Delta(r_n \Delta(x_n + p_n x_{n-k})) + q_n x_{n-\tau(n)} = 0, \quad n \geq 0, \tag{2.1}$$

where the integer k, and the sequences $\{r_n\}$, $\{p_n\}$, $\{q_n\}$ and $\{\tau(n)\}$ satisfy the conditions (including (1.2)) imposed in Section 1.

Lemma 2.1. *Suppose* $\{x_n\}$ *is an eventually positive solution of* (2.1). *Let*

$$z_n = x_n + p_n x_{n-k}, \quad n \geq 0.$$

Then $\{\Delta z_n\}$ *is eventually positive, and*

$$\Delta(r_n \Delta z_n) + q_n(1 - p_{n-\tau(n)}) z_{n-\tau(n)} \leq 0 \tag{2.2}$$

for all large n.

Proof. From (2.1), $\Delta(r_n \Delta z_n) \leq 0$ for all large n. If $\{r_n \Delta z_n\}$ is eventually nonpositive, then since $\{q_n\}$ has infinitely many nonzero terms, $\{r_n \Delta z_n\}$ is eventually negative. But then (1.2) would lead to $\lim z_n = -\infty$, which is a contradiction.

Next, we rewirite (2.1) as

$$\Delta(r_n \Delta z_n) + q_n z_{n-\tau(n)} - q_n p_{n-\tau(n)} x_{n-\tau(n)-k} = 0, \quad n \geq 0.$$

Since $z_n \geq x_n$ for large n, hence

$$\Delta(r_n \Delta z_n)+q_n z_{n-\tau(n)}-q_n p_{n-\tau(n)} z_{n-\tau(n)-k} \leq 0$$

for all large n. But $\Delta z_n > 0$ implies $z_{n-\tau(n)} \geq z_{n-\tau(n)-k}$ and thus

$$\Delta(r_n \Delta z_n)+q_n z_{n-\tau(n)}-q_n p_{n-\tau(n)} z_{n-\tau(n)} \leq 0$$

for all large n. Q.E.D.

Lemma 2.2. *Suppose* $\{x_n\}$ *is an eventually positive solution of* (2.1). *Let*

$$w_n = \frac{r_n \Delta z_n}{z_{n-\tau(n)-1}},$$

for large n, *then*

(2.3) $$\Delta w_n \leq -q_n(1-p_{n-\tau(n)})-\frac{w_{n+1} w_n}{r_{n-\tau(n)-1}}$$

for large n.

Proof. Since for large n,

$$\Delta w_n = r_n \Delta z_n \Delta\left\{\frac{1}{z_{n-\tau(n)-1}}\right\}+\frac{1}{z_{n-\tau(n)}}\Delta(r_n \Delta z_n) \leq$$

$$\leq \frac{-r_n \Delta z_n \Delta z_{n-\tau(n)-1}}{z_{n-\tau(n)} z_{n-\tau(n)-1}} - q_n(1-p_{n-\tau(n)}) =$$

$$= -w_n \frac{\Delta z_{n-\tau(n)-1}}{z_{n-\tau(n)}} - q_n(1-p_{n-\tau(n)}).$$

But $\Delta(r_n \Delta z_n) \leq 0$ for large n, hence

(2.4) $$\frac{r_{n-\tau(n)-1} \Delta z_{n-\tau(n)-1}}{z_{n-\tau(n)}} \geq \frac{r_{n+1} \Delta z_{n+1}}{z_{n-\tau(n)}} \geq \frac{r_{n+1} \Delta z_{n+1}}{z_{n-\tau(n+1)}} = w_{n+1},$$

since $n-\tau(n) \leq n-\tau(n+1) < n+1-\tau(n+1)$ and $\{\Delta z_k\}$ is eventually positive by Lemma 2.1.
Thus

$$\Delta w_n \leq -q_n(1-p_{n-\tau(n)})-\frac{w_{n+1} w_n}{r_{n-\tau(n)-1}}$$

for all large n. Q.E.D.

For convenience, let

$$Q_n = q_n(1-p_{n-\tau(n)}) \geq 0, \text{ for all large } n.$$

Suppose (2.1) has an eventually positive solution, then by Lemma 2.2, (2.3) has a solution $\{w_n\}$ which is positove for n greater than or equal to some N. Summing (2.3) from s to n where $s \geq N$,

$$w_{n+1} - w_s \leq -\sum_{i=s}^{n} Q_i - \sum_{i=s}^{n} \frac{w_{i+1} w_i}{r_{i-\tau(i)-1}}. \tag{2.5}$$

Thus

$$\sum_{i=s}^{n} Q_i \leq \sum_{i=s}^{n} Q_i + w_{n+1} \leq w_s - \sum_{i=s}^{n} \frac{w_{i+1} w_i}{r_{i-\tau(i)-1}} \leq w_s,$$

which implies

$$\sum_{i=s}^{\infty} Q_i < \infty.$$

Hence we obtain our first oscillation cirterion for equation (2.1).

Theorem 2.1. *Assume that*

$$\sum_{i=s}^{\infty} Q_i < \infty.$$

holds, then every solution of equation (2.1) *is oscillatory.*

In other words, if (2.1) has an eventually positive solution, then the sequence φ_k^0 defined by

$$\varphi_k^0 = \sum_{i=k}^{\infty} Q_i, \quad k = 0, 1, 2, \ldots \tag{2.6}$$

exists and in view of (2.5), $\varphi_k^0 \leq w_k$ for k larger than or equal to some integer N. If we now define

$$\varphi_k^1 = \varphi_k^0 + \sum_{i=k}^{\infty} \frac{\varphi_{i+1}^0 \varphi_i^0}{r_{i-\tau(i)-1}}, \quad k = 0, 1, 2, \ldots,$$

then by (2.5) again,

$$\varphi_s^1 = \varphi_s^0 + \sum_{i=s}^{\infty} \frac{\varphi_{i+1}^0 \varphi_i^0}{r_{i-\tau(i)-1}} \leq \sum_{i=s}^{\infty} Q_i + \sum_{i=s}^{\infty} \frac{w_{i+1} w_i}{r_{i-\tau(i)-1}} \leq w_s < \infty, \quad s \geq N.$$

Inductively, if we define for $j \geq 0$,

$$\varphi_k^{j+1} = \varphi_k^0 + \sum_{i=k}^{\infty} \frac{\varphi_{i+1}^j \varphi_i^j}{r_{i-\tau(i)-1}}, \quad k = 0, 1, 2, \dots . \tag{2.7}$$

then, assume by induction that $\varphi_k^j \leq w_k$ for $k \geq N$, we have

$$\varphi_s^{j+1} = \varphi_s^0 + \sum_{i=s}^{\infty} \frac{\varphi_{i+1}^j \varphi_i^j}{r_{i-\tau(i)-1}} \leq \sum_{i=s}^{\infty} Q_i + \sum_{i=s}^{\infty} \frac{w_{i+1} w_i}{r_{i-\tau(i)-1}} \leq w_s < \infty, \quad s \geq N.$$

We have thus shown $\varphi_k^j \leq w_k$ for all large k and $j \geq 0$.

Theorem 2.2. *If* (2.1) *has an eventually positive solution, then the double sequence defined by* (2.6) *and* (2.7) *satisfies* (*i*) $\varphi_k^j < \infty$ *for every* $j \geq 0$ *and all large* k; *and* (*ii*) $\limsup_{j\to\infty} \varphi_k^j < \infty$ *for all large* k.

The above theorem is similar to an oscillation cirterion of Erbe and Zhang [5, Theorem 3.1].

As a consequence of Theorem 2.2, we have the following oscillation theorem for the equation

$$\Delta(r_n \Delta(x_n + p_n x_{n-k})) + q_n x_{n-\tau} = 0, \quad n \geq 0 \tag{2.8}$$

where the integer k, the sequences $\{r_n\}$, $\{p_n\}$ and $\{q_n\}$ satisfy the conditions as before, but $\tau(n)$ has been specialized to a fixed integer τ greater than or equal to -1.

Theorem 2.3. *Let*

$$\Gamma_n = \sum_{i=0}^{n-1} \frac{1}{r_i}, \quad n \geq 1. \tag{2.9}$$

Suppose

$$\Gamma_{n-\tau-1} \sum_{i=n}^{\infty} Q_i \geq \beta \text{ for all large } n, \tag{2.10}$$

where $\beta > 1/4$, *then every solution of* (2.8) *is oscillatory.*

Proof. In riew of (2.10) and (2.6),

$$\varphi_n^0 = \sum_{i=n}^{\infty} Q_i \geq \frac{\beta}{\Gamma_{n-\tau-1}}$$

for all large n. Hence

$$\varphi_n^1 = \varphi_n^0 + \sum_{i=n}^{\infty} \frac{\varphi_{i+1}^0 \varphi_i^0}{r_{i-\tau-1}} \geq \frac{\beta}{\Gamma_{n-\tau-1}} + \beta^2 \sum_{i=n}^{\infty} \frac{1}{r_{i-\tau-1}\Gamma_{i-\tau-1}\Gamma_{i-\tau}} \geq$$

$$\geq \frac{\beta}{\Gamma_{n-\tau-1}} + \beta^2 \sum_{i=n}^{\infty} \left\{ \frac{1}{\Gamma_{i-\tau-1}} - \frac{1}{\Gamma_{i-\tau}} \right\} = \frac{\beta+\beta^2}{\Gamma_{n-\tau-1}} = \frac{\beta_1}{\Gamma_{n-\tau-1}}$$

for large n, where $\beta_1 = \beta+\beta^2$.

Proceeding inductively, we see that for all large n,

$$\varphi_n^j \geq \frac{\beta}{\Gamma_{n-\tau-1}}, \tag{2.11}$$

where $\beta_j = \beta+\beta_{j-1}^2, j = 1, 2, \ldots$. Clearly, $\beta = \beta_0 < \beta_1 < \beta_2 < \ldots$. If β_j converges to some positive number α, then

$$\alpha = \beta_0 + \alpha^2 .$$

But there is no real positive solution for such an equation when $\beta_0 > 1/4$. Thus $\lim \beta_j = \infty$. Hence, every solution of (2.8) oscillates in view of (2.11) and Theorem 2.2. Q.E.D.

Section 3

In this section, we shall present an application of Theorem 2.3 by deriving an oscillation theorem for the following nonlinear difference equation

$$\Delta(r_n \Delta y_n) + q_n f(y_{n-\tau}) = 0, \quad n = 0, 1, 2, \ldots \tag{3.1}$$

where $\{r_n\}$, $\{q_n\}$, and f satisfy the same assumptions including (1.2) as imposed in Section 1, but $\tau(n)$ has been specialized to a fixed integer greater than or equal to -1.

Lemma 3.1. *Let $\{q_j\}_{j=0}^{\infty}$ be a sequence of nonnegative numbers with infinitely many nonzero terms and $\Sigma^{\infty} q_j < \infty$. Then*

$$\lim_{n\to\infty} \sum_{j=0}^{n} q_j \left\{ \sum_{i=j+1}^{\infty} q_i \right\}^{-1} = \infty .$$

Proof. Let

$$\rho_j = \sum_{i=j+1}^{\infty} q_i, \quad j \geq 0.$$

Then $\{\rho_j\}$ is a nonincreasing sequence and $\Delta\rho_j = -q_{j+1}$ for $j \geq 0$. Let $h(t) = \rho_j + (t-j)\Delta\rho_j$ for $j \leq t \leq j+1$. Then $h(j) = \rho_j$, $h'(t) = \Delta\rho_j$ for $j < t < j+1$ and

$$-\frac{\Delta\rho_j}{\rho_{j+1}} \geq -\frac{h'(t)}{h(t)}, \quad j < t < j+1.$$

Thus

$$\sum_{j=0}^{n} q_j \left\{ \sum_{i=j+1}^{\infty} q_i \right\}^{-1} = -\sum_{j=0}^{n} \frac{\Delta\rho_{j-1}}{\rho_j} \geq -\sum_{j=1}^{n} \int_{j-1}^{j} \frac{h'(t)}{h(t)}\,dt =$$

$$= -\int_0^n \frac{h'(t)}{h(t)}\,dt = -\ln h(n) + \ln h(0) \to \infty$$

as $n \to \infty$. Q.E.D.

The proof of the above Lemma can be found in Zhang and Cheng [12, Lemma 5] but is included here for the sake of completeness.

Theorem 3.1. *Under the assumption that* $\tau \geq -1$ *and the conditions (including* (1.2)) *on* $\{r_n\}$, $\{q_n\}$, *and* f *imposed in Section* 1, *assume further that*

$$(3.2) \qquad \Gamma_{n-\tau-1} \sum_{i+n+1}^{\infty} q_i > \frac{1}{4} \max\left\{ \limsup_{y \to \infty} \frac{y}{f(y)}, \limsup_{y \to +\infty} \frac{y}{f(y)} \right\} > 0,$$

for all large n, then every solution of (3.1) *is oscillatory (where* Γ_i *is defined by* (2.9)).

Proof. For convenience, let us define

$$\Pi_f = \max\left\{ \limsup_{y \to -\infty} \frac{y}{f(y)}, \limsup_{y \to +\infty} \frac{y}{f(y)} \right\}$$

Suppose to the contrary that $\{y_n\}$ is an eventually positive solution of (3.1). Then as in the proff of Lemma 2.1, we may show that $\{\Delta y_n\}$ is eventually positive. We have two cases to consider. First, suppose that $\{y_n\}$ is bounded, then for n greater than or equal to some integer N, there exist positive numbers α and β such that $\alpha \le y_n \le \beta$ and $\alpha \le y_{n-\tau} \le \beta$. From (3.1),

$$r_n \Delta y_n \ge \sum_{i=n}^{\infty} q_i f(y_{i-\tau}),$$

thus

$$y_{n+1} - y_N \ge \sum_{j=N}^{n} \frac{1}{r_j} \sum_{i=j}^{\infty} q_i f(y_{i-\tau}).$$

Let $\lambda = \min_{\alpha \le u \le \beta} f(u)$, then

$$\beta \ge y_{n+1} \ge \lambda \sum_{j=N}^{n} \frac{1}{r_j} \sum_{i=j}^{\infty} q_i \ge \lambda \sum_{j=N}^{n} q_j \sum_{i=N}^{j} \frac{1}{r_i} = \lambda \sum_{j=N}^{n} q_j (\Gamma_{j+1} - \Gamma_N). \tag{3.3}$$

If $\Sigma^\infty q_i = \infty$, then since Γ_j is increasing, we would arrive at a contradiction. If $\Sigma^\infty q_i < \infty$, then in view of (3.2) and Lemma 3.1,

$$\sum_{j=N}^{n} q_j \Gamma_{j+1} \ge \sum_{j=N}^{n} q_j \Gamma_{j-\tau-1} \ge \frac{1}{4} \Pi_f \sum_{j=N}^{n} q_j \left\{ \sum_{i=j+1}^{\infty} q_i \right\}^{-1} \to \infty$$

as $n \to \infty$, which is contradictory to (3.3).

Next, we consider the case that $\{y_n\}$ is unbounded. Let M be an integer such that $y_{n-\tau} > 0$ for $n \ge M$. Let integer $T > M$ be such that $n - \tau \ge M$ for $n \ge T$. We rewrite (3.1) in the form

$$\Delta(r_n \Delta y_n) + \hat{q}_n y_{n-\tau} = 0, \quad n \ge M \tag{3.4}$$

where

$$\hat{q}_n = \begin{cases} q_n(y_{n-\tau})/y_{n-\tau} & n \ge T \\ \hat{q}_T & M \le n \le T \end{cases}.$$

Since equation (3.4) has a positive solution, by Theorem 2.3, we have

$$\Gamma_{n-\tau-1} \sum_{i=n}^{\infty} \hat{q}_i \le 1/4.$$

But then for all large n,

$$\sum_{i=n}^{\infty} q_i \sum_{i=n}^{\infty} \frac{\hat{q}_i y_{i-\tau}}{f(y_{i-\tau})} \le \left\{ \sup_{i \ge n} \frac{y_{i-\tau}}{f(y_{i-\tau})} \right\} \sum_{i=n}^{\infty} \hat{q}_i \le \left\{ \sup_{z \ge y_{n-\tau}} \frac{z}{f(z)} \right\} \sum_{i=n}^{\infty} \hat{q}_i.$$

Hence,

$$\Gamma_{n-\tau-1}\sum_{i=n+1}^{\infty} q_i \le \Gamma_{n-\tau-1}\sum_{i=n}^{\infty} q_i \le \left\{\sup_{z\ge y_{n-\tau}} \frac{z}{f(z)}\right\}\Gamma_{n-\tau-1}\sum_{i=n}^{\infty}\hat{q}_i \le \Pi_f/4,$$

which contradicts conditon (3.2). The proof is complete.

We remark that if in Theorem 3.1, we assume that

$$\liminf_{n\to\infty} \Gamma_{n-\tau-1}\sum_{i=n+1}^{\infty} q_i > 0, \tag{3.5}$$

instead of condition (3.2), then the first part of the proof of Theorem 3.1 shows that any nonoscillatory solution of (3.1) must be unbounded.

Section 4

In this section and in the next, we shall be concerned with comparison theorems for the following delay-type difference equation

$$\Delta(r_n \Delta x_n) + q_n x_{n-m+1} = 0, \quad n \ge 0$$

where the sequences $\{q_n\}$ satisfies the assumptions imposed in Section 1, m is a nonnegative integer, and $\{r_n\}$ is a positive sequence.

We begin by considering difference equations of the form

$$\Delta(\rho_{k-1}\Delta x_{k-1}) + s_k x_{k-m} = 0, \qquad k = 0,1,2,\dots,n. \tag{4.1}$$

where m is a nonnegative integer, $\{s_k\}_1^n$ is a sequence of real numbers, $\rho_k > 0$ for $\sigma \le k \le n$, and $\sigma = \min\{0, 1-m\}$. A solution of (4.1) is a real sequence $\{x_k\}_\sigma^{n+1}$ which satisfies (4.1). It is clear that a solution of (4.1) is determined uniquely by the initial values $x_\sigma, x_{\sigma+1}, \dots, x_1$.

Lemma 4.1. *If the sequence* $\{x_k\}_\sigma^{n+1}$ *is a positive* (*or negative*) *solution of* (4.1), *then the sequence* $\{z_k\}_\sigma^n$ *defined by*

$$z_k = \frac{-\rho_k \Delta x_k}{x_k}, \quad \sigma \le k \le n \tag{4.2}$$

satisfies $z_k < \rho_k$ for $\sigma \le k \le n$ *and*

$$\Delta z_k = \frac{z_k^2}{\rho_k - z_k} + \tag{4.3}$$

$$+ s_{k+1} \left\{ \frac{\rho_{k-m+1}}{\rho_{k-m+1} - z_{k-m+1}} \right\} \left\{ \frac{\rho_{k-m+2}}{\rho_{k-m+2} - z_{k-m+2}} \right\} \cdots \left\{ \frac{\rho_k}{\rho_k - z_k} \right\}$$

for $0 \le k \le n-1$.

Proff. If $\{x_k\}_\sigma^{n+1}$ is a positive solution of (4.1), then

$$z_k = -\rho_k \left\{ \frac{x_{k+1}}{x_k} - 1 \right\}, \quad \sigma \le k \le n$$

so that

$$1 - \frac{z_k}{\rho_k} = \frac{x_{k+1}}{x_k} > 0, \quad \sigma \le k \le n.$$

Furthermore,

$$\Delta z_k = -\left\{ \frac{1}{x_{k+1}} \right\} \Delta(\rho_k \Delta x_k) + \frac{\rho_k (\Delta x_k)^2}{x_k x_{k+1}} = \frac{s_{k+1} x_{k-m+1}}{x_{k+1}} + \frac{z_k^2}{\rho_k - z_k} =$$

$$= s_{k+1} \left\{ \frac{x_{k-m+1}}{x_{k-m+2}} \right\} \left\{ \frac{x_{k-m+2}}{x_{k-m+3}} \right\} \cdots \left\{ \frac{x_k}{x_{k+1}} \right\} + \frac{z_k^2}{\rho_k - z_k} =$$

$$= s_{k+1} \left\{ \frac{\rho_{k-m+1}}{\rho_{k-m+1} - z_{k-m+1}} \right\} \cdots \left\{ \frac{\rho_k}{\rho_k - z_k} \right\} + \frac{z_k^2}{\rho_k - z_k}$$

for $0 \le k \le n-1$ as required. Q.E.D.

Note that in the above Lemma, we have assumed and will assume the convention that empty product equals one. Also, a solution of (4.3) will be a sequence $\{z_k\}_\sigma^n$ which satisfies (4.3). It is clear that a solution of (4.3) is determined uniquely by the initial values $z_\sigma, \ldots, z_0$.

Lemma 4.2. *If* $\{z_k\}_\sigma^n$ *is a solution of* (4.3) *and satisfies* $z_k < \rho_k$ *for* $\sigma \le k \le n$, *then the sequence* $\{x_k\}_\sigma^{n+1}$ *defined by* $x_\sigma = \alpha$ *and*

$$x_{k+1} = \left\{ 1 - \frac{z_k}{\rho_k} \right\} x_k, \quad \sigma \le k \le n$$

is a solution of (4.1).

The proof is a straight-forward verification and is thus omitted.

Lemma 4.3. *If* $|w| \leq z < p$, *then*

$$\left|\frac{w^2}{p-w}\right| \leq \frac{z^2}{p-z}.$$

Proof. Note that the derivative of the function $f(x) = x^2/(p-x)$ is $(2px-x^2)/(p-x)^2$ which is nonnegative for $0 \leq x < p$. Thus

$$\left|\frac{w^2}{p-w}\right| \leq \frac{w^2}{p-|w|} = f(|w|) \leq f(z) = \frac{z^2}{p-z}$$

as required. Q.E. D.

Note that equation (4.3) is equivalent to

$$z_{k+1} = z_0 + \sum_{i=0}^{k} \frac{z_i^2}{\rho_i - z_i} + \sum_{i=0}^{k} s_{i+1} \left\{\frac{\rho_{i-m+1}}{\rho_{i-m+1} - z_{i-m+1}}\right\} \cdots \left\{\frac{\rho_i}{\rho_i - z_i}\right\}, \quad 0 \leq k \leq n-1. \tag{4.4.}$$

Consider now another equation of the form

$$\Delta w_k = \frac{w_k^2}{\rho_k - w_k} + S_{k+1} \left\{\frac{\rho_{k-m+1}}{\rho_{k-m+1} - w_{k-m+1}}\right\} \cdots \left\{\frac{\rho_k}{\rho_k - w_k}\right\}, \quad 0 \leq k \leq n-1, \tag{4.5}$$

where $S_k \leq s_k$ *for* $1 \leq k \leq n$.

Theorem 4.1. *Suppose* (4.3) *has a solution* $\{z_k\}_\sigma^n$ *which satisfies* $z_k < \rho_k$ *for* $\sigma \leq k \leq n$. *Suppose further that* $w_i \leq z_i$ *for* $\sigma \leq i \leq -1$,

$$0 \leq w_0 < z_0 \text{ and } s_k \geq S_k \geq 0 \text{ for } 1 \leq k \leq n. \tag{4.6}$$

Then the solution $\{w_k\}_\sigma^n$ *of* (4.5) *determined by* $w_\sigma, \ldots, w_0$ *is well defined and satisfies*

$$0 \leq w_k < z_k \text{ for } 1 \leq k < n. \tag{4.7}$$

Proof. It is clear from (4.5) that

$$0 \leq w_1 = w_0 + \frac{w_0^2}{\rho_0 - w_0} + S_1 \left\{ \frac{\rho_{-m+1}}{\rho_{-m+1} - w_{-m+1}} \right\} \cdots \left\{ \frac{\rho_0}{\rho_0 - w_0} \right\} <$$

$$< z_0 + \frac{z_0^2}{\rho_0 - z_0} + s_1 \left\{ \frac{\rho_{-m+1}}{\rho_{-m+1} - z_{-m+1}} \right\} \cdots \left\{ \frac{\rho_0}{\rho_0 - z_0} \right\} = z_1 .$$

We now assume by induction that $w_k < z_k$ for $1 \leq k \leq j$. Then

$$0 \leq w_{j+1} = w_0 + \sum_{i=0}^{j} \frac{w_i^2}{\rho_i - w_i} + \sum_{i=0}^{j} S_{i+1} \left\{ \frac{\rho_{i-m+1}}{\rho_{i-m+1} - w_{i-m+1}} \right\} \cdots \left\{ \frac{\rho_i}{\rho_i - w_i} \right\} <$$

$$< z_0 + \sum_{i=0}^{j} \frac{z_i^2}{\rho_i - z_i} + \sum_{i=0}^{j} s_{i+1} \left\{ \frac{\rho_{i-m+1}}{\rho_{i-m+1} - z_{i-m+1}} \right\} \cdots \left\{ \frac{\rho_i}{\rho_i - z_i} \right\} = z_{j+1}$$

as required. Q.E.D.

We remark that there are several possible variants of Theorem 4.1. For instance, the Theorem still holds verbatim et literatim if the inequality signs in (4.6) and (4.7) are replaced by $\leq$. We may also change the condition (4.6) to

$$|w_0| < z_0 \text{ and } s_k \geq |S_k| \text{ for } 1 \leq k \leq n,$$

provided that the conclusion (4.7) is changed to

$$|w_k| < z_k \text{ for } 1 \leq k \leq n.$$

Similar remarks can be made on later theorems.

As a consequence of Theorem 4.1, we have the following comparsion theorem.

Theorem 4.2. *Suppose equation* (4.1) *has a positive solution* $\{x_k\}_\sigma^{n+1}$. *Suppose* $0 \leq S_k \leq s_k$ *for* $1 \leq k \leq n$ *and suppose* $\{y_k\}_\sigma^{n+1}$ *is a solution of*

$$\Delta(\rho_{k-1} \Delta y_{k-1}) + S_k y_{k-m} = 0, \quad 1 \leq k \leq n \tag{4.8}$$

determined by the conditions $y_\sigma, y_{\sigma+1}, \ldots, y_1$ *such that* $y_i \neq 0$ *for* $\sigma \leq i \leq 0$ *and*

$$-\Delta y_i / y_i \leq -\Delta x_i / x_i \quad \text{for } \sigma \leq i \leq -1; \; 0 \leq \Delta y_0 / y_0 < -\Delta x_0 / x_0. \tag{4.9}$$

Then

$$0 \leq -\Delta y_i / y_i < -\Delta x_i / x_i, \quad 1 \leq i \leq n \tag{4.10}$$

and $\{y_k\}_1^{n+1}$ *is either a positive or a negative sequence. The same theorem holds verbatim et literatim if the inequality signs in* (4.9) *and* (4.10) *are replaced by* $\leq$.

Proof. Let $\{z_k\}_\sigma^n$ be defined by (4.2), then $\{z_k\}_\sigma^n$ is a solution of (4.3) and satisfies $z_k < \rho_k$ for $\sigma \leq k \leq n$. Let $w_i = -\rho_i \Delta y_i / y_i$ for $\sigma \leq i \leq 0$. Then $w_i \leq z_i$ for $\sigma \leq i \leq -1$ and $0 \leq w_0 < z_0$. Thus by Theorem 4.1, the solution $\{w_k\}_\sigma^n$ of (4.5) determined by $w_\sigma, ..., w_0$ satisfies $0 \leq w_k < z_k < \rho_k$ for $1 \leq k \leq n$. Note that the sequence $\{u_k\}_\sigma^{n+1}$ defined by $u_i = y_i$ for $\sigma \leq i \leq 1$ and

$$u_{k+1} = \left\{1 - \frac{w_k}{\rho_k}\right\} u_k \quad 1 \leq k \leq n,$$

is a solution of (4.8). Since $u_i = y_i$ for $\sigma \leq i \leq 1$, by uniqueness, $u_k = y_k$ for $1 \leq k \leq n$, so that $0 \leq -\Delta y_i / y_i < -\Delta x_i / x_i$ for $1 \leq k \leq n$, and $\{y_k\}_1^{n+1}$ is either a positive or a negative sequence depending on $y_1 > 0$ or $y_1 < 0$ respectively. Q.E.D.

It is possible to compare equation (4.1) with a similar equation with a different value of delay. Consider an equation of the form

$$(4.11) \quad \Delta w_k = \frac{w_k^2}{\rho_k - w_k} + s_{k+1} \left\{\frac{\rho_{k-M+1}}{\rho_{k-M+1} - w_{k-M+1}}\right\} \cdots \left\{\frac{\rho_k}{\rho_k - w_k}\right\}, \quad 0 \leq k \leq n-1,$$

where $m \geq M \geq 0$. Let $\mu = \min\{0, 1-M\}$ which of course is greater than or equal to σ. By reasoning similar to that used in the proofs ot Theorems 4.1 and 4.2, we have the following results.

Theorem 4.3. *Suppose equation* (4.3) *has a solution* $\{z_k\}_\sigma^n$ *which satisfies* $z_k < \rho_k$ *for* $0 \leq k \leq n$. *Suppose further that*

$$w_i \leq z_i \text{ for } \mu \leq i \leq 0 \text{ and } 0 \leq w_0 < z_0.$$

Then the solution $\{w_k\}_\mu^n$ *of* (4.11) *determined by* $w_\mu, ..., w_0$ *is well defined and satisfies* $0 \leq w_k < z_k$ *for* $1 \leq k \leq n$.

Proof. It is clear from (4.11) that

$$0 \leq w_1 = w_0 + \frac{w_0^2}{\rho_0 - w_0} + s_1 \left\{\frac{\rho_{-M+1}}{\rho_{-M+1} - w_{-M+1}}\right\} \cdots \left\{\frac{\rho_0}{\rho_0 - w_0}\right\} \leq$$

$$\leq s_1 \left\{ \frac{\rho_{-m+1}}{\rho_{-m+1} - z_{-m+1}} \right\} \cdots \left\{ \frac{\rho_{-M}}{\rho_{-m} - z_{-M}} \right\} \left\{ \frac{\rho_{-M+1}}{\rho_{-M+1} - z_{-M+1}} \right\} \cdots \left\{ \frac{\rho_0}{\rho_0 - z_0} \right\} +$$

$$+ z_0 + \frac{z_0^2}{\rho_0 - z_0} = z_1 .$$

We now assume by induction that $w_k < z_k$ for $1 \leq k \leq j$. Then

$$0 \leq w_{j+1} = w_0 + \sum_{i=0}^{j} \frac{w_i^2}{\rho_i - w_i} + \sum_{i=0}^{j} s_{i+1} \left\{ \frac{\rho_{i-M+1}}{\rho_{i-M+1} - w_{i-M+1}} \right\} \cdots \left\{ \frac{\rho_i}{\rho_i - w_i} \right\} <$$

$$< z_0 + \sum_{i=0}^{j} \frac{z_i^2}{\rho_i - z_i} + \sum_{i=0}^{j} s_{i+1} \left\{ \frac{\rho_{i-m+1}}{\rho_{i-m+1} - z_{i-m+1}} \right\} \cdots \left\{ \frac{\rho_i}{\rho_i - z_i} \right\} = z_{j+1}$$

as required. Q.E.D.

Theorem 4.4. *Suppose equation* (4.1) *has a positive solution* $\{x_k\}_\sigma^{n+1}$. *Suppose* $0 \leq M \leq m$, $\mu = \min\{0, 1-M\}$ *and suppose* $\{y_k\}_\mu^{n+1}$ *is a solution of*

$$(4.12) \qquad \Delta(\rho_{k-1} \Delta y_{k-1}) + s_k y_{k-M} = 0, \quad 1 \leq k \leq n$$

determined by the initial conditions $y_\mu, y_{\mu+1}, \ldots, y_1$ *such that* $y_i \neq 0$ *for* $\mu \leq i \leq 0$ *and*

$$(4.13) \quad -\Delta y_i / y_i \leq -\Delta x_i / x_i \quad \text{for } \mu \leq i \leq -1; \ 0 \leq \Delta y_0 / y_0 < -\Delta x_0 / x_0.$$

Then

$$(4.14) \qquad 0 \leq -\Delta y_i / y_i < -\Delta x_i / x_i, \quad 1 \leq i \leq n$$

and $\{y_k\}_1^{n+1}$ *is either a positive or a negative sequence. The same theorem holds verbatim et literatim if the inequality signs in* (4.13) *and* (4.14) *are replaced by* $\leq$.

By means of Theorems 4.2 and 4.4, the following comparison theorem can easily be deduced.

Theorem 4.5. *Suppose* m *and* M *are nonnegative integers such that* $0 \leq M \leq \mathrm{m}$. *Let* $\sigma = \min\{0, 1-\mathrm{m}\}$, $\{\rho_k\}_\sigma^\infty$ *is a positive sequence*, $s_k \geq S_k \geq 0$ for $k \geq 1$. *Suppose further that*

$$(4.15) \qquad \Delta(\rho_{k-1} \Delta x_{k-1}) + s_{k-1} x_{k-m} = 0. \quad k \geq 1$$

has a positive solution $\{x\}_\sigma^T$ *such that* $\Delta x_0 < 0$. *Then* $\{x_k\}$ *is decreasing for* $1 \le k \le T-1$, *and*

$$\Delta(\rho_{k-1}\,\Delta y_{k-1}) + S_{k-1}\,y_{k-M} = 0, \quad k \ge 1 \tag{4.16}$$

has a solution $\{y_k\}$ *which is positive for* $1 \le k \le T$ *and decreasing for* $1 \le k \le T-1$.

Similarly, under the same assumptions of Theorem 4.5, if equation (4.15) has an eventually positive solution $\{x_k\}$ such that $x_k > 0$ for $k \ge N+\sigma$ and $\Delta x_N < 0$, then equation (4.16) has an eventually positive solution.

As an example, it can be calculated that

$$\{x_n\}_0^\infty = \{1/10, 1/11, 0.081727, 0.072462, \ldots, 0.006007, -0.00361, \ldots\}$$

is a solution of the equation

$$\Delta^2 x_n + \frac{1}{1100} x_n = 0, \quad n = 0, 1, 2, \ldots,$$

where $x_n > 0$ for $0 \le n \le 10$ and $\Delta x_n < 0$ for $0 \le n \le 9$. By Theorem 4.5, the equation

$$\Delta^2 y_n + Q_n y_n = 0, \quad n \ge 0$$

will have a solution $\{y_n\}$ which is also positive for $0 \le n \le 10$ and decreasing for $0 \le n \le 9$, provided that

$$0 \le Q_n \le 1/1100$$

for $n \ge 0$.

The question naturally arises whether the condition that $\Delta x_N < 0$ can be omitted from the above remark without jeopardizing the existence of an eventually positive solution of equation (4.16). The case that $M = m = 0$ is well known to be true, see for example Cheng [2]. As we shall see in the next section, we shall also show that for the case $M = m = 1$, the answer is affirmative. The general case, however, remains unsolved.

Section 5

In this section, we consider the class of delay difference equation of the form

$$\Delta(r_n\,\Delta x_n) + q_n\,x_{n-m+1} = 0, \quad n \ge 0 \tag{5.1}$$

where the sequences $\{r_n\}$ and $\{q_n\}$ satisfy the assumptions including (1.2) as imposed in Section 1, and m is a nonnegative integer. We shall focus our attention on the question raised in the last Section by considering the case that $m=1$.

Let $\{x_n\}$ be an eventually positive solution of (5.1). As seen in the proof of Lemma 2.1, we may prove that $\{\Delta x_n\}$ is eventually positive. Furthermore, letting

$$w_n = \frac{r_n \Delta x_n}{x_n} \tag{5.2}$$

for all large n, we see that

$$\frac{x_{n+1}}{x_n} = \frac{r_n + w_n}{r_n}$$

and

$$\Delta w_n = \left\{\frac{1}{x_{n+1}}\right\} \Delta(r_n \Delta x_n) + r_n \Delta x_n \Delta \left\{\frac{1}{x_n}\right\} = \tag{5.3}$$

$$= -q_n \frac{x_{n-m+1}}{x_{n+1}} - \frac{r_n(\Delta x_n)^2}{x_n x_{n+1}} =$$

$$= -q_n \left\{\frac{r_{n-m+1}}{r_{n-m+1}+w_{n-m+1}}\right\}\left\{\frac{r_{n-m+2}}{r_{n-m+2}+w_{n-m+2}}\right\} \cdots \left\{\frac{r_n}{r_n+w_n}\right\} - \frac{w_n^2}{r_n+w_n}$$

for large n. Note that we are still adopting the convention that empty product equals one.

By (5.3), $\{\Delta w_n\}$ is eventually nonpositive so that $\{w_n\}$ converges to some nonnegative number β. If $\beta > 0$ then

$$\beta - w_n + \sum_{i=n}^{\infty} \frac{w_i^2}{r_i + w_i} < 0 \tag{5.4}$$

for all large n. But this is impossible since we have assumed the validity of (1.2). Thus $\{w_n\}$ converges to zero.

We summarize these as follows.

Lemma 5.1. *Let $\{x_n\}$ be an eventually positive solution of* (5.1), *then the sequence $\{\Delta x_n\}$ is eventually positive. Furthermore, the sequence $\{w_n\}$ defined by* (5.2) *converges to zero and satisfies*

(5.5) $$\Delta w_n + \frac{w_n^2}{r_n+w_n} + q_n\left\{\frac{r_{n-m+1}}{r_{n-m+1}+w_{n-m+1}}\right\}\cdots\left\{\frac{r_n}{r_n+w_n}\right\} = 0$$

for all large n.

Lemma 5.2. *Suppose* $m = 1$. *Then equation* (5.1) *has a nonoscillatory solution if and only if there exists an eventually positive sequence* $\{w_n\}$ *which satisfies*

(5.6) $$\Delta w_n + \frac{w_n^2}{r_n+w_n} + q_n\left\{\frac{r_n}{r_n+w_n}\right\} \le 0$$

for all large n.

Proof. Necessity follows from Lemma 5.1. Now suppose (5.6) has an eventually positive solution $\{w_n\}$, then $\{\Delta w_n\}$ is eventually nonpositive so that $\{w_n\}$ converges to some nonnegative β. If $\beta > 0$, then

$$\beta - w_n + \sum_{i=n}^{\infty} \frac{w_i^2}{r_i+w_i} \le 0$$

for all large n. But this is impossible since we have assumed the validity of (1.2). Thus $\{w_n\}$ converges to zero. It follows from (5.6) that

(5.7) $$w_n > \sum_{i=n}^{\infty} \frac{w_i^2+q_i r_i}{r_i+w_i}$$

for all large n. Furthermore, if $q_j > w_j$ for some large j, then from (5.7),

$$w_n > \frac{w_j^2+q_j r_j}{r_j+w_j} > \frac{w_j(w_j+r_j)}{r_j+w_j} = w_j$$

which is a contradiction. Thus $q_n \le w_n$ for all large n. Next, note that if we define

$$f_i(y) = \frac{y^2+q_i r_i}{r_i+y} \quad \text{for all large } i,$$

then we have

(5.8) $$f_i'(y) = \frac{y^2+2r_i(y-q_i/2)}{(r_i+y)^2}.$$

Thus for $q_i \le y \le w_i$, we have $f_i' \ge 0$.

We shall seek a positive solution of equation (5.1) by means of a successive approximation argument. First, let N be so large that all our previous inequalities hold. Define a sequence of successive approximations $\{u^j\}_{j=N}^{\infty}$ as follows:

$$u_n^0 = 0, \quad n \geq N$$

$$u_n^{j+1} = (Tu^j)_n \equiv \sum_{i=n}^{\infty} f_i(u_i^j) = \sum_{i=n}^{\infty} \frac{(u_i^j)^2 + q_i r_i}{r_i + u_i^j}, \quad n \geq N. \tag{5.9}$$

Then, since $q_i \leq f_i(w_i)$ for $n \geq N$, we have

$$0 \leq q_n < u_n^1 = \sum_{i=n}^{\infty} q_i \leq (Tw)_n \leq w_n, \quad n \geq N.$$

Furthermore, we have, in view of the monotonicity of f_i, that

$$u_n^2 = (Tu^1)_n \leq (Tw)_n \leq w_n, \quad n \geq N$$

and

$$(Tu^1)_n \geq (Tq)_n = \sum_{i=n}^{\infty} \frac{q_i^2 + q_i r_i}{r_i + q_i} = \sum_{i=n}^{\infty} q_i = u_n^1.$$

Proceeding inductively, we see that

$$0 \leq q_n < u_n^1 \leq u_n^2 \leq \ldots \leq w_n, \quad n \geq N.$$

Thus u_n^j is nonnegative, bounded and nondecreasing in j for each $n \geq N$ so that we may define

$$u_n = \lim_{j \to \infty} u_n^j, \quad n \geq N.$$

Since the convergence of u^j is uniform, we may take limit on both sides of (5.9) and arrive at

$$u_n = \sum_{i=n}^{\infty} \frac{u_i^2 + q_i r_i}{r_i + u_i}, \quad n \geq N.$$

Now if we define $x_N = 1$ and

$$x_{n+1} = \left\{\frac{r_n + u_n}{r_n}\right\} x_n, \quad n \geq N$$

and x_n for $0 \leq n \leq N-1$ by induction through (5.1), then $\{x_n\}_0^{\infty}$ is an eventually positive solution of equation (5.1). Q.E.D.

As an immediate consequence, we have the following partial converse of Theorem 4.5.

Theorem 5.1. *If equation* (5.1) *has a nonoscillatory solution for* $m=0$, *then for* $m=1$, *the corresponding equation* (5.1) *has also a nonoscillatory solution.*

Proof. It is know [12] that when $m=0$, the corresponding equation (5.1) has a nonoscillatory solution if and only if the exists an eventually positive sequence $\{w_n\}$ such that

$$\Delta w_n + \frac{w_n^2}{r_n + w_n} + q_n \le 0$$

for all large n. Now the proof follows from

$$\Delta w_n + \frac{w_n^2}{r_n + w_n} + q_n \left\{\frac{r_n}{r_n + w_n}\right\} \le \Delta w_n + \frac{w_n^2}{r_n + w_n} + q_n \le 0 .$$

Q.E.D.

As another immediate consequence, we have the following comparison theorem.

Theorem 5.2. *Suppose* $q_n \ge Q_n$ *for* $n \ge 0$. *If*

$$\Delta(r_n \Delta x_n) + q_n x_n = 0, \quad n \ge 0$$

has a nonoscillatory solution, so does equation

$$\Delta(r_n \Delta x_n) + Q_n x_n = 0, \quad n \ge 0 .$$

The proof follows from

$$\Delta w_n + \frac{w_n^2}{r_n + w_n} + Q_n \left\{\frac{r_n}{r_n + w_n}\right\} \le \Delta w_n + \frac{w_n^2}{r_n + w_n} + q_n \left\{\frac{r_n}{r_n + w_n}\right\} \le 0 .$$

We remark that the comparison theorem 5.2 is weak in the sense that not all of the coefficient sequence is allowed to vary. This is due to the fact that the monotonicities of the function $u^2/(r+u)$ and $r/(r+u)$ with respect to r are not compatible. The same situation arises in the case of delay differential equations (see for example [4]).

Finally, we remark that several nonoscillation criteria for the linear equation

$$\Delta(r_n \Delta x_n) + q_n x_{n+1} = 0, \quad n \geq 0$$

can be found in the literature (see for example [3, 5, 10]). In view of Theorem 5.1, these nonoscillation criteria are also criteria for the existence of nonosocillatory solutions of equation (5.1) when $m = 1$.

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