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SOME CHARACTERIZATIONS OF DIFFERENTIABILITY OF CONVEX FUNCTIONS ON SMALL SET

Consider a real-valued locally Lipschitz function f defined on a nonempty closed convex set C (possibly int $C = \phi$), with nonempty nonsupport points $N(C)$, of a Banach space E, the relationship of differentability (subdifferntiability) between f and the Minkowski gauge by epi (f) and a characterization of Gateaux (Fréchet) differentiability of Minkowski gauge on a small set are given.

Let E be a real Banach space, C a closed convex subset of E and f a real-valued convex function defined on C. Rainwater [1] and Verona [13] generalized Asplund's theorem [14], resp., Mazur's theorem to closed convex set C which may have empty interior. They did so by substituting the set $N(C)$ of non support points of set C to the interior of C. In § 1 of the present paper, an eqivalence of the differentiability (subdifferentiability) betweeen f and the Minkowski gauge by f is established, and, in § 2, a characterization of Gateaux (Fréchet) differentiability of f is given.

Key words: Gateaux differentiability, Fréchet differentiability, real-valued locally Lipschitz function, convex function, subdifferential, convex set of Banach space.

First, recall a sequence of definitions.

Definition

a) A point $x \in C$ is called a support point of C provided there exists a nonzero $x^* \in E^*$ such that

$$\langle x^*, x\rangle = \sup\{\langle x^*, y\rangle; y \in C\}$$

The set of all points in C which are not support points is denoted by $N(C)$ [see [1] and [5] for the properties of $N(C)$].

b) If $x \in C$, we denote by C_x the cone generated by C from x, that is, $y \in C_x$ provided there exists some $t > 0$ such that $x + ty \in C$.

c) The subdifferential $\partial f(x)$ of the convex function f at the point $x \in C$ is defined to be the set of all $x^* \in E^*$ satisfying

$$\langle x^*, y - x\rangle \leqslant f(y) - f(x) \text{ for all } y \in C.$$

d) We say that f is Gateaux differentiable at $x \in N(C)$ provided $\partial f(x)$ is single-valued, equivalently, provided there is a unique $x^* \in E^*$ such that

$$\langle x^*, y - x\rangle \leqslant f(y) - f(x) \text{ for all } y \in C.$$

e) We say that f is Fréchet differentiable at $x \in N(C)$ provided that there exists $x^* \in E^*$ such that for any $\varepsilon > 0$, there is $\delta > 0$ such that both $y \in C$ and $\|x-y\| < \delta$ imply that

$$0 \leqslant f(y)-f(x)-\langle x^*, y-x\rangle \leqslant \varepsilon\|y-x\|.$$

The letter C will always denote a closed convex subset of E with $N(C) \neq \phi$, and we denote by dom(f) the essential domain of f, that is, dom$(f) = \{x \in E;\ f(x) < \infty\}$.

1. In this section, we will see a close relationship of the differentiability (subdifferentiability) between f and the Minkowski gauge by f.

Theorem 1.1. *If f is locally Lipschitzian on $N(C)$ and convex on C. Then for any $x_0 \in N(C)$ there exists Minkowski gauge P by epi(g) such that the subdifferential map ∂f exists at $x \in N(C)$ if an only if the subdifferential map ∂P exists at the point $(x-x_0,\ f(x)-r_0)$, where $g(x) = f(x-x_0)-r_0$, $r_0 = 1+f(x_0)$ and epi$(g) = \{(y,t) \in E\times R, t \geqslant g(y)\}$.*

Proof. We pair $E \times R$ with its dual $E^* \times R$ by

$$\langle (x^*, r^*),\ (x,r)\rangle = \langle x^*, x\rangle + r^*\cdot r$$

The definition of g implies that g is locally Lipschitzian and convex on the convex set $N(K)$, where $K = C-x_0$ with $0 \in N(K)$ and $g(0) = -1$. Thus, epi(g) is closed and convex and contains the origin of $E \times R$.

Suppose that the subdifferential map ∂P of P exists at $(u, g(u)) \in$ egi(g) with $u \in N(K)$, choose a subdifferential of P at $(u, g(u))$, (u^*, r^*), say, that is

$$\langle (u^*, r^*),\ (y,r)-(u, g(u)) \leqslant P(y,r)-P(u, y(u))$$

for all $(y,r) \in$ dom(P), or equivalently,

$$(*) \qquad \langle u^*, y-u\rangle + r^*(r-y(u)) \leqslant p(y,r)-1,\ (y,r) \in \mathrm{dom}(P).$$

Note that for each $y \in K$, $P(y,r) \leqslant 1$ whenever $r \geqslant g(y)$, this and $(*)$ imply $r^* \leqslant 0$ by letting r which tends to the positive intinity, and further we have $r^* < 0$. Suppose, to the contrary, that $r^* = 0$, then either $u^* \neq 0$, in which case $\langle u^*, y-u\rangle \leqslant 0$ for all $y \in K$ by taking $r = g(y)$, a contradiction to $u \in N(K)$; or $u^* = 0$, in which case we have $p(y,r) \geqslant 1$ for all $(y,r) \in$ dom(P), a contradiction to $P(0,0) = 0$. Since $P(y, g(y)) = 1$, by taking $r = g(y)$ in $(*)$ we have

$$(**) \qquad \langle u^*, y-u\rangle \leqslant -r^*(g(y)-g(u)) \text{ for all } y\in K$$

that is, $-(1/r^*)\,u^*\in\partial g(u)$. Hence the theorem is shown in one direction.

Conversely, suppose that the subdifferential map ∂g exists at $u\in N(K)$ and let $u^*\in\partial g(u)$, that is,

$$\langle u^*, y-u\rangle \leqslant g(y)-g(u) \text{ for all } y\in K$$

Taking $y=0$, we see that $\langle u^*, u\rangle - g(u)\geqslant 1$, define $s(r)=r/(\langle u^*, u\rangle - g(u))$ for all $r\in R$. By an argumentation which is close to the one given in [2, p13] we know that $s(1)$ $(u^*, -1)\in E^*\times R$ is a subdifferential to P *at* the point $(u, \mathrm{g}(u))$.

Proposition 1.2. *Suppose that f is convex on C, then f is locally Lipschitzian on $N(C)$ if and only if ∂f is locally bounded on $N(C)$.*

Proof. The necessity follows immediately from [1].

Sufficiency. Suppose that ∂P is locally bounded on $N(C)$, that is, for any $z\in N(C)$, there is a neighbourhood U of z in $N(C)$ such that

$$M = 3\sup\{\|x^*\|;\ x^*\in\partial f(x),\ x\in U\} < \infty .$$

For any x, $y\in U$ and for all $x^*\in\partial f(x)$ and $y^*\in\partial f(y)$, we have $\langle y^*, y-x\rangle \geqslant f(y)-f(x)\geqslant\langle x^*, y-x\rangle$.
This implies

$$\frac{2}{3}M\,\|y-x\| \geqslant |\langle y^*-x^*, y-x\rangle| \geqslant |f(y)-f(x)-\langle x^*, y-x\rangle| \geqslant$$

$$\geqslant |f(y)-f(x)| - |\langle x^*, y-x\rangle \geqslant f(y)-f(x)| - \frac{M}{3}\|y-x\|.$$

Thus we have

$$|f(y)-f(x)| \leqslant M\|y-x\| \text{ for all } x,\ y\in U$$

and this explains that f is locally Lipschitzian on $N(C)$.

Lemma 1.3. *Suppose that f is locally Lipschitzian on $N(C)$ and convex on C and suppose that $0\in N(C)$ with $f(0)=-1$. Then the Minkowski gauge P by $epi(f)$ is also locally Lipschitzian on $N(dom(P))$.*

Proof. It is easy to observe that for any $(x,r)\in N(\mathrm{dom}(P))$ there is $\lambda>0$ such that $\lambda\ x\in N(C)$. By positive homogeneity of P and the proof of Proposition 1.2, it suffices to show that there is a selection ϕ for ∂P on $N(\mathrm{dom}(P))$ which is locally bounded on $\{(x, f(x));\ x\in N(C)\} = \mathrm{grap}\, h_{N(C)}\ f$.

Let ψ be any selection for ∂f on $N(C)$, then ψ is locally bounded on $N(C)$ [1], since f is locally Lipschitzian. For any $x\in N(C)$, by the proof of the necessity of Theorem 1.1, $\phi\ (x,f(x))\equiv S(x,1)\cdot(\psi(x),-1)\in\partial P(x,f(x))$, where $S(x,1)=1/(\langle\psi(x),x\rangle-f(x))\leqslant 1$. Clearly, the local boundedness of ψ on $N(C)$ implies the local boundedness of ϕ on $\mathrm{graph}_{N(C)}\ f$ and which completes our proof.

Theorem 1.4. *If f is locally Lipschitzian on $N(C)$ and convex on C, then for any $x_0\in N(C)$ there exists a Minkowski gauge P by epi(g) such that f is Gateaux differentiable at $x\in N(C)$ if and only if P is Gateaux differentiable at $(x-x_0, f(x)-r_0)$, where $g(y)=f(y+x_0)-r_0$ for all $y\in C-x_0$ and $r_0=1+f(x_0)$.*

Proof. Suppose that f is Gateaux differentiable at $x\in N(C)$, equivalently, g is Gateaux differentiable at $u=x-x_0\in N(K)$ (where $K=C-x_0$), suppose that (u_j^*,r_j^*) for $j=1,2$ are two subdifferentials to P *at* the point $(u,g(u))$, by using the similar argument of Theorem 1.1, we see that $r_j^*<0$ for $j=1,2$ and that $-\ (1/r_j^*)\cdot u_j^*\in\partial g(u)$ for $j=1,2$. By the uniqueness of $\partial g(u)$, we have

$$(\Delta)-\frac{1}{r_1^*}(u_1^*,r_1^*)=-\frac{1}{r_2^*}(u_2^*,r_2^*).$$

Since $\langle(u_j^*,r_j^*),(u,g(u))\rangle=P(u,g(u))=1$ for $j=1,2$, combining this and (Δ), we obtain that $r_1^*=r_2^*$ which in turn implies $(u_1^*,r_1^*)=(u_2^*,r_2^*)$. It explains that ∂P is unique at $(u,g(u))$ and hence P is Gateaux differentiable at $(u,g(u))$.

Conversely, suppose that u_j^* for $j=1,2$ are distinct subdifferentials to g at u. Let $s_j(r)=r/(\langle u_j^*,u\rangle-g(u))$ for $j=1,2$ and $r\in R$, again by an argument simillar to [1, P13], we obtain that $s_j(1)(u_j^*,-1)\in\partial P(u,g(u))$ for $j=1,2$. It is evident that $s_1(1)(u_1^*,-1)\neq s_2(1)(u_2^*,-1)$ if $u_1^*=u_2^*$ and which established Theorem 1.4.

Theorem 1.5. *If f is convex on C and locally Lipschitzian on $N(C)$ then for any $x_0\in N(C)$ there is Minkowski gauge P by* epi(*g*) *such that f is Fréchet differentiable at $x\in N(C)$ if and olny if P is Fréchete differentiable at $(x-x_0, f(x)-r_0)$,* where *$g(x)\equiv f(x+x_0)-r_0$ and $r_0=1+f(x_0)$.*

Proof. Clearly, $N(\text{epi}(g)) \neq \phi$ (containing, say, the origin of $E \times R$) and by Lemma 1.3, P is locally Lipschitzian on $\text{dom}(P)$. By [1], the subdifferential map $\partial g(\partial P)$ exists on $N(K)$ $[N(\text{dom}(P)]$. Suppose that g is Fréchet differentiable at $u \in N(K)$, by the uniqueness of $\partial g(u)$, we can assume that $u^* = \partial g(u) \in E^*$, and there is a selection ϕ for the subdifferential map ∂g on $N(K)$ with $\psi(u) = u^*$ which is norm-to-norm continuous at u [1]. From the proof of necessity of Theorem 1.1, for each such $\phi(y)$, there exists $r(y)\left[=\dfrac{1}{\langle\phi(y), y\rangle - g(y)}\right]$ with $0 < r(y) \leqslant 1$ such that $r(y)(\phi(y), -1) \in \partial P(y, g(y))$. Thus $\psi(y) = r(y)(\phi(y), -1)$ is a selection for ∂P on $N(\text{dom(P)})$. Obviously, both $y \to u$ and $\phi(y) \to \phi(u)$ imply that $\psi(y) \to \psi(u)$, that is, there is a selection for ∂P on $N(\text{dom}(P))$ which is norm-to-norm continuous at $(u, g(u))$ and hence P is Fréchet differentiable at $(u, g(u))$ [1].

Conversely, if P is Fréchet differentiable at $(u, g(u))$, then there is a selection ψ for ∂P on $N(\text{dom}(P))$, of course, on $\{(y, g(y));\ y \in N(K)\}$, which is norm-to-norm continuous at $(u, g(u))$. Let $\psi(y, g(y)) = (y^*, s^*)$ and let $\psi(u, g(u)) = \partial P(u, g(u)) = (u^*, r^*)$. By Theorem 1.1 we have $s^* < 0$ and $r^* < 0$, for any $y \in N(K)$. By the continuity of ψ, we have $y^* \to u^*$, $s^* \to r^*$ whenever $y \to u$ in $N(K)$. Equivalently, $-\dfrac{1}{s^*}y^* \to -\dfrac{1}{r^*}u^*$ whenever $y \to u$ in $N(K)$. Note that there is a selection ψ with $\psi(\mathrm{y}) = -\dfrac{1}{s^*}y^*$ for ∂g on $N(K)$ which is norm-to-norm continuous at $u \in N(K)$. Thus, g is Fréchet differentiable at u.

Remark. Theorems 1.1 and 1.4 still hold if f is assumed to be lower semi continuous and real valued on C.

2. Now, we consider the differentiability of Minkowski gauge by a closed convex set C.

Theorem 2.1. Let P be a Minkowski gauge on E by C with $0 \in N(C)$. Suppose that P is locally Lipschitzian on $N(C)$, then for $x \in N(C)$ $[P(x) > 0]$ it is Gateaux differentiable at x if and only if for each finite dimensional subspace $F \subset E$ which contains x there holds

$$\lim_{r \to 0+} \sup_{\substack{u, v \in S(x, P) \cap U_F(x, r) \\ u \neq v}} \frac{P(x) - P\left(\dfrac{u+v}{2}\right)}{\|u - v\|} = 0 \tag{a}$$

where $U_F(x,r) = U(x,r) \cap F$, $U(x,r) = \{y \in E;\ \|x-y\| < r\}$ *and* $S(x,P)$ *denotes the level set* $\{y \in E;\ P(y) = P(x)\}$.

If $P(x) = 1$, *we shall also write the level set* S *instead of* $S(x,P)$ *for fixed* $x \in E$.

Proof. Since P is locally Lipschitzian on $N(C)$, by [1], ∂P exists on $N(\mathrm{dom}(P))$. By positive homogeneity we can assume that $P(x) = 1$.

Suppose that P is not Gateaux differentiable at x and suppose that x_j^* for $j = 1,2$, are two distinct subdifferentials to P at x, then we have $\langle x_j^*, x\rangle = p(x) = 1$ for $j = 1,2$. Let $h = \dfrac{x_1^* + x_2^*}{2}$, one can choose $z \in h_{(0)}^{-1}$ with $\|z\| = 1$ such that

$$\langle x_1^*, z\rangle = \langle -x_2^*, z\rangle = 2a > 0 \text{ for some } a > 0.$$

Note that $-C_x$ is also dense in E, since C_x is dense in E, one can also choose $y_1 \in C_x$ and $y_2 \in -C_x$ with $\|y_j\| = 1$ for $j = 1,2$ and with $\|y_1 - y_2\| \leqslant a/M$ such that

$$\langle x_1^*, y_1\rangle \geqslant a \quad \text{and } \langle x_2^*, -y_2\rangle \geqslant a,$$

where M is Lipschitz constant of P at x. Thus, for sufficiently small $t > 0$, we have

$$x + ty_1 \in C \quad \text{and } x - ty_2 \in C.$$

So we obtain that

$$P(x+ty_1) - P(x) \geqslant \langle x_1^*, ty_1\rangle \geqslant at$$

and

$$P(x-ty_2) - P(x) \geqslant \langle x_2^*, -ty_2\rangle \geqslant at.$$

For each such t, set $\alpha_t = 1/P(x+ty_1)$, $\beta_t = 1/P(x-ty_2)$ and set $u_t = \alpha_t(x+ty_1)$ and $v_t = \beta_t \cdot (x-ty_2)$. Then u_t, $v_t \in S$ and

$$|a_t - \beta_t| \leqslant |P(x+ty_1) - P(x-ty_2)|/(1+at)^2 \leqslant$$

$$\leqslant |P(x+ty_1) - P(x-ty_2)| \leqslant Mt\|y_1 + y_2\| \leqslant 2Mt,$$

$$\|u_t - v_t\| = \|(\alpha_t - \beta_t)x + (\alpha_t y_1 + \beta_t y_2)t\| \leqslant$$

$$\leqslant |\alpha_t - \beta_t|\,\|x\| + (\alpha_t + \beta_t)t \leqslant (2M\|x\| + 2)t \equiv Lt,$$

$$P\left(\frac{u_t+v_t}{2}\right)=P\left(\frac{\alpha_t+\beta_t}{2}x+\frac{1}{2}t(\alpha_t y_1+\beta_t y_2)\right)\leqslant P\left(\frac{\alpha_t+\beta_t}{2}x\right)+M\left\|\frac{t}{2}(\alpha_t y_1-\beta_t y_2)\right\|\leqslant$$

$$\leqslant\frac{\alpha_t+\beta_t}{2}+\frac{M\cdot t}{2}[|\alpha_t-\beta_t|+|\beta_t|\,\|y_1-y_2\|]\leqslant\frac{1}{1+at}+(Mt)^2+\frac{a}{2}t.$$

Thus

$$\frac{1-P\left(\frac{u_t+v_t}{2}\right)}{\|u_t-v_t\|}\geqslant\frac{1-\left(\frac{1}{1+at}+(Mt)^2+\frac{a}{2}t\right)}{L\cdot t}\to\frac{a}{2L}>0$$

whenever $t\to 0^+$. By taking $F=\operatorname{span}\{x,y_j;\text{ for } j=1,2\}$ we have

$$\lim_{r\to 0^+}\sup_{\substack{u,v\in S\cap U_F(x,r)\\ u\neq v}}\frac{P(x)-P\left(\frac{u+v}{2}\right)}{\|u-v\|}\geqslant\overline{\lim_{t\to 0^+}}\frac{1-P\left(\frac{u_t+v_t}{2}\right)}{\|u_t-v_t\|}\geqslant\frac{a}{2L}>0.$$

Conversely, suppose that P is Gateaux differentiable at x. We can assume, again as before, that $P(x)=1$. By [1] there is a selection ψ for ∂P on $N(C)$ which is norm-to-weak* continuous at x. For every finite dimensional subspace F of E which contains x and for $u,\ v\in S\cap F$ with $u\neq v$, we have

$$0\leqslant\frac{1-P\left(\frac{u+v}{2}\right)}{\|u-v\|}=\frac{P(u)-P\left(\frac{u+v}{2}\right)}{\|u-v\|}=$$

$$=\frac{P(u)-P\left(u+t\,\frac{v-u}{\|v-u\|}\right)}{2t}\left[\text{where } t=\frac{1}{2}\|v-u\|\right]\leqslant$$

$$\leqslant-\frac{1}{2}\langle\psi(u),\ \frac{v-u}{\|v-u\|}\rangle=$$

$$=-\frac{1}{2}\langle\psi(u)-\psi(v),\ \frac{v-u}{\|v-u\|}\rangle-\frac{1}{2\|v-u\|}[\langle\psi(v),v\rangle-\langle\psi(v),u\rangle]\leqslant$$

$$\leqslant\frac{1}{2}\langle\psi(u)-\psi(v),\ \frac{u-v}{\|u-v\|}\rangle\to 0$$

whenever $u,\ v\in S\cap F$, $u\neq v$ and $u,\ v\to x$, i.e.

$$\lim_{r\to 0+} \sup_{\substack{u,v\in S\cap U_F(x,r)\\ u\neq v}} \frac{P(x)-P\left(\frac{u+v}{2}\right)}{\|u-v\|}=0.$$

Theorem 2.2. With the same assumptions as in Theorem 2.1 on the set C, the function P and the point x, if P Fréchet differentiable at x, then we have

$$\lim_{r\to 0+} \sup_{\substack{u,v\in S\cap U(x,r)\\ u\neq v}} \frac{P(x)-P\left(\frac{u+v}{2}\right)}{\|u-v\|}=0 \tag{b}$$

if, in addition, $\operatorname{int} C \neq \phi$, *then P is Fréchet differentiable at x if and only if (b) holds.*

Proof. Apply Fréchet differentiability of P at x in place of Gateaux differentiability in the proof of necessity of theorem 2.1 and recall that P is Fréchet differentiable at x if and only if there is a selection for the subdifferential map ∂P which is norm-to-norm continuous at x, so the first part of this theorem is proved, immediately.

If $\operatorname{int} C=\phi$, then $\operatorname{int} C=N(C)$ [1]. Let (b) hold. Then P is Gateaux differentiable at x. Suppose that f is not Fréchet differentiable at x, then there exists $\varepsilon>0$ and sequences $\{x_n\}\subset E\cap x^{*-1}(0)$ with $\|x_n\|=1$ and $\{t_n\}\subset R^+$, $t_n\to 0$ such that

$$\frac{P(x+t_n x_n)-P(x)}{t_n}>\varepsilon \quad \text{where } x^*=\partial P(x).$$

It is easy to see that

$$P(x+t_n x_n)\geqslant 1+t_n\varepsilon \quad \text{and } P(x-t_n x_n)\geqslant 1$$

for all sufficiently large n such that $x\pm t_n x_n\in C$.

Let $\alpha_n=1/P(x+t_n x_n)$, $\beta_n=1/P(x-t_n x_n)$, $u_n=\alpha_n(x+t_n x_n)$ and $v_n=\beta_n(x-t_n x_n)$. Then u_n, $v_n\in S$. Now we have

$|\alpha_n-\beta_n|\leqslant|\alpha_n-1|+|1-\beta_n|\leqslant 2Mt_n$

(where M is locally Lipschitz constant of P at x),

$$\|u_n-v_n\|=\|(\alpha_n-\beta_n)x+(\alpha_n+\beta_n)t_n x_n\|\leqslant 2M\|x\|t_n+2\|x_n\|t_n=2(M\|x\|+1)t_n,$$

$$P\left(\frac{u_n+v_n}{2}\right)=P\left(\frac{\alpha_n+\beta_n}{2}x+\frac{1}{2}(\alpha_n-\beta_n)t_nx_n\leqslant P\left(\frac{\alpha_n+\beta_n}{2}x\right)+M\left\|\frac{1}{2}(\alpha_n-\beta_n)t_nx_n\right\|\leqslant\right.$$

$$\leqslant\frac{\alpha_n+\beta_n}{2}+M^2t_n^2\|x_n\|\leqslant\frac{1}{2}+\frac{1}{2(1+\varepsilon t_n)}+M^2t_n^2,$$

$$1-P\left(\frac{u_n+v_n}{2}\right)\geqslant\frac{1}{2}-\frac{1}{2(1+\varepsilon t_n)}-M^2t_n^2=\frac{\varepsilon\cdot t_n}{2(1+\varepsilon\cdot t_n)}-M^2t_n^2.$$

Thus

$$\lim_{r\to 0+}\sup_{\substack{u,v\in S\cap U(x,r)\\ u\neq v}}\frac{1-\mathrm{P}\left(\frac{u+v}{2}\right)}{\|u-v\|}\geqslant\overline{\lim_{n\to\infty}}\frac{1-P\left(\frac{u_n+v_n}{2}\right)}{\|u_n-v_n\|}\geqslant\frac{\varepsilon}{4(M\|x\|+1)}.$$

This is a contradiction which completes our proof.

We know that the set of Fréchet differentiability points of locally Lipschitz convex function f defined on nonempty, open convex set D is always G_δ – set of D. Rainwater [1] proved that if $N(C)\neq\phi$, then the above conclusion still holds in Asplund space. Wu and Cheng [6] pointed out if $N(C)\neq\phi$ then the set of Gateaux differentiability points is always a G_δ – set of $N(C)$ in weak Asplund space. For any Banach space E, we have

Theorem 2.3. Let f be convex C and locally Lipschitzian on $N(C)$. Then a set $G\subset N(C)$ on which f is Fréchet differentiable is contained in some G_δ – set of $N(C)$ on which f is Gateanx differentiable.

Proof. Without loss of generality we assume that $0\in N(C)$ and $f(0)=-1$. Let P be the Minkowski functional by epi(f). Since f is locally Lipschitzian on $N(C)$, by Lemma 1.3, P is locally Lipschitzian on $N(\mathrm{dom}(P))$. By Theorem 1.4 and Theorem 1.5, f is Gateaux (Fréchet) differentiable at $x\in N(C)$ if and only if P is Fréchet differentiable at $(x,f(x))$. Therefore, it suffices to show that the set $Q\subset\mathrm{graph}(f)\cap N(\mathrm{dom}(P))$ on which P is Fréchet differentiable is contained in some G_δ – set of $\mathrm{graph}(f)\cap N(\mathrm{dom}(p))$ of Gateaux differentiability points.

Let $\mathrm{graph}(f)=S$. For each intiger $n\geqslant 1$, let

$$G_n=\left\{x\in N(\mathrm{dom}(P))\cap S;u,v\in S\bigcap_{u\neq v}^{\mathrm{SUP}}U(x,r)\frac{1-P\left(\frac{u+v}{2}\right)}{\|u-v\|}<\frac{1}{n},\text{ for some }r>\mathrm{Q}\right\}$$

and let

$$G = \bigcap G_n.$$

By Theorems 2.1 and 2.2, $Q \subset G$ and f is Gateaux differentiable on G. It remains to show that G_n is open set of S for every n. For any $x \in G_n$, let $r(>0)$ satisfy

$$\sup_{\substack{u,v \in S \bigcap U(x,r) \\ u \neq v}} \frac{P(x) - P\left(\frac{u+v}{2}\right)}{\|u-v\|} < \frac{1}{n}.$$

Note that $S \cap U\left(y, \frac{r}{2}\right) \subset S \cap U(x,r)$ whenever $y \in S \cap U\left(x, \frac{r}{2}\right)$, so we have

$$\sup_{\substack{u,v \in S \bigcap U\left(y, \frac{r}{2}\right) \\ u \neq v}} \frac{P(y) - P\left(\frac{u+v}{2}\right)}{\|u-v\|} \leqslant \sup_{\substack{u,v \in S \bigcap U(x,r) \\ u \neq v}} \frac{P(x) - P\left(\frac{u+v}{2}\right)}{\|u-v\|} < \frac{1}{n}$$

whenever $y \in S \cap U\left(x, \frac{r}{2}\right)$, i.e. $S \cap U\left(x, \frac{r}{2}\right) \subset G_n$. Since $S \cap U\left(x, \frac{r}{2}\right)$, is open subset of S, this says that G is G_δ – set of S.

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