FASCICULI MATHEMATICI

Nr 25 1995

MARIA ŻAREK

SOME GENERALIZED VARIABILITY ORDERINGS AMONG LIFE DISTRIBUTIONS WITH APPLICATION TO WEIBULL'S AND GAMMA DISTRIBUTION

We consider generalized variability orderings amogn non-negative random variables (lifetimes) defined in [1]. Our results include:

1) necessary and sufficient conditions which justify the generalized variability interpretation of these domianance relations between various Weibull's and the gamma distributions,

2) applications of closure properties of the discussed ordering to the sequences of order statistics.

Key words: Partial ordering, linear ordering, life distribution, Weibull's distribution, series and parallel systems, gamma distrubution.

1. Introduction and definitions

A lifetime is one of the basic notion of the theory of reliability. Non-negative random variable is the simplest model of lifetime and it is the model we will be using.

Let X and Y denote non-negative random variables (lifetimes) with distributions F and G, respectively, and survival function $\bar{F} = 1-F$ and $\bar{G} = 1-G$, respectively. We shall define partial orderings $<^p$, $<^{(p)}$ and $<_{(p)}$ among life distributions, with a finite moment of order $p > 0$.

Definition 1.1. *We say that:*
(*i*) $F <^p G$ (*or* $X <^p Y$),
(*ii*) $F <^{(p)} G$ (*or* $X <^{(p)} Y$),
(*iii*) $F <_{(p)} G$ (*or* $X <_{(p)} Y$),

if the survival functions $\bar{F}$ *and* $\bar{G}$ *satisfy*:

(1) (*i*) $\int_t^\infty x^{p-1}[\bar{G}(x)-\bar{F}(x)]\,dx \geq 0$ *for all* $t > 0$, *with equality at* $t = 0$,

(2) (ii) $$\int_t^{\infty} x^{p-1}\bar{F}(x)\,dx \le \int_t^{\infty} x^{p-1}\bar{G}(x)\,dx < \infty \text{ for all } t \ge 0,$$

(3) (iii) $$\int_0^{t} x^{p-1}\bar{F}(x)\,dx \ge \int_0^{t} x^{p-1}\bar{G}(x)\,dx \text{ for all } t \ge 0.$$

We notice, that equality at $t = 0$ in (1) denotes equality of p^{th} moments of random variables X and Y. Now we define $<^L$, $\le^{st}$ and $<^{\mathrm{HNBUE}}$ – orderings.

Definition 1.2. *We say that:*
(i) $F <^L G$ (or $X <^L Y$),
(ii) $F \le^{st} G$ (or $X \le^{st} Y$),
if the survival functions $\bar{F}$ *and* $\bar{G}$ *satisfy:*

(4) (i) $$\int_0^{\infty} e^{-ax}[\bar{F}(x) - \bar{G}(x)]\,dx \ge 0 \text{ for all } s > 0,$$

(5) (ii) $$\bar{F}(x) \le \bar{G}(x) \quad \text{pointwise.}$$

Let F_1 and G_1 be defined as follows:

$$F_1(x) = \frac{1}{EX}\int_0^x \bar{F}(t)\,dt, \quad G_1(x) = \frac{1}{EY}\int_0^x \bar{G}(t)\,dt.$$

Definition 1.3. *We say that* $F <^{\mathrm{HNBUE}} G$ *(or* $X <^{\mathrm{HNBUE}} Y$*) if:*

(6) $$G_1^{-1}[F_1(x)] \ge \left(\frac{EY}{EX}\right)x \text{ for all } x > 0$$

where G_1^{-1} *is the usual left inverse of* G_1.

These definitions originate from [5].

Definition 1.4. *The relation* R *is called:*
(i) *reflexive, if:* xRx *for all* x,
(ii) *symmetric, if:* $xRy \Rightarrow yRx$ *for all* x *and* y,
(iii) *weekly asymmetric, if:* $(xRy \wedge yRx) \Rightarrow x = y$ *for all* x *and* y,
(iv) *transitive, if:* $(xRy \wedge yRz) \Rightarrow xRz$ *for all* x, y *and* z,

(v) *connected, if: $xRy \vee yRx$ for all x and y,*
(vi) *equivalence relation if it is reflexive, symmetric and transitive,*
(vii) *partial ordering if it is reflexive, weekly asymmetric and transitive,*
(viii) *linear ordering if it is partial ordering and connected,*
(ix) *inverse to realtion R^{-1} if $xRy \Leftrightarrow yR^{-1}x$ for all x and y.*

Our purpose in this paper is to define these orderings in the class of Weibull's and the gamma distributions. In section 3 we apply so called closure properties under formation of series and parallel systems to the sequences of positional statistics.

2. Generalized variability orderings and Weibull's distributions with the same fixed shape parameter

Let now X and Y denote non-negative random variables specified on the common probability space $(\Omega, \mathscr{F}, P)$ with the survival functions $\bar{F}(x) = \exp(-\lambda_F x^\alpha)$, $\bar{G}(x) = \exp(-\lambda_G x^\alpha)$, (for all $x \geq 0$), respectively. It is easy to notice that these variables have the Weibull's distributions with the same fixed shape parameter $\alpha > 0$ and with the scale parameters $\lambda_F > 0$ and $\lambda_G > 0$, respectively.

Theorem 2.1. *The ordering $X <^p Y$ holds if and only if $\lambda_F = \lambda_G$.*

Proof. Consider the p^{th} moments of X a and Y:

$$EX^p = p \int_t^\infty x^{p-1}\bar{F}(x)\,dx = p\int_t^\infty x^{p-1}\exp(-\lambda_F x^\alpha)\,dx = p\alpha^{-1}\lambda_F^{-p/\alpha}\Gamma(p/\alpha).$$

Similarly $EY^p = p\alpha^{-1}\lambda_G^{-p/\alpha}\Gamma(p/\alpha)$. Then $EX^p = EY^p$ if and only if $\lambda_F = \lambda_G$.

Consider now the condition (1). Let $t > 0$ denote any fixed number. Then

$$\int_t^\infty x^{p-1}[\bar{G}(x) - \bar{F}(x)]\,dx = \int_t^\infty x^{p-1}[\exp(-\lambda_G x^\alpha) - \exp(-\lambda_F x^\alpha)]\,dx. \tag{7}$$

We introduce the notation

$$h(x) = \exp(-\lambda_G x^\alpha) - \exp(-\lambda_F x^\alpha). \tag{8}$$

It is easy to remark that $h(0)$ and $\lim_{x\to\infty} h(x) = 0$. If $\lambda_F = \lambda_G$, then $h(x) \equiv 0$. It follows from the properties of exponential function:

– if $\lambda_F < \lambda_G$, then $h(x)$ assumes only negative values for all $x > 0$,

– if $\lambda_F > \lambda_G$, then $h(x)$ assumes only positive values for all $x > 0$.

Let us return to (7). W know that $x^{p-1} > 0$ for all $x > 0$. Therefore $h(x) \geq 0$ is a necessary and sufficient condition that the inequality

$$\int_t^{\infty} x^{p-1}[\exp(-\lambda_G x^{\alpha}) - \exp(-\lambda_F x^{\alpha})]\, dx \geq 0$$

holds, and it is true if $\lambda_F \geq \lambda_G$. Therefore the equality of p^{th} moments and condition (1) is true if and only if $\lambda_F = \lambda_G$. The proof is complete.

It is possible to verify that $<^p$–relation is reflexive, symmetric and transitive. Therefore this relation is an equivalence relation for the family of random variables, which have the Weibull's distrubutions. The ordering $<^p$ divides this family into equivalence classes like this: two random variables belong to the same class if and only if they have an identical scale parameter. Furthermore, this relation is weekly asymmetric and so it is a partial ordering.

Theorem 2.2. *The ordering* $X <^{(p)} Y$ *holds if and only if* $\lambda_F \geq \lambda_G$.

Theorem 2.3. *The ordering* $X <_{(p)} Y$ *holds if and only if* $\lambda_F \leq \lambda_G$.

Proof. As before, it suffices to consider the function $h(x)$.

The orderings $<^{(p)}$ and $<_{(p)}$ are reflexive, weakly asymmetric, transitive and connected, and so they are linear orderings. We also notice that $<^{(p)}$–relation sets in order Weibull's distributions in accordance with non-increasing scale parameters, and $<_{(p)}$–relation in accordance with no-non-decreasing scale paremeters. Furthermore these two relations are inverse because

$$X <^{(p)} Y \Leftrightarrow \lambda_F \geq \lambda_G \Leftrightarrow Y <_{(p)} X.$$

Theorem 2.4. *The ordering* $X <^{L} Y$ *holds if and only if* $\lambda_F \leq \lambda_G$.

Proof. Consider the condition (4). Let $s > 0$ denote any fixed real; then

$$(9)\qquad \int_0^\infty e^{-sx}[\bar{F}(x)-\bar{G}(x)]\,dx = \int_0^\infty e^{-sx}[\exp(-\lambda_F x^\alpha)-\exp(-\lambda_G x^\alpha)]\,dx = $$

$$= \int_0^\infty e^{-sx} h(x)\,dx,$$

where $h(x)$ is defined as in (8). We know that $e^{-sx} > 0$ for all $x \geq 0$. From the properties of $h(x)$ it follows that the integral (9) is non-negative if only if $\lambda_F \leq \lambda_G$. The proof is complete.

The ordering $<^L$ is a linear ordering. It sets in order Weibull's distributions in accordance with non-decreasing scale parameters as it is in the case of $<_{(p)}$. Therefore $<_{(p)}$ and $<^L$–relations are mutually equivalent and $<^{(p)}$ and $<^L$ are inverse relations.

Theorem 2.5. The ordering $X \leq^{st} Y$ holds if and only if $\lambda_F \geq \lambda_G$.

Proof. Let us consider condition (5). Let $x \geq 0$ denote any fixed number. Then

$$(10)\qquad \bar{F}(x) \leq \bar{G}(x) \Leftrightarrow \exp(-\lambda_F x^\alpha) \leq \exp(-\lambda_G x^\alpha) \Leftrightarrow h(x) \geq 0,$$

where $h(x)$ is defined as in (8). The equivalency (10) holds if and only if $\lambda_F \geq \lambda_G$. The proof is complete.

The relation $\leq^{st}$ is a linear ordering. It sets in order Weibull's distributions in accordance with non-increasing scale parameters. The relations $<_{(p)}$ and $\leq^{st}$ are mutually equivalent. The orderings $\leq^{st}$ and $<_{(p)}$, $\leq^{st}$ and $<^L$ are inverse relations.

Theorem 2.6. The ordering $X <^{\mathrm{HNBUE}} Y$ holds for all λ_F and λ_G.

Proof. At the beginning let us count the values of EX, EY and functions $F_1(x)$, $G_1(x)$. From the formula for the p^{th} moment of Weibull's distribution we get $EX = \lambda_F^{-1/\alpha}\Gamma\left(1+\frac{1}{\alpha}\right)$, $EY = \lambda_G^{-1/\alpha}\Gamma\left(1+\frac{1}{\alpha}\right)$. Let $x > 0$ denote any

fixed number. Then the right side of inequality (6) takes the form $\left(\frac{\lambda_F}{\lambda_G}\right)^{1/\alpha} x$.
We can see that

$$F_1(x) = \lambda_F^{1/\alpha}\left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^{-1}\int_0^x \exp(-\lambda_F t^\alpha)\,dt = \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^{-1} A(\lambda_F^{1/\alpha}x),$$

where

$$A(x) = \int_0^x \exp(-u^\alpha)\,du. \tag{11}$$

Similarly

$$G_1(x) = \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right] A(\lambda_G^{1/\alpha}x).$$

The function $A(x)$ is a increasing function, because the integrand in (11) is positive for all $u > 0$. Therefore there exists an inverse function $A^{-1}(x)$ and $G_1^{-1}(x) = \lambda_G^{-1/\alpha}A^{-1}\left[\Gamma\left(1+\frac{1}{\alpha}\right)x\right]$. Then the left side of the inequality (6) takes the form $\left(\frac{\lambda_F}{\lambda_G}\right)^{1/\alpha} x$. It is easy to notice that the condition (6) holds with equality for any scale parameters λ_F and λ_G. The proof is complete.

This relation is reflexive, symmetric and transitive and so it is an equivalence relation, but there exist only one equivalence class. All Weibull's distributions with the same fixed shape parameter $\alpha > 0$ belong to this class.

3. Sequences of the minimum and maximum order statistics for Weibull's distributions

Let X_i (Y_i) with life distribution F_i (G_i respectively), $i = 1,...,n$, be independent and let $X_{(1)n} = \min(X_1,...,X_n)$, $X_{(n)n} = \max(X_1,...,X_n)$ be the respective times to failure of series and parallel systems constituted of these components; $Y_{(1)n}$ and $Y_{(n)n}$ are defined similarly.

Theorem 3.1 [1]. *The ordering* $<_{(p)}$ $(<^{(p)})$ *is closed under formation of series (parallel) systems; moreover, we have* (*i*) $X_i <_{(p)} X_i,\ i = 1,...,n \Rightarrow X_{(1)n} <_{(r)} Y_{(1)n}$ *for all* $0 < r \leq p$,

(*ii*) $X_i <^{(p)} Y_i,\ i = 1,...,n \Rightarrow X_{(n)n} <^{(r)} Y_{(n)n}$ *for all* $r \geq p$.

Let now (X_n) and (Y_n), $n = 1, 2, ...$, denote two sequences of non-negative independent random variables specified on the same probability space $(\Omega, \mathscr{F}, P)$. Let X_n (Y_n), $n = 1, 2, ...$, have the Weibull's distribution with the shape parameter $\alpha > 0$ and the scale parameter $\lambda_F > 0$ (λ_G respectively). For (X_n) and (Y_n) we form sequences of minimum order statistics $X_{(1)1}, X_{(1)2}, ..., X_{(1)n}, ...$ and $Y_{(1)1}, Y_{(1)2}, ..., Y_{(1)n}, ...$ The distribution of $X_{(1)n}$ $(Y_{(1)n})$ has the form
$F_n(x) = 1 - \exp(-n\lambda_F x^\alpha)$ for $x \geq 0$, $\quad (G_n(x) = 1 - \exp(-n\lambda_G x^\alpha)$ for $x \geq 0$ respectively).

Theorem 3.2. *If sequences* (X_n), (Y_n), $(X_{(1)n})$, $(Y_{(1)n})$ *are defined as above, then* $(X_{(1)n})$ *and* $(Y_{(1)n})$ *are convergent in distribution, i.e. there exists distribution* $F(x)$ *such that:*

$$\lim_{n \to \infty} F_n(x) = \lim_{n \to \infty} G_n(x) = F(x)$$

for every point of continuity of distribution $F(x)$.

It is easy to verify that $(X_{(1)n})$ and $(Y_{(1)n})$ are convergent to the random variables X and Y with the same distribution

$$F(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } x > 0. \end{cases}$$

Theorem 3.3. *If* $X_i <_{(p)} Y_i$, $i = 1, 2, ...$, *then limit random variables* X *and* Y *of sequences* $(X_{(1)n})$ *and* $(Y_{(1)n})$ *are* $<_{(r)}$ – *ordered for all* $r > 0$.

Similar theorems cannot be formulated for sequences of maximum order statistics, because these sequences are not convergent in distribution.

4. Generalized variability orderings and the gamma distributions with the same fixed shape parameter

Let now X and Y denote non-negative random variables specified on the same probability space $(\Omega, \mathscr{F}, P)$ with the gamma distributions. Let X and Y have the same fixed shape parameter $\alpha > 0$ and the scale parameters $\lambda_F > 0$ and $\lambda_G > 0$, respectively. Furthermore, let $\alpha \in \mathbb{N}$, i.e. X and Y have Erlang's distributions with distribution functions respectively

$$F(x) = \sum_{i=\alpha}^{\infty} \frac{(\lambda_F x)^i}{i!} \exp(-\lambda_F x), \quad G(x) = \sum_{i=\alpha}^{\infty} \frac{(\lambda_G x)^i}{i!} \exp(-\lambda_G x) \quad \text{for } x > 0.$$

Densities of probability of these random variables have the forms respectively

$$f(x) = \frac{\lambda_F^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\lambda_F x), \quad g(x) = \frac{\lambda_G^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\lambda_G x) \text{ for } x > 0.$$

We check if it is possible to define variability orderings in this class.

Theorem 4.1. *The ordering* $X <^p Y$ *holds if and only if* $\lambda_F = \lambda_G$.

Proof. From formulas $\bar{F}(x) = 1 - F(x)$, $\bar{G}(x) = 1 - G(x)$ and from expansion of the exponential function into power series we obtain

$$\bar{F}(x) = \sum_{i=0}^{\alpha-1} \frac{(\lambda_F x)^i}{i!} \exp(-\lambda_F x) \text{ and } \bar{G}(x) = \sum_{i=0}^{\alpha-1} \frac{(\lambda_G x)^i}{i!} \exp(-\lambda_G x) \text{ for } x > 0.$$

Let $t > 0$ denote any fixed number. Then

$$\int_t^\infty x^{p-1} \bar{F}(x)\, dx = \int_t^\infty x^{p-1} \left[\sum_{i=0}^{\alpha-1} \frac{(\lambda_F x)^i}{i!} \right] \exp(-\lambda_F x)\, dx =$$

$$= \int_t^\infty \left[\sum_{i=0}^{\alpha-1} \frac{\lambda_F^i}{i!} x^{i+p-1} \exp(-\lambda_F x) \right] dx.$$

It is easy to verify that, for all $i = 0, 1, \dots, \alpha - 1$, the function

$$\frac{\lambda_F^i}{i!} x^{i+p-1} \exp(-\lambda_F x)$$

is integrable in each interval $\langle t, \infty) \subseteq \langle 0, \infty)$. From propety 4, p. 515 in [3] we can change the order of summation and integration. We can see that

$$\int_t^{\infty} x^{p-1}\bar{F}(x)\,dx = \frac{1}{\lambda_F^p}\sum_{i=0}^{\alpha-1}\frac{1}{i!}\int_{\lambda_F t}^{\infty} u^{i+p-1}e^{-u}du.$$

Similarly,

$$\int_t^{\infty} x^{p-1}\bar{G}(x)\,dx = \frac{1}{\lambda_G^p}\sum_{i=0}^{\alpha-1}\frac{1}{i!}\int_{\lambda_G t}^{\infty} u^{i+p-1}e^{-u}du.$$

Let now $t = 0$. We make use of the definition of the gamma function (formula (6), p. 645 in [3]) in the above formulas. We obtain

$$\int_0^{\infty} x^{p-1}\bar{F}(x)\,dx = \frac{1}{\lambda_F^p}\sum_{i=0}^{\alpha-1}\frac{1}{i!}\Gamma(i+p)$$

and $\int_0^{\infty} x^{p-1}\bar{G}(x)\,dx = \frac{1}{\lambda_G^p}\sum_{i=0}^{\alpha-1}\frac{1}{i!}\Gamma(i+p).$

It is easy to notice that the condition (1) holds if and only if $\lambda_F \geq \lambda_G$ and the equality for $t=0$ – if and only if $\lambda_F = \lambda_G$. The proof is complete.

This relation is an equivalence relation for the family of random variables, which have the Erlang's distributions with the same fixed shape parameter and has the same properties as $<^p$–ordering in section 2.

We can prove similarly:

Theorem 4.2. *The ordering* $X <^{(p)} Y$ *holds if and only if* $\lambda_F \geq \lambda_G$.

This relation has the same properties as $<^{(p)}$–ordering in section 2.

Let us disregard the assumption that $\alpha \in I\!N$, i.e. let us consider a general case of gamma distribution.

Theorem 4.3. *The ordering* $X <^L Y$ *holds if and only if* $\lambda_F \leq \lambda_G$.

Proof. We use the equivalent condition for (4):

(12) $$F <^L G \Leftrightarrow F^* \leq^{st} G^* \Leftrightarrow \bar{F}^*(x) \leq \bar{G}^*(x) \text{ for all } x,$$

where $\bar{F}^*(s) = \int_0^{\infty} e^{-sx}dF(x)$, $\bar{G}^*(s)$ is defined similarly. Let $s > 0$ denote any fixed real. Then

$$\bar{F}^*(s) = \int_0^\infty e^{-sx} dF(x) = \int_0^\infty e^{-sx} f(x)\,dx = \left(1+\frac{s}{\lambda_F}\right)^{-\alpha}.$$

Similarly $\bar{G}^*(s) = \left(1+\frac{s}{\lambda_G}\right)^{-\alpha}$. It is easy to notice that the condition (6) holds if and only if $\lambda_F \leq \lambda_G$. The proof is complete.

This relation is a partial ordering, which sets in order gamma distributions in accordance with non-decreasing scale parameters. Furthermore, this relation is inverse to the relation $<^{(p)}$.

5. Generalized variability orderings and the gamma distributions with the same fixed scale parameter

Letl now random variables X and Y have gamma distributions with the same fixed scale parameter $\lambda > 0$ and the shape parameters $\alpha_F < 0$ and $\alpha_G > 0$, respectively. Furthermore, let α_F and $\alpha_G \in \mathbb{N}$, i.e. X and Y have Erlang's distributions with distribution functions and densities of probability respectively

$$\bar{F}(x) = \sum_{i=0}^{\alpha_F-1} \frac{(\lambda x)^i}{i!} e^{-\lambda x}, \quad \bar{G}(x) = \sum_{i=0}^{\alpha_G-1} \frac{(\lambda x)^i}{i!} e^{-\lambda x},$$

$$f(x) = \frac{\lambda^{\alpha_F}}{\Gamma(\alpha_F)} x^{\alpha_F-1} e^{-\lambda x}, \quad g(x) = \frac{\lambda^{\alpha_G}}{\Gamma(\alpha_G)} x^{\alpha_G-1} e^{-\lambda x}, \text{ for } x > 0.$$

Theorem 5.1. *The ordering $X <^p Y$ holds if and only if $\alpha_F = \alpha_G$.*

Proof. Similarly as before we obtain

$$\int_t^\infty x^{p-1} \bar{F}(x)\,dx = \frac{1}{\lambda^p} \sum_{i=0}^{\alpha_F-1} \frac{1}{i!} \int_{\lambda t}^\infty u^{i+p-1} e^{-u} du \text{ for all } t > 0,$$

$$\int_t^\infty x^{p-1} \bar{G}(x)\,dx = \frac{1}{\lambda^p} \sum_{i=0}^{\alpha_G-1} \frac{1}{i!} \int_{\lambda t}^\infty u^{i+p-1} e^{-u} du \text{ for all } t > 0,$$

$$\int_0^\infty x^{p-1} \bar{F}(x)\,dx = \frac{1}{\lambda^p} \sum_{i=0}^{\alpha_F-1} \frac{1}{i!} \Gamma(i+p)$$

and $$\int_0^\infty x^{p-1} \bar{G}(x)\,dx = \frac{1}{\lambda^p} \sum_{i=0}^{\alpha_G-1} \frac{1}{i!} \Gamma(i+p).$$

It is easy to notice that the condition (1) holds if and only if $\alpha_F \leq \alpha_G$ and the equality for $t=0$ – if and only if $\alpha_F = \alpha_G$. The proof is complete.

Similarly

Theorem 5.2. *The ordering* $X <^{(p)} Y$ *holds if and only if* $\alpha_F \leq \alpha_G$.

These relations have the same properties as $<^p$ and $<^{(p)}$ in section 4, but $<^{(p)}$ – ordering sets in order Erlangs distributions in accordance with non-decreasing shape parameteres.

Theorem 5.3. *The ordering* $X <_{(p)} Y$ *holds if and only if* $\alpha_F \geq \alpha_G$.

Proof. We consider the condition (3). We use again the possibility of changing the order of summation and integration. We obtain

$$\int_0^t x^{p-1}\bar{F}(x)\,dx = \frac{1}{\lambda^p}\sum_{i=0}^{\alpha_F-1}\frac{1}{i!}\int_0^{\lambda t} u^{i+p-1}e^{-u}du,$$

$$\int_0^t x^{p-1}\bar{G}(x)\,dx = \frac{1}{\lambda^p}\sum_{i=0}^{\alpha_G-1}\frac{1}{i!}\int_0^{\lambda t} u^{i+p-1}e^{-u}du \quad \text{for all } t>0.$$

We notice that condition (3) holds if and only if $\alpha_F \geq \alpha_G$. The proof is complete.

This ordering is a linear ordering, which sets in order Erlang's distributions in accordance with non-increasing shape parameters. Furthermore, relations $<^{(p)}$ and $<_{(p)}$ are inverse.

Theorem 5.4. *The ordering* $X \leq^{st} Y$ *holds if and only if* $\alpha_F \leq \alpha_G$.

This theorem is easy to prove when we compare distribution functions of X and Y. This relation has the same properties as $<^{(p)}$ from theorem 5.2.

Let us disergard the assumption that α_F and $\alpha_G \in I\!N$, i.e. let us consider a general case of gamma distribution.

Theorem 5.5. *The ordering* $X <^L Y$ *holds if and only if* $\alpha_F \geq \alpha_G$.

Proof. We use condition (12). Let $s > 0$ denote any fixed real. Then

$$\bar{F}^*(s) = \left(1+\frac{s}{\lambda}\right)^{-\alpha_F} \text{ and } \bar{G}^*(s) = \left(1+\frac{s}{\lambda}\right)^{-\alpha_G} \text{ for all } s > 0.$$

It is easy to notice that condition (6) holds if and only if $\alpha_F \geq \alpha_G$. The proof is complete.

This relation has the same properties as $<_{(p)}$ and it is inverse ro $\leq^{st}$ and $<^{(p)}$.

6. Final remarks

We notice that the consideration of conditions (3) and (6) in the family of the gamma distributions with the same shape parameter does not give us results as in other relations. To obtain a partial result we use the property from [1]:

$$X <^{p} Y \Rightarrow X <_{(p)} Y \tag{13}$$

Let now X and Y be random variables defined as in section 4. From (13) we obtain

Theorem 6.1. *If* $\lambda_F = \lambda_G$ *then the relation* $X <_{(p)} Y$ *holds.*

But we do not know if for other parameters λ_F and λ_G this relation also holds. We compare our results from sections 2 and 4. Let X and Y denote random variables with distribution functions respectively:

(i) $\bar{F}(x) = \exp(-\lambda_F x^{\alpha})$ and $\bar{G}(x) = \exp(-\lambda_G x^{\alpha})$ in the case of family of Weibull's distributions,

(ii) $\bar{F}(x) = \sum_{i=0}^{\alpha-1} \frac{(\lambda_F x)^i}{i!} \exp(-\lambda_F x)$ and $\bar{G}(x) = \sum_{i=0}^{\alpha-1} \frac{(\lambda_G x)^i}{i!} \exp(-\lambda_G x)$

in the case of Erlang's distributions.

Relations	Assumptions about scale parameters for:	
	Weibull's distributions	Erlang's distributions
$X <^{p} Y$	$\lambda_F = \lambda_G$	$\lambda_F = \lambda_G$
$X <^{(p)} Y$	$\lambda_F \geq \lambda_G$	$\lambda_F \geq \lambda_G$
$X <_{(p)} Y$	$\lambda_F \leq \lambda_G$	
$X <^{L} Y$	$\lambda_F \leq \lambda_G$	$\lambda_F \leq \lambda_G$
$X \leq^{st} Y$	$\lambda_F \geq \lambda_G$	

We notice that the exponential distribution is a special case of Weibull's and Erlang's distributions for $\alpha = 1$. Maybe it is true for random variables with Erlang's distributions with the same shape parameters that:

(i) the ordering $X <_{(p)} Y$ holds if and only if $\lambda_F \leq \lambda_G$,

(ii) the ordering $X \leq^{st} Y$ holds if and only if $\lambda_F \geq \lambda_G$.

References

[1] Bhattacharjee M. C., *Some Generalized Variability Orderings among Life Distributions with Reliability Applications*, J. Appl. Prob., 28, 1991, pp. 374–384.

[2] Bobrowski D., *Modele i metody matematyczne w teorii niezawodności*, WNT, Warszawa 1985.

[3] Fichtenholz G. M., *Rachunek różniczkowy i całkowy*, tom 2, PWN, Warszawa 1980.

[4] Fisz M., *Rachunek prawdopodobieństwa i statystyka matematyczna*, PWN, Warszawa 1969.

[5] Kuratowski K., *Wstęp do teorii mnogości i topologii*, PWN, Warszawa 1980.

(Poznań University of Technology, Institute of Mathematics, 60-965 Poznań, Poland)
Received on 15.3.1994.