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STRONG SUMMABILITY OF INTERPOLATORY POLYNOMIALS IN GENERALIZED HÖLDER NORMS

We consider the problem of strong summability of trigonometric interpolatory polynomials in generalized Hölder spaces.

This note is strictly connected with results given in the papers [1] – [5].

Key words: strong summability, interpolatory polynomials, Hölder spaces.

1. Preliminaries

1.1. Let $C = C_{2\pi}$ be the space of 2π-periodic real-valued functions, continuous on $\langle -\pi, \pi \rangle$ with the norm

(1) $$\|f\|_C : \max_{|x| \le \pi} |f(x)|.$$

Denote as in [4] by Ω the set of all functions of modulus of continuity type ([6], i.e. Ω is the set of all functions ω having the following properties:

a) ω is a function defined and continuous on $<0, +\infty)$,

b) ω is monotonically increasing and $\omega(0) = 0$,

c) $\omega(h)h^{-1}$ is monotonically decreasing for $h > 0$.

For a given $\omega \in \Omega$ denote by H^ω the class of all functions $f \in C$ for which

(2) $$\|f\|_\omega := \sup_{h>0} \frac{\|\Delta_h f\|_C}{\omega(h)} < +\infty,$$

where

(3) $$\Delta_h f(x) := f(x+h) - f(x).$$

In H^ω we define the norm:

(4) $$\|f\|_{H^\omega} := \|f\|_C + \|f\|_\omega.$$

H^ω with the norm (4) is a Banach space and is called generalized Hölder space. If $\omega(h) = h^\alpha$, $0 < \alpha \leq 1$, then we have the classical Hölder space H^α.

Similarly as in [4], let $\bar{H}^\omega$ be the subspace of H^ω defined by

$$\bar{H}^\omega := \left\{ f \in H^\omega : \lim_{h->0+} \frac{\|\Delta_h f\|_C}{\omega(h)} = 0 \right\},$$

with the norm $\|\cdot\|_{\bar{H}^\omega}$ defined by (4). If ω, $\mu \in \Omega$ and $q(h) := \frac{\omega(h)}{\mu(h)}$, $h > 0$, is a monotonically increasing function, then

$$H^\omega \subset H^\mu \quad \text{and} \quad \bar{H}^\omega \subset \bar{H}^\mu. \tag{5}$$

1.2 Let X be one of the spaces C, H^ω and $\bar{H}^\omega$. For $f \in X$ denote by $E_n(f; X)$, $n \in N := \{0, 1, 2, ...\}$, the best approximation of a function f trigonometric polynomials of degree $\leq$ n in the sense of X, i.e.

$$E_n(f; X) := \inf_{T_n} \|f - T_n\|_X,$$

where the infimum is taken over all trigonometric polynomials T_n of degree $\leq n$.

It is known ([6], [4]) that if $f \in H^\omega$, then

$$E_n(f; C) \leq 3\omega\left(\frac{1}{n+1}\right)\|f\|_\omega, \quad n \in N. \tag{6}$$

If $f \in \bar{H}^\omega$, then

$$E_n(f; C) = o\left(\omega\left(\frac{1}{n+1}\right)\right) \quad \text{as } n \to \infty. \tag{7}$$

In [4] there is given the following

Theorem A. *If* $f \in H^\omega$ *and* $\mu \in \Omega$ *is a function such that* $q(h) = \frac{\omega(h)}{\mu(h)}$ *is monotonically increasing for* $h > 0$, *then*

$$E_n(f; H^\mu) \leq 648\, q\left(\frac{1}{n}\right)\|f\|_\omega, \quad n \geq 1.$$

In the case of $f \in \bar{H}^\omega$,

$$E_n(f; \bar{H}^\mu) \equiv E_n(f; \mathrm{H}^\mu) = o\left(q\left(\frac{1}{n}\right)\right) \quad \textit{as } n \to \infty.$$

1.3. Let $D_n(\cdot)$, $n \in N$, be the Dirichlet kernel defined by

$$D_n(t): \frac{1}{2} + \sum_{k=1}^{n} \cos kt.$$

For a given $f \in C$ we consider the trigonometric interpolatory polynomial of degree n

$$I_n(x:f) := \frac{2}{2n+1} \sum_{j=-n}^{n} f(x_j) D_n(x-x_j) \equiv \tag{8}$$

$$\equiv \frac{a_o^{(n)}}{2} + \sum_{k=1}^{n} (a_k^{(n)} \cos kx + b_k^{(n)} \sin kx),$$

$n \in N$, $x \in R := (-\infty, +\infty)$, which for

$$x_j := \frac{2\pi j}{2n+1} \quad (j = 0, \pm 1, \pm 2, ...)$$

satisfies the condition $I_n(x_j:f) = f(x_j)$ ([7], [2], [3], [5]).

Denote by $I_{n,k}(x;f)$ the k-th partial sum of polynomial (8), i.e.

$$I_{n,k}(x:f) := \frac{a_o^{(n)}}{2} + \sum_{v=1}^{k} (a_v^{(n)} \cos vx + b_v^{(n)} \sin vx) \equiv \tag{9}$$

$$\equiv \frac{2}{2n+1} \sum_{j=-n}^{n} f(x_j) D_k(x-x_j)$$

for $0 \leq k \leq n$ and $n \in N$.

Let $\lambda = [\lambda_{n,k}]$ be an infinite triangular matrix of real numbers such that

$$1^\circ \quad \lambda_{n,k} > 0 \quad \text{for} \quad 0 \leq k \leq n, \quad n \in N,$$

$$\lambda_{n,k} = 0 \quad \text{for} \quad k > n, \quad n \in N,$$

$$2^\circ \quad \lim_{n \to \infty} \lambda_{n,k} = 0 \quad \text{for every fixed } k.$$

Consider the following strong means of polynomial (8):

$$L_n(x; f, \lambda := \sum_{k=0}^{n} \lambda_{n,k} |I_{n,k}(x;f) - f(x)|, \tag{10}$$

for $x \in R$ and $n \in N$. In particular, we have the strong Cesàro means of (8):

(11) $$L_n(x;f)=\frac{1}{n+1}\sum_{k=0}^{n}|I_{n,k}(x;f)-f(x)|$$

($n\in N$, $x\in R$) examined in [5] for $f\in C$. Clearly, if $f\in C$, then $L_n(\cdot;f,\lambda)\in C$ also.

The purpose of this note is to estimate the generalized Hölder norms of $L_n(\cdot;f,\lambda)$ for $f\in H^{\omega}$ and $f\in \bar{H}^{\omega}$.

By $M_k(\cdot)$, $k=1,2,...$, we shall denote suitable positive constans depending only on indicated parameters.

2. Auxiliary results

For the polynomial (8) of $f\in C$ and the mean (10) we introduce the functions:

(12) $$U_m(x;f):=\frac{1}{m+1}\sum_{k=0}^{m}|I_{n,k}(x;f)|,\quad 0\le m\le n,$$

(13) $$U_{n_1,n_2}(x;f,\lambda):=\frac{1}{n_2-n_1+1}\sum_{k=n_1}^{n_2}\lambda_{n,k}|I_{n,k}(x;f)|$$

and

(14) $$U_{n_1,n_2}(x;f,\lambda):=\frac{1}{n_2-n_1+1}\sum_{k=n_1}^{n_2}\lambda_{n,k}|I_{n,k}(x;f)-f(x)|$$

for $0\le n_1\le n_2\le n$, $x\in R$; the last is called strong generalized de la Vallée Poussin mean of (8). Moreover let

(15) $$B_{p,q}(n:\lambda);=\max_{p\le k\le q}\lambda_{n,k}$$

for $0\le p\le q\le n$, $n\in N$.

The following lemmas hold.

Lemma 1. *Let X be the space C or H^{ω}. If $f\in X$ and $n\in N$, then*

(16) $$\|U_m(\cdot;f)\|_X\le M_1\|f\|_X$$

for all $0\le m\le n$, which proves that $U_m(\cdot;f)$ belongs to X also.

Proof. The proof of (16) for $X = C$ is given in [5]. If $f \in H^\omega$, then by (1)–(4) we have

$$\|U(\cdot\,;f)\|_{H^\omega} = \|U_m(\cdot\,;f)\|_C + \|U_m(\cdot\,;f)\|_\omega$$

and

$$\|U_m(\cdot\,;f)\|_\omega = \sup_{h>0} \frac{\|\Delta_h U_m(\cdot\,;f)\|_C}{\omega(h)}.$$

Using (16) with $X = C$, we have

$$\|U_m(\cdot\,;f)\|_C \le M_1 \|f\|_C, \quad 0 \le m \le n.$$

For the sum (9) we have

$$\Delta_h I_{n,k}(x\,;f) = I_{n,k}(x; \Delta_h f) \quad 0 \le k \le \mathrm{n}, \quad n \in N.$$

Hence

$$|\Delta_h U_m(x\,;f)| = \left|\frac{1}{m+1} \sum_{k=0}^{m} \Delta_h |I_{n,k}(x\,;f)|\right| \le$$

$$\le \frac{1}{m+1} \sum_{k=0}^{m} |\Delta_h I_{n,k}(x\,;f)| = \frac{1}{m+1} \sum_{k=0}^{m} |I_{n,k}(x\,;\Delta_h f)| = U_m(x\,;\Delta_h f)$$

and, by (16) with $X = C$, we get

$$\|U_m(\cdot\,;f)\|_\omega = \sup_{h>0} \frac{\|U_m(\cdot\,;\Delta_h f)\|_C}{\omega(h)} \le$$

$$\le M_1 \sup_{h>0} \frac{\|\Delta_h f\|_C}{\omega(h)} = M_1 \|f\|_\omega .$$

Summing up, we obtain

$$\|U_m(\cdot\,;f)\|_{H^\omega} \le M_1 (\|f\|_C + \|f\|_\omega,$$

which completes the proof of (16) for $f \in H^\omega$. ■

Analogously we obtain the following

Lemma 2. *If* $f \in \bar{H}^\omega$ *and* $n \in N$, *then* $U_m(\cdot\,;f) \in \bar{H}^\omega$, $0 \le m \le n$. ■

Lemma 3. *Let X be the space C or H^{ω} and $n \in N$. If $f \in X$ and $0 \le n_1 \le n_2 \le n$, then*

$$\|U_{n_1,n_2}(\cdot;f,\lambda)\|_X \le M_2 B_{n_1,n_2}(n;\lambda) \frac{n_2+1}{n_2-n_1+1} \|f\|_X,$$

where $B_{p,q}(n;\lambda)$ is defined by (15). ■

Lemma 4. *If $f \in \bar{H}^{\omega}$, $n \in N$ and $0 \le n_1 \le n_2 \le n$, then $U_{n_1,n_2}(\cdot;f,\lambda)$ belongs to $\bar{H}^{\omega}$ also.* ■

Now we shall prove three lemmas on strong means (14).

Lemma 5. *If $f \in C$, then*

$$\|W_{n_1,n_2}(\cdot;f,\lambda)\|_C \le M_3 B_{n_1,n_2}(n;\lambda) \frac{n_2+1}{n_2-n_1+1} E_{n_1}(f;C)$$

for all $0 \le n_1 \le n_2 \le n$ and $n \in N$.

Proof. In [5] it is proved that if $f \in C$, then

$$\left\| \frac{1}{n_2-n_1+1} \sum_{k=n}^{n_2} |I_{n,k}(\cdot;f)-f(\cdot)| \right\|_C \le M_4 \frac{n_2+1}{n_2-n_1+1} E_{n_1}(f;C)$$

for $0 \le n_1 \le n_2 \le n$ and $n \in N$. Using this inequality, we obtain our assertion.

Lemma 6. *If $f \in H^{\omega}$ and $H^{\omega} \subset H^{\mu}$, then*

$$\|W_{n_1,n_2}(\cdot;f,\lambda)\|_{H^{\mu}} \le M_5 B_{n_1,n_2}(n;\lambda) \frac{n_2+1}{n_2-n_1+1} E_{n_1}(f;H^{\mu}) \tag{17}$$

for all $0 \le n_1 \le n_2 \le n$ and $n \in N$.

Proof. By (1)–(5) we have

$$\|W_{n_1,n_2}(\cdot;f,\lambda)\|_{H^{\mu}} = \|W_{n_1,n_2}(\cdot;f,\lambda)\|_C + \|W_{n_1,n_2}(\cdot;f,\lambda)\|_{\mu}. \tag{18}$$

Denote by $T_{n_1}^*$ the trigonometric polynomial of the best approximation of function f by trigonometric polynomials of degree $\le n_1$ in the sense of H^{μ}, i.e.

(19) $$E_{n_1}(f;H^{\mu}):=\|f-T^*_{n_1}\|_{H^{\mu}}.$$

Then we have

$$I_n(x;T^*_{n_1})=T^*_{n_1}(x) \text{ and } I_{n,k}(x;T^*_{n_1})=T^*_{n_1}(x)$$

for $n_1 \le k \le n$ and $x \in R$. Hence

$$|I_{n,k}(x;f)-f(x)| \le |I_{n,k}(x;f)-T^*_{n_1}(x)|+$$
$$+|T^*_{n_1}(x)-f(x)|=|I_{n,k}(x;f-T^*_{n_1})|+|T^*_{n_1}(x)-f(x)|$$

for $n_1 \le k \le n$ and $x \in R$. From this and by (12)–(15) we get

$$\|W_{n_1,n_2}(\cdot;f,\lambda)\|_C \le \|U_{n_1,n_2}(\cdot;f,-T^*_{n_1},\lambda)\|_C+B_{n_1,n_2}(n;\lambda)\|T^*_{n_1}-f\|_C.$$

Now using Lemma 3, we obtain

(20) $$\|W_{n_1,n_2}(\cdot;f,\lambda)\|_C \le \left\{M_2\frac{n_2+1}{n_2-n_1+1}+1\right\}B_{n_1,n_2}(n;\lambda)\|T^*_{n_1}-f\|_C.$$

Arguing as in the proof of Lemma 1, we get

$$\|\Delta_h W_{n_1,n_2}(\cdot;f,\lambda)\|_C \le \|W_{n_1,n_2}(\cdot;\Delta_h f,\lambda)\|_C \le$$
$$\le \left\|\frac{1}{n_2-n_1+1}\sum_{k=n}^{n_2}\lambda_{n,k}|I_{n,k}(\cdot:\Delta_h f)-\Delta_h T^*_{n_1}(\cdot)|\right\|_C+$$
$$+\left\|\frac{1}{n_2-n_1+1}\sum_{k=n}^{n_2}\lambda_{n,k}|\Delta_h T^*_{n_1}(\cdot)-\Delta_h f(\cdot)|\right\|_C \le$$
$$\le \|U_{n_1,n_2}(\cdot;\Delta_h(f-T^*_{n_1}),\lambda)\|_C+B_{n_1,n_2}(n;\lambda)\|\Delta_h(f-T^*_{n_1})\|_C.$$

Hence, using Lemma 3, we obtain

$$\|\Delta_h W_{n_1,n_2}(\cdot;f,\lambda)\|_C \le \left\{M_2\frac{n_2+1}{n_2-n_1+1}+1\right\}B_{n_1,n_2}(n;\lambda)\|\Delta_h(f-T^*_{n_1})\|_C.$$

which, by (2), gives

(21) $$\|W_{n_1,n_2}(\cdot;f,\lambda)\|_{\mu} \le \left\{M_2\frac{n_2+1}{n_2-n_1+1}+1\right\}B_{n_1,n_2}(n;\lambda)\|T^*_{n_1}-f\|_{\mu}.$$

Using (19)–(21) to (18), we obtain (17). ■

Analogously as Lemma 6 we obtain the following

Lemma 7. *If* $f \in \bar{H}^{\omega}$ *and* $\bar{H}^{\omega} \subset \bar{H}^{\mu}$, *then*

$$\|W_{n_1,n_2}(\cdot;f,\lambda)\|_{\bar{H}^{\mu}} \le M_5 B_{n_1,n_2}(n;\lambda) \frac{n_2+1}{n_2-n_1+1} E_{n_1}(f;\bar{H}^{\mu})$$

for all $0 \le n_1 \le n_2 \le n$ *and* $n \in N$. ∎

3. Main theorems

Now we shall given the main theorems on generalized Hölder norms of the strong means $L_n(\cdot;f,\lambda)$ defined by (10). We observe that the formula (10) can be written in the form

$$(22) \qquad L_n(\cdot;f,\lambda) = \sum_{p=0}^{s-1} 2^p W_{\bar{p},2\bar{p}}(x;f,\lambda)+(n-\bar{s}+1)W_{\bar{s},n}(x;f,\lambda), \quad n \in N,$$

where s is an integer such that $2^s \le n+1 < 2^{s+1}$, $\bar{p} = 2^p-1$ and $\bar{s} = 2^s-1$.

Theorem 1. *If* $f \in C$, *then*

$$(23) \qquad \|L_n(\cdot;f,\lambda)\|_C \le M_7 \sum_{p=0}^{s} 2^p B_{\bar{p},2\bar{p}}(n;\lambda) E_{\bar{p}}(f;C)$$

for all $n \in N$, *where* s *an is integer such that* $2^s \le n+1 < 2^{s+1}$ *and* $\bar{p} = 2^p-1$. *In particular, if* $f \in H^{\omega}$, *then*

$$(24) \qquad \|L_n(\cdot;f,\lambda)\|_C \le 3M_7 \|f\|_{\omega} \sum_{p=0}^{s} 2^p B_{\bar{p},2\bar{p}}(n;\lambda)\,\omega(2^{-p})$$

for all $n \in N$.

Proof. By (22) and Lemma 5 we have

$$\|L_n(\cdot;f,\lambda)\|_C \le \sum_{p=0}^{s-1} 2^p \|W_{\bar{p},2\bar{p}}(\cdot;f,\lambda)\|_C+(n-\bar{s}+1)\|W_{\bar{s},n}(\cdot;f,\lambda)\|_C \le$$

$$\le M_4 2 \sum_{p=0}^{s} 2^p B_{\bar{p},2\bar{p}}(n;\lambda) E_{\bar{p}}(f;C)$$

for all $n \in N$. Thus the proof of (23) is completed. Using the inequality (6) to (23), we obtain (24). ∎

From Theorem 1 we obtain the following

Corollary 1. *Suppose that the matrix* λ *satisfies the conditions* $1° - 2°$ *and moreover,*

$$\sum_{p=0}^{[\log_2(n+1)]} 2^p B_{\bar{p},2\bar{p}}(n;\lambda) \le M_{10} \tag{25}$$

for all $n \in N$. *If* $f \in C$, *then*

$$\|L_n(\cdot; f, \lambda)\|_C \to 0 \text{ as } n \to \infty.$$

In particular, for the strong means defined by (11), *we have*

$$\|L_n(\cdot; f, \lambda)\|_C \to 0 \text{ as } n \to \infty,$$

which is proved in [5]. ■

Theorem 2. *If* $f \in H^\omega$ *and* $H^\omega \subset H^\mu$, *then*

$$\|L_n(\cdot; f, \lambda)\|_{H^\mu} \le M_8 \sum_{p=0}^{s} 2^p B_{\bar{p},2\bar{p}}(n;\lambda) E_{\bar{p}}(f; H^\mu) \tag{26}$$

for all $n \in N$, *where* s *and* $\bar{p}$ *are integers as in Theorem* 1. *In particular, if* μ *is a function such that* $q(h) = \omega(h)/\mu(h)$ *is monotonically increasing for* $h > 0$, *then*

$$\|L_n(\cdot; f, \lambda)\|_{H^\mu} \le M_g \|f\|_\omega \sum_{p=0}^{s} 2^p B_{\bar{p},2\bar{p}}(n;\lambda) q(2^{-p}) \tag{27}$$

for all $n \in N$.

Proof. From (22) and Lemma 6 it follows that

$$\|L_n(\cdot; f, \lambda)\|_{H^\mu} \le \sum_{p=0}^{s-1} 2^p \|W_{\bar{p},2\bar{p}}(\cdot; f, \lambda)\|_{H^\mu} + (n - \bar{s} + 1)\|W_{\bar{s},n}(\cdot; f, \lambda)\|_{H^\mu} \le$$
$$\le 2M_5 \sum_{p=0}^{s} 2^p B_{\bar{p},2\bar{p}}(n;\lambda) E_{\bar{p}}(f; H^\mu)$$

for all $n \in N$, which gives (26).

Using Theorem *A* to (26), we obtain (27). ■

Theorem 2 implies the following

Corollary 2. *Suppose that the matrix* λ *satisfies the conditions* 1°, 2° *and* (25).

If $f \in H^{\omega}$ *and* μ, q *are functions as in Theorem* 2 *and* $q(h) \to 0$ *if* $h \to 0^{+}$, *then*

$$\|L_n(\cdot\,; f, \lambda)\|_{H^{\mu}} \to 0 \text{ as } n \to \infty. \blacksquare$$

Applying (22), Lemma 7 and Theorem A, and arguing as in the proof of Theorem 2, we obtain

Theorem 3. *If* $f \in \bar{H}^{\omega}$ *and* $\bar{H}^{\omega} \subset \bar{H}^{\mu}$, *then*

$$\|L_n(\cdot\,; f, \lambda)\|_{\bar{H}^{\mu}} \le M_8 \sum_{p=0}^{s} 2^p B_{\bar{p},2\bar{p}}(n; \lambda)\, E_{\bar{p}}(f; \bar{H}^{\mu})$$

for all $n \in N$, *where* s *and* $\bar{p}$ *are integers as in Theorem* 1.

4. Applications

We shall examine, for example, the strong Riesz and Cesàro means of polynomial (8) of $f \in C$.

4.1. Consider the strong Riesz means of (8) defined by

$$R_n x; f, \beta) := \frac{1}{\beta(n+1)^{\beta}} \sum_{k=0}^{n} \{(k+1)^{\beta} - k^{\beta}\} |I_{n,k}(x; f) - f(x)|, \tag{28}$$

$n \in N$, $x \in R$, where β is a fixed positive number. The strong means (28) contain strong Cesàro means defined by (11).

From Theorems 1–2, (6), (7) and Theorem A we easily obtain the following three simple facts.

Corollary 3. *If* $f \in H^{\omega}$ *with* $\omega(h) = h^{\alpha}$, $0 < \alpha \le 1$ *and* $\beta > 0$, *then*

$$\|R_n(\cdot\,; f, \beta)\|_C \le$$

$$\le M_{10}(\alpha, \beta) \|f\|_{\omega} \begin{cases} (n+1)^{-\alpha} & \text{if } \alpha < \beta, \\ (n+1)^{-\alpha} \log(n+2) & \text{if } \alpha = \beta, \\ (n+1)^{-\beta} & \text{if } \alpha > \beta, \end{cases}$$

for all $n \in N$. $\blacksquare$

Corollary 4. *Suppose that the assumptions of Theorem* 2 *are fulfilled and* $q(h) \le M_{11} h^{\gamma}$ *for* $h > 0$, *with some* $0 < \gamma < 1$. *Then we have*

$$\|R_n(\cdot\,; f, \beta)\|_{H^{\mu}} \le \tag{29}$$

$$\le M_{12}^{*} \|f\|_{\omega} \begin{cases} (n+1)^{-\gamma} & \text{if } \gamma < \beta, \\ (n+1)^{-\gamma}\log(n+2) & \text{if } \gamma = \beta, \\ (n+1)^{-\beta} & \text{if } \gamma > \beta, \end{cases}$$

for all $n \in N$ *and* $\beta > 0$, *where* $M_{12}^{*} = M_{12}(\beta, \gamma)$. ■

Corollary 5. *Suppose that* $f \in \bar{H}^{\omega}$ *and* $\bar{H}^{\omega} \subset \bar{H}^{\mu}$, *where* ω, μ, q *satisfy the assumptions of Corollary* 4. *Then, for every* $\beta > 0$, *we have*

$$\|R_n(\cdot\,; f, \beta)\|_{\bar{H}^{\mu}} = \begin{cases} o((n+1)^{-\gamma}) & \text{if } \gamma < \beta, \\ o((n+1)^{-\gamma}\log(n+2)) & \text{if } \gamma = \beta, \\ o((n+1)^{-\beta}) & \text{if } \gamma > \beta, \end{cases}$$

as $n \to \infty$. ■

4.2. Now, we shall consider the strong Cesàro (C, δ) means, $\delta \ge 1$, of the polynomial (8) of $f \in C$, i.e.

$$t_n(x; f, \delta) := \frac{1}{A_n^{\delta}} \sum_{k=0}^{n} A_{n-k}^{\delta-1} |I_{n,k}(x; f) - f(x)|,$$

$n \in N$, $x \in R$, where

$$A_0^{\delta} = 1 \text{ and } A_k^{\delta} = \frac{(\delta+1)(\delta+2)\dots(\delta+k)}{k!} \text{ for } k \ge 1.$$

We have $t_n(x; f, 1) \equiv R_n(x; f, 1)$ and

$$t_n(x; f, \delta) := \frac{1}{A_n^{\delta}} \sum_{k=0}^{n} A_{n-k}^{\delta-1} A_k^{1} t_k(x; f, 1) \tag{30}$$

for all $n \in N$, $x \in R$ and $\delta > 1$. Using the estimations of the norms of $R_n(\cdot\,; f, 1)$ given in Corollaries 3–5 to the norms of $t_n(\cdot\,; f, \delta)$ defined by (30), we obtain the analogues of Corollaries 3–5, formulated below

Corollary 6. *If* $f \in H^{\omega}$ *with* $\omega(h) = h^{\alpha}$, $0 < \alpha \le 1$ *and* $\delta > 1$, *then*

$$\|t_n(\cdot\,; f, \delta)\|_C \le M_{13}(\alpha, \delta)\, \|f\|_{\omega} \begin{cases} (n+1)^{-\alpha} & \text{if } \alpha < 1, \\ (n+1)^{-1}\log(n+2) & \text{if } \alpha = 1, \end{cases}$$

for all $n \in N$. ∎

Corollary 7. *Suppose that the assumptions of Corollary* 4 *are fulfilled. Then we have*

$$\|t_n(\cdot\,; f, \delta)\|_{H^{\mu}} \le M_{14}(\gamma, \delta)\, \|f\|_{\omega}(n+1)^{-\gamma} \tag{31}$$

for all $n \in N$, $\delta > 1$ *and* $0 < \gamma < 1$. ∎

Corollary 8. *Suppose that the assumptions of Corollary* 5 *are satisfied. Then, for every* $\delta > 1$ *we have*

$$\|t_n(\cdot\,; f, \delta)\|_{\bar{H}^{\omega}} = o((n+1)^{-\gamma}) \text{ as } n \to \infty. \ \blacksquare$$

Indeed, e.g. inequality (31) we obtain with the application of (29) and (30) as follows:

$$\|t_n(\cdot\,; f, \delta)\|_{H^{\mu}} \le \frac{1}{A_n^{\delta}} \sum_{k=0}^{n} A_{n-k}^{\delta-2}(k+1)\, \|t_k(\cdot\,; f, 1)\|\|_{H^{\mu}} \le$$

$$\le M_{12}(\gamma)\, \|f\|_{\omega} \frac{1}{A_n^{\delta}} \sum_{k=0}^{n} A_{n-k}^{\delta-2}(k+1)^{1-\gamma} \le$$

$$\le M_{12}(\gamma)\, \|f\|_{\omega}(n+1)^{1-\gamma} \frac{1}{A_n^{\delta}} \sum_{k=0}^{n} A_{n-k}^{\delta-2} \le$$

$$\le M_{14}(\gamma, \delta))\, \|f\|_{\omega}(n+1)^{1-\gamma} \quad \text{for } n \in N.$$

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