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WALDEMAR GRYGIEL

THE PROBLEM OF OPTIMIZATION FOR SOME FUNCTIONALS DEFINED ON THE PERMUTATION SET AND THEIR RELIABILITY INTERPRETATION

An optimization problem for some functionals being applied in the reliability theory is discussed in this paper. These functionals are defined on the permutation set S_n. In particular the properties of the optimal permutations are analyzed. The properties are talked about in Theorems 2.1 and 2.2. of Section 2. Besides, a conclusion from these propositions for the permutation set S_5 is derived. Section 4 includes the reliability interpretation corresponding to the a.m. functionals.

Key words: permutation, reliability, consecutive k-out-of-n system, optimization.

1. Symbols

Let $n \geq 1$ be an integer and let k be a fixed integer such that $1 \leq k \leq n$. Every bijection $f: N_n \to N_n$, where $N_n = \{1, 2, ..., n\}$, is called a permutation on the set N_n. Every permutation f can be identified with the sequence $\langle f(1), f(2), ..., f(n) \rangle$. The class of all permutations on the set N_n is denoted by S_n. For a given function:

$$p: N_n \to [0,1] \tag{1.1}$$

its value on the element i will be denoted shortly by p_i. Moreover, by the symbol $\bar{p}(m)$ we will denote the vector:

$$\bar{p}(m) = \langle p_1, p_2, ..., p_m \rangle; \quad 1 \leq m \leq n \tag{1.2}$$

We define the function q by the function p in the following way:

$$q = 1 - p \tag{1.3}$$

$$q_i = 1 - p_i \qquad i = 1, 2, ..., n$$

As $\bar{p}(f, m)$, where $f \in S_n$, we understand the vector

(1.4) $$\bar{p}(f,m) = \langle (p\circ f)(1),\ (p\circ f)(2), ..., (p\circ f)(m)\rangle,$$

where the symbol $\circ$ denotes the superposition of given functions. Similarly

(1.5) $$\bar{q}(f,m) = \langle (q\circ f)(1),\ (q\circ f)(2), ..., (q\circ f)(m)\rangle.$$

According to [2] or [3] on the permutation class S_n we define functionals F and R, such that:

(1.6) $$F(p,k,n): S_n \to [0,1];$$

(1.7) $$R(p,k,n): S_n \to [0,1];$$

The value of the functional $F(p,k,n)$ on an element $f\in S_n$ can be found in the following way:

(1.8) $$F(p,k,n)(f) = F(\bar{p}(f,n),k,n).$$

Writing $F(\bar{p}(f,n),k,n)$ out there is derived:

(1.9) $$\begin{gathered} F(\bar{p}(f,i),k,i) = 0 \quad \text{for } i = 0,1,2,...,k-1; \\ F(\bar{p}(f,k),k,k) = q\circ f(1)\, q\circ f(2)\, q\circ f(3) \dots q\circ f(k) \\ F(\bar{p}(f,n),k,n) = F(\bar{p}(f,n-1),k,n-1) + \\ +[1-F(\bar{p}(f,n-k-1),k,n-k-1)]\, p\circ f(n-k)\, q\circ f(n-k+1)\cdot \\ \cdot q\circ f(n-k+2) \dots q\circ f(n) \end{gathered}$$

The value of the functional $R(p,k,n)$ on an element $f\in S_n$ is found in the following way:

(1.10) $$R(p,k,n)(f) = R(\bar{p},k,n),k,n) = 1 - F(\bar{p}(f,n),k,n)$$

2. Problem formulation. Theorems

In Section 1 we defined the functional $R(p,k,n)$ on the function set S_n, such that its values are in the interval $[0,1]$. Since the set S_n is finite, there must exist one or more optimal elements of the functional R. The problem of finding the optimum is solved in some particular cases. (see [2], [3]). Basing on the previous functional the following relation of partial ordering is put in the group S_n:

(2.1) $$\forall f, q \in S_n \quad f \leq q \Leftrightarrow R(p,k,n)(f) \leq R(p,k,n)(q);$$

A set is partially ordered if a reflexive, transitive and anti-symmetric realation is defined in it. The relation S is anti-symmetric in the X when:

(2.2) $$\forall\, x, y \in X\, [(xSy \wedge ySx) \Rightarrow x = y];$$

Theorem 2.1. *Let n, k be integers such that,* $n \geq 2k-1 \geq 0$ *and p, q are functions given by* (1.1) (1.3). *If the functional* $R(p,k,n)$ *has the maximal value for the permutation f then the following conditions are satisfied:*

(2.3) $$p(f(n-k+1)) \geq p(f(n-k+2)) \geq \ldots \geq p(f(n))$$

Proof. The problem of finding a maximum of the functional $R(p,k,n)$ is equal to the problem of finding a minimum of the functional $F(p,k,n)$ on the group S_n. According to (1.9) the value of the functional $F(p,k,n)$ can be formulated as follows:

(2.4) $$F(p,k,n)(f) = F(\bar{p}(f,n),k,n)$$

writing $F(\bar{p}(f,n),k,n)$ out next,

$$F(\bar{p}(f,i),k,i) = 0 \quad \text{for } i = 0,1,2,\ldots,k-1;$$

$$F(\bar{p}(f,k),k,k) = q \circ f(1)\, q \circ f(2)\, q \circ f(3) \ldots q \circ f(k)$$

(2.5) $$F(\bar{p}(f,n),k,n) = F(\bar{p}(f,n-1),k,n-1) +$$

$$+ [1 - F(\bar{p}(f,n-k-1),k,n-k-1)]\, p \circ f(n-k)\, q \circ f(n-k+1) \cdot$$

$$\cdot\, q \circ f(n-k+2) \ldots q \circ f(n) = F(\bar{p}(f,n-2,k,n-2) +$$

$$+ [1 - F(\bar{p}(f,n-k-2),k,n-k-2)]\, p \circ f(n-k-1)\, q \circ f(n-k)\, q \circ f(n-k-1) \ldots$$

$$\ldots q \circ f(n-1) + [1 - F(\bar{p}(f,n-k-1),k,n-k-1)]\, p \circ f(n-k)\, q \circ f(n-k+1) \cdot$$

$$\cdot\, q \circ f(n-k+2) \ldots q \circ f(n)$$

Denoting $h = n-k-i-1$, we have

$$F(\bar{p}(f,n),k,n) = F(\bar{p}(f,n-i-1),k,n-i-1) +$$

$$+ [1 - F(\bar{p}(f,h),k,h)]\, p \circ f(n-k-i)\, q \circ f(n-k-i+1)\, q \circ f(n-k-i+2) \ldots q \circ f(n-i) +$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$+[1-F(\bar{p}(f,n-k-3),k,n-k-3)]p\circ f(n-k-2)q\circ f(n-k-1)q\circ f(n-k)\ldots q\circ f(n-2)+$$

$$+[1-F(\bar{p}(f,n-k-2),k,n-k-2)]p\circ f(n-k-1)q\circ f(n-k)q\circ f(n-k+1)\ldots q\circ f(n-1)+$$

$$+[1-F(\bar{p}(f,n-k-1),k,n-k-1)]p\circ f(n-k)q\circ f(n-k+1)q\circ f(n-k+2)\ldots q\circ f(n)$$

Denoting $w = n-2k-1$, $z = n-k-i-1$, we have

$$F(\bar{p}(f,n),k,n) = F(\bar{p}(f,n-k-1),k,n-k-1)+$$

$$+[1-F(\bar{p}(f,w),k,w)]\,p\circ f(n-2k)\,q\circ f(n-2k+1)\,q\circ f(n-2k+2)\ldots q\circ f(n-k)+$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+$$
$$+[1-F(\bar{p}(f,z),k,z)]\,p\circ f(n-k-i)\,q\circ f(n-k-i+1)\,q\circ f(n-k-i+2)\ldots q\circ f(n-i)+$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+$$
$$+[1-F(\bar{p}(f,n-k-3),k,n-k-3)]p\circ f(n-k-2)q\circ f(n-k-1)q\circ f(n-k)\ldots q\circ f(n-2)+$$

$$+[1-F(\bar{p}(f,n-k-2),k,n-k-2)]p\circ f(n-k-1)q\circ f(n-k)q\circ f(n-k+1)\ldots q\circ f(n-1)+$$

$$+[1-F(\bar{p}(f,n-k-1),k,n-k-1)]p\circ f(n-k)q\circ f(n-k+1)q\circ f(n-k+2)\ldots q\circ f(n)$$

Let a permutation $f\in S_n$ be given for which the functional $F(p,k,n)$ has minimal value. If $p(f,(n))$ is greater than elements $p(f(n-1)), p(f(n-2)),\ldots, p(f(n-k+1))$ then a new permutation g_1 will be derived from the permutation f in the following way:

i) we check which of elements $p(f(n-1))$, $p(f(n-2))$, $p(f(n-3)),\ldots, p(f(n-k+1))$ is the smallest one and we remember the index of this element;

ii) we make a transposition in the permutation f, changing the element standing on the position n, with element having an index expressed in i);

It is observed that for the permutation g_1 obtained in this way, the functional $F(p,k,n)$ has lower value as for the permutation f, because according to the formula the last element of the permutations f and g_1 is the same, and the previous components in the permutation g_1 are smaller than in case of f. Thus it is proved that for the optimal permutation f the element $p(f(n))$ is smaller than elements:

$$p(f(n-1)),\ p(f(n-2)),\ p(f(n-3)),\ldots, p(f(n-k+1)).$$

Next, we assume that for the optimal permutation f, the element $p(f(n-1))$ has a greater value than any of elements:

$$p(f(n-2)),\ p(f(n-3)),\ldots, p(f(n-k+1)).$$

A new permutation is derived from the permutation f in the following way:

i) it is checked which one of elements $p(f(n-2))$, $p(f(n-3)),\dots,p(f(n-k+1))$ has the smallest value and the index of this element is recorded. ("index" means in this case argument);

ii) in the permutation f the element on the position $(n-1)$ is transposed with the element having the index as in i).

Note that in case of permutations f and g_2: the last and the penultimate components are the same, while the previous components are greater than in case of permutation f. Conclusion: the value of the functional $F(p,k,n)$ is greater for the argument f than for g_2, what is in contradiction with the optimum of f. It is proved that

$$p(f(n-1)) \geq p(f(n)).$$

Next, suppose that in the optimal permutation f the value $p(f(n-i))$ $(i < k-1)$ is greater than any of the following numbers:

$$p(f(n-i-1)),\ p(f(n-i-2)),\dots,p(f(n-k+1)).$$

A new permutation g_{i+1} is derived from the permutation f in the following way:

i) it will be checked which of elements $p(f(n-i-1))$, $p(f(n-i-2)),\dots,p(f(n-k+1))$ has the smallest value and index of this element (word index means in this case argument) will be remembered;

ii) in the permutation f the element being on position $(n-i)$ is exchanged with an element having index as the one in i).

Observe that in case of permutation f and g_{i+1} the last $(i+1)$ components are the same while earlier components are greater in case of permutation f. The proof is complete.

Theorem 2.2. *Let there be given integers n, k such that* $n \geq 2k-1$ *and let there be given functions p and q by* (1.1) (1.3). *If the functional* $R(p,k,n)$ *has a maximal value for a permutation f then following conditions are satisfied:*

$$p(f(1) \leq p(f(2)) \leq \dots \leq p(f(k)) \tag{2.6}$$

Proof. Let there be given integers n, k such that $n \geq 2k-1 \geq 0$ and let there be given functions p and q described earlier. Let f be an optimal permutation for the functional $R(p,k,n)$. Next, the permutation f^* derived from the permutation f, through changing the order of elements is considered; more exactly, if $f = \langle f(1),\ f(2),\ f(3),\dots,f(n)\rangle$ then $f^* = \langle f(n),\ f(n-1),\dots,f(3),$ $f(2), f(1)\rangle$. Because of the functional definition (see [2] and [3] the permutation f^* is optimal too. According to Theorem 2.1.:

$$p(f^*(n-k+1)) \geq p(f^*(n-k+2)) \geq \dots \geq p(f^*(n)). \tag{2.7}$$

Since

$$f^*(n-k+1) = f(k),\ f^*(n-k+2) = f(k-1), \dots, f^*(n) = f(1); \tag{2.8}$$

we have

$$p(f(k)) \geq p(f(k-1)) \geq \dots \geq p(f(1)). \tag{2.9}$$

This proves Theorem 2.2.

3. Example

Let $k = 3$, $n = 5$ and let there be given function p having only three different values, where for two elements from set N_5 let it have an "a" value, for two elements let it have a "b" value and for one element let it have a value "c", and at the same time let the condition:

$$0 \leq a \leq b \leq c \leq 1 \tag{3.1}$$

be fulfilled. If $p(f(s)) = a$ where $s \in N_n$ then the element $f(s)$ of permutation f is denoted by S, whereas if $p(f(s)) = b$, then the element $f(s)$ is denoted by $\not{S}$ and in case when $p(f(s)) = c$ then the element $f(s)$ is denoted by M.

Corollary. 3.1. (*The conclusion from Theorems* 2.1. *and* 2.2.)
Let there be given the permutation set S_5 *and the functional* $R(p,3,5)$ *and let there be satisfied the earlier described requirements. The functional* $R(p,3,5)$ *has the maximal value for such a permutation, which corresponds to the sequence:*

$$(S, \not{S}, M, \not{S}, S); \tag{3.2}$$

Proof. On the basis of Theorems 2.1. and 2.2 it is obvious that the only element M in the permutation has to be in the middle. Taking both Theorems into account there are theree possibilities:

$$(S, S, M, \not{S}, \not{S});\ (\not{S}, \not{S}, M, S, S);\ (S, \not{S}, M, \not{S}, S); \tag{3.3}$$

The class of permutations corresponding to the first sequence is denoted by f_1, to the second sequence by f_2, and corresponding to the third one by f_3. It is sufficient to show that for a permutation corresponding to the third sequence the functional $F(p,3,5)$ has the least value. Let there be put the following denotation $A = 1-a$, $B = 1-b$, $C = 1-c$; Values of the functional $F(p,3,5)$ on these permutations are the following:

$$F(\bar{p}(f_1,5),3,5) = AAC+(1-A)ACB+(1-A)CBB;$$

$$F(\bar{p}(f_2,5),3,5) = BBC+(1-B)BCA+(1-B)CAA;$$

$$F(\bar{p}(f_3,5),3,5) = ABC+(1-A)BCB+(1-B)CBA;$$

Thus:

$$F(\bar{p}(f_1,5),3,5)-F(\bar{p}(f_3,5),3,5) = AC(A-B)(1-B) > 0; \tag{3.4}$$

$$F(\bar{p}(f_2,5),3,5)-F(\bar{p}(f_3,5),3,5) = AC(A-B)(1-B) > 0; \tag{3.5}$$

is obtained. This proves Conclusion 3.1.

4. Reliability interpretation

Functionals $F(p,k,n)$ and $R(p,k,n)$ have a reliability interpretation connected with well known consecutive *k-out-of-n* systems. The number n on which functionals depend means the number of elements of the system, whereas the k shows the number of consecutive components of the system which put the system out of order. Element p_i of the vector $\bar{p}(m)$ (see section 1) can be interpreted as probability that the component i of the system works properly. It is known that components of the system are permutable. The vector $\bar{p}(f,n)$ is explained as the readiness vector, which corresponds to the new system made of the old system by arrangement of elements of the system according to permutation f, i.e. that at the place i there is a component of the previous system denoted by number $f(i)$ $(i = 1,2,...,n)$. The value of the functional $R(p,k,n)$ (f) can be interpreted as the reliability consecutive *k-out-of-n* which has the following readiness vector: $\bar{p}(f,m) = \langle p\circ f(1),\ p\circ f(2),...,p\circ f(m)\rangle$. Adequately functional $F(p,k,n)$ (f) denotes deceptiveness of the new system. Theorems 2.1 nad 2.2 in the reliability language can be put into words:

Theorem 4.1. *Let there be given a consecutive k-out-of-n system, such that $n \geq 2k-1$. Then in the optimal system (in the reliability sense) the last k-components are ordered not in an increasing manner. (reliabilities of elements are ordered not in an increasing manner).*

Theorem 4.2. Let there be given a consecutive k-out-of-n system satisfying the condition $n \geq 2k-1$, then the optimal arrangement in a reliability sense has on the first k-places the elements ordered not in an decreasing manner (in the sense of reliabilities of components).

Corollary 3.1. can be expressed in the form:

Remark. 4.1. (On basis of Theorem 2.1. and Theorem 2.2.)
Let there be givena consecutive 3-out-of-5 system and reliability of system elements satisfies conditions:

$$p_1 = p_2 < p_3 = p_4 < p_5; \tag{4.1}$$

Moreover, let there be set: a) that the system is out of order if and only if three consecutive system elements are out of order; b) these elements of the system are permutable.
The optimal arrangement of elements (in the reliability sense) of the above system is the following:

$$(S, \not{s}, M, \not{s}, S); \tag{4.2}$$

where elements of reliability p_1 or p_2 denoted by S, $\not{s}$ correspond to elements of reliability of p_3 or p_4 whereas M means an element of reliability p_5.

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(Poznań University of Technology, Institute of Mathematics, 60-965 Poznań, Poland)
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