FASCICULI MATHEMATICI

Nr 25 1995

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ON THE CHARACTERIZATION OF THE EXPONENTIAL DISTRIBUTION BY DISTRIBUTIONAL PROPERTIES OF SPACING BETWEEN RECORD VALUES WITH RANDOM INDEX

In this paper we give some characterizations of the exponential distribution by distributional properties of spacing between record values. These characterizations are considered for distribution functions belonging to *NBU* or *NWU*. The index of record values has the geometric distribution.

Key words: characteristic function, record values, *NBU*, *NWU*.

1. Introduction

Let X be a nonnegative random variable, and let $F(x) = P(X < x)$ be its distribution function. Let $\bar{F}(x) = 1 - F(x)$ be a survival function of X.

We say that F is an increasing failure rate ($F \in IFR$) distribution if $\frac{\bar{F}(t+x)}{\bar{F}(t)}$ is nonincreasing in $t \in (-\infty, \infty)$ for each $x \geqslant 0$ ([3]). Similarly, F is a decreasing failure rate ($F \in DFR$) distribution if $\frac{\bar{F}(t+x)}{\bar{F}(t)}$ is nondecreasing in $t \geqslant 0$ for each $x \geqslant 0$ ([3]).

A distribution F is *NBU* (*NWU*) if $\bar{F}(x+y) \leqslant (\geqslant) \bar{F}(x)\bar{F}(y)$ for $x \geqslant 0$, $y \geqslant 0$ ([3]).

We say that X is exponentially distributed if

(1) $$F(x) = 1 - e^{-\lambda x} \ (x > 0) \quad \text{for some } \lambda > 0.$$

We say that v is geometrically distributed if

(2) $$P(v = k) = p(1-p)^{k-1}, \qquad k = 1, 2, \ldots, \text{ for some } 0 < p < 1.$$

Let $(X_n, n \geqslant 1)$ be a sequence of independent and identically distributed random variables. Define the sequence of record times $(L(n), n \geqslant 1)$ in the following way

$$L(1) = 1, \ L(n) = \min\{j: X_j > X_{L(n-1)}\}, \quad n \geqslant 2.$$

Then the sequence $(R_n, n \geq 1)$, where $R_n = X_{L(n)}$, is called the sequence of record values of $(X_n, n \geq 1)$.

The following theorem is given in [2] (Theorem 8.1, p. 63):

Let $(X_n, n \geq 1)$ be a sequence of independent and identically distributed nonnegative and nondegenerate random variables with a distribution function F. Assume that ν is a geometric random variable independent of the sequence $(X_n, n \geq 1)$, and the condition (2) holds. The random variables X_1 and $p \sum_{j=1}^{\nu} X_j$ are identically distributed if and only if F is of the form (1).

Ahsanullah (e.g. [2], Theorem 5.5, p. 43) proved that for a sequence of independent and identically distributed nonnegative, and nondegenerate random variables $(X_n, n \geq 1)$ with an absolutely continuous distribution function $F \in NBU$ (or $F \in NWU$), the equality of the distributions of X_1 and $R_n - R_{n-1}$ (for some fixed $n > 1$) characterizes the exponential distribution.

Moreover, the following theorem given in [5] (Theorem 1, p. 160) is valid:

Let $(X_n, n \geq 1)$ be a sequence of independent and identically distributed random variables having absolutely continuous distribution function $F \in IFR$ (or $F \in DFR$) such that $\inf\{x\colon F(x) > 0\} = 0$. Moreover, let the density function f of X_n's satisfy the condition $f(x) > 0$ for $x > 0$. Then for some fixed m, n, $1 \leq m < n$, the random variables R_{n-m} and $R_n - R_m$ are identically distributed if and only if F is a distribution function of the exponential law.

2. Results

Theorem 1. *Let* $(X_n, n \geq 1)$ *be a sequence of independent and identically distributed nonnegative random variables with an absolutely continuous distribution function* $F \in NBU$ *such that* $\inf\{x\colon F(x) > 0\} = 0$. *Moreover, let the density function* f *of* X_n*'s satisfy the condition* $f(x) > 0$ *for* $x > 0$. *Assume that* ν *is a geometric random variable independent of the sequence* $(X_n, n \geq 1)$, *and the condition* (2) *holds. Then for some fixed* $n \in \mathbb{N}$ *the random variables*

$$p \sum_{j=1}^{\nu} X_j \quad \textit{and} \quad R_{n+1} - R_n$$

are identically distributed if and only if F *is of the form* (1).

Proof. It is known that the density function of $R_{n+1} - R_n$ is of the form

$$(3) \qquad f_{R_{n+1}-R_n}(z) = \frac{1}{(n-1)!} \int_0^{\infty} \frac{f(u)}{\bar{F}(u)} [-\log \bar{F}(u)]^{n-1} f(u+z)\, du, \quad z > 0.$$

Let φ_1 be the characteristic function of $R_{n+1}-R_n$. For $t\in\mathbb{R}$ we have

$$\begin{aligned}\varphi_1(t) &= \frac{1}{(n-1)!}\int_0^\infty\int_0^\infty e^{itv}\frac{f(u)}{\bar F(u)}[-\log \bar F(u)]^{n-1}f(u+v)\,du\,dv = \\ &= 1+\frac{1}{(n-1)!}\,it\int_0^\infty\int_0^\infty e^{itv}\frac{f(u)}{\bar F(u)}[-\log \bar F(u)]^{n-1}\bar F(u+v)\,du\,dv.\end{aligned} \tag{4}$$

Let φ and φ_2 be the characteristic functions of X_1 and $p\sum_{j=1}^{\nu} X_j$, respectively. For $t\in\mathbb{R}$ we get

$$\begin{aligned}\varphi_2(t) &= E\left[\exp(itp\sum_{j=1}^{\nu} X_j)\right] = \sum_{k=1}^{\infty} p(1-p)^{k-1}[\varphi(pt)]^k = \\ &= \frac{p\varphi(pt)}{1-q\varphi(pt)}, \quad \text{where } q=1-p.\end{aligned} \tag{5}$$

Suppose that $p\sum_{j=1}^{\nu} X_j$ and $R_{n+1}-R_n$ are identically distributed for some fixed $n\in\mathbb{N}$. Then $\varphi_1(t)=\varphi_2(t)$ for all $t\in\mathbb{R}$. Comparing (5) with (4) we obtain on simplification

$$\frac{\varphi(pt)-1}{1-q\varphi(pt)}\cdot\frac{1}{it} = \frac{1}{(n-1)!}\int_0^\infty\int_0^\infty e^{itv}\frac{f(u)}{\bar F(u)}[-\log \bar F(u)]^{n-1}\bar F(u+v)\,du\,dv. \tag{6}$$

Taking limits of both sides of (6) as t goes to 0, we have

$$\frac{\varphi'(0)}{i} = \frac{1}{(n-1)!}\int_0^\infty\int_0^\infty f(u)[-\log \bar F(u)]^{n-1}\frac{\bar F(u+v)}{\bar F(u)}\,du\,dv. \tag{7}$$

Writing $EX_1 = \dfrac{\varphi'(0)}{i} = \int_0^\infty \bar F(v)\,dv$ and $(n-1)! = \int_0^\infty f(u)[-\log \bar F(u)]^{n-1}du$, we get from (7)

$$\int_0^\infty\int_0^\infty f(u)[-\log \bar F(u)]^{n-1}\left[\bar F(v)-\frac{\bar F(u+v)}{\bar F(u)}\right]du\,dv = 0. \tag{8}$$

Since $F\in NBU$, we have $\bar F(v)-\dfrac{\bar F(u+v)}{\bar F(u)}\geqslant 0$ for $u,v>0$.

Further $f(x) > 0$ for $x > 0$. Therefore from (8) it follows that

$$\bar{F}(u+v) = \bar{F}(u)\bar{F}(v) \quad \text{for almost all } u, v > 0. \tag{9}$$

The only solution of (9) among nondegenerate distribution functions is the exponential distribution (1).

Now suppose that X_1 has the distribution function (1). It is known ([4], p. 70) that the statistic $\sum_{j=1}^{\nu} X_i$ is exponentially distributed with the scale parameter $p\lambda$. Hence the statistic $p\sum_{j=1}^{\nu} X_j$ has the exponential distribution with the scale parameter λ. From (3) we obtain $f_{R_{n+1}-R_n}(z) = \lambda e^{-\lambda z}$ for $z > 0$. Therefore $p\sum_{j=1}^{\nu} X_j$ and $R_{n+1}-R_n$ are identically distributed ($n \in \mathbb{N}$).

Theorem 2. *Assume that there are satisfied the assumptions of Theorem* 1. *Then the random variables*

$$p\sum_{j=1}^{\nu} X_j \quad \text{and} \quad R_{\nu+1}-R_\nu$$

are identically distributed if and only if F is the exponential distribution function (1).

Proof. By (3), the probability density function of $R_{\nu+1}-R_\nu$ is given by

$$\begin{aligned} f_{R_{\nu+1}-R_\nu}(z) &= \sum_{n=1}^{\infty} p(1-p)^{n-1} f_{R_{n+1}-R_n}(z) = \\ &= p\int_0^{\infty} f(u)[\bar{F}(u)]^{p-2} f(u+z)\,du \quad \text{for } z > 0. \end{aligned} \tag{10}$$

Let φ_3 be the characteristic function of $R_{\nu+1}-R_\nu$. For $t \in \mathbb{R}$ we have

$$\begin{aligned} \varphi_3(t) &= p\int_0^{\infty}\int_0^{\infty} e^{itv} f(u)\,[\bar{F}(u)]^{p-2} f(u+v)\,du\,dv = \\ &= 1 + pit\int_0^{\infty}\int_0^{\infty} e^{itv} f(u)\,[\bar{F}(u)]^{p-2}\,\bar{F}(u+v)\,du\,dv. \end{aligned} \tag{11}$$

First suppose that $p\sum_{j=1}^{\nu} X_j$ and $R_{\nu+1}-R_\nu$ are identically distributed. Then $\varphi_2(t)=\varphi_3(t)$ for $t\in\mathbb{R}$. Comparing (11) with (5) we obtain on simplification

$$\frac{1}{i}\cdot\frac{\varphi(pt)-1}{pt}\cdot\frac{p}{1-q\varphi(pt)}=p\int_0^\infty\int_0^\infty e^{itv}f(u)\,[\bar F(u)]^{p-2}\,\bar F(u+v)\,du\,dv.$$

Taking limits of both sides of the above equality as t goes to 0, and writing $\frac{\varphi'(0)}{i}=EX_1=\int_0^\infty \bar F(v)\,dv$, we have

$$\int_0^\infty \bar F(v)\,dv=\int_0^\infty\int_0^\infty pf(u)\,[\bar F(u)]^{p-2}\bar F(u+v)\,du\,dv. \tag{12}$$

Since $\int_0^\infty pf(u)\,[\bar F(u)]^{p-1}du=1$, we get from (12)

$$\int_0^\infty\int_0^\infty pf(u)\,[\bar F(u)]^{p-2}\,[\bar F(u)\,\bar F(v)-\bar F(u+v)]\,du\,dv=0. \tag{13}$$

Since $F\in NBU$, we have $\bar F(u)\,\bar F(v)-\bar F(u+v)\geqslant 0$ for $u,v>0$. By the assumption $f(x)>0$, $x>0$, and by (13) it follows the equation (9). Therefore F is of the form (1).

If X_1 has the distribution function (1), then the statistics $p\sum_{j=1}^{\nu} X_j$ and $R_{\nu+1}-R_\nu$ have the same distribution function as X_1.

Theorem 3. *Assume that there are satisfied the assumptions of Theorem* 1. *The random variables*

$$X_1 \quad \textit{and} \quad R_{\nu+1}-R_\nu$$

are identically distributed if and only if F is of the form (1).

Proof. If X_1 has an exponential distribution (1), then from the proof of Theorem 2 we infer that $R_{\nu+1}-R_\nu$ has the same distribution as X_1.

Now we assume that X_1 and $R_{\nu+1}-R_\nu$ are identically distributed. Then $\varphi(t)=\varphi_3(t)$ for $t\in\mathbb{R}$. Comparing $\varphi(t)$ with (11) we obtain on simplification

$$\frac{1}{i}\cdot\frac{\varphi(t)-1}{t}=\int_0^\infty\int_0^\infty pe^{itv}f(u)\,[\bar F(u)]^{p-2}\,\bar F(u+v)\,du\,dv.$$

Taking limits of both sides of the above equality as t goes to 0, we get the equation (12). Hence, by the same way as in the proof of Theorem 2, we conclude that F is of the form (1).

Remark 1. Theorems 1, 2, 3 are true if the condition "$F \in NBU$" is replaced by "$F \in NWU$ and $EX_1 < \infty$".

Theorem 4. *Assume that there are satisfied the assumptions of Theorem* 1. *Then for some fixed* $k \in \mathbb{N}$ *the random variables*

$$R_\nu \quad \text{and} \quad R_{k+\nu} - R_k$$

are identically distributed if and only if X_1 *has the distribution function of the form* (1).

Proof. We know that the density functions of $R_{k+n} - R_k$ $(k, n \in \mathbb{N})$ is given by

$$f_{R_{k+n}-R_k}(z) = \frac{1}{(n-1)!\,(k-1)!} \int_0^\infty \frac{f(u)}{\bar{F}(u)} [-\log \bar{F}(u)]^{k-1} \cdot \left[-\log \frac{\bar{F}(u+z)}{\bar{F}(u)}\right]^{n-1} f(u+z)\,du \quad \text{for } z > 0.$$

So the probability density function of $R_{k+\nu} - R_k$ can be written by

$$f_{R_{k+\nu}-R_k}(z) = \sum_{n=1}^{\infty} p(1-p)^{n-1} f_{R_{k+n}-R_k}(z) =$$

$$(14) \qquad = \frac{1}{(k-1)!} p \int_0^\infty \frac{f(u)}{\bar{F}(u)} [-\log \bar{F}(u)]^{k-1} \left[\frac{\bar{F}(u+z)}{\bar{F}(u)}\right]^{p-1} f(u+z)\,du, \quad z > 0.$$

Let φ_4 be the characteristic functions of $R_{k+\nu} - R_k$. From (14) we have

$$\varphi_4(t) = \int_0^\infty \int_0^\infty e^{itv} \frac{p}{(k-1)!} \frac{\mathrm{f}(u)}{[\bar{F}(u)]^p} [-\log \bar{F}(u)]^{k-1} f(u+v) \cdot$$

$$(15) \qquad \cdot [\bar{F}(u+v)]^{p-1} du\,dv = 1 + \frac{it}{(k-1)!} \int_0^\infty \int_0^\infty e^{itv} f(u) \cdot$$

$$\cdot [-\log \bar{F}(u)]^{k-1} \left[\frac{\bar{F}(u+v)}{\bar{F}(u)}\right]^p du\,dv, \quad t \in \mathbb{R}.$$

Since for $n \in \mathbb{N}$ the probability density function of R_n is of the form

$$f_{R_n}(z) = \frac{1}{(n-1)!}[-\log \bar{F}(z)]^{n-1} f(z), \quad z > 0,$$

we get

$$\text{(16)} \qquad f_{R_\nu}(z) = \sum_{n=1}^{\infty} p(1-p)^{n-1} f_{R_n}(z) = p[\bar{F}(z)]^{p-1} f(z) \quad \text{for } z > 0.$$

Let us denote by φ_5 the characteristic function of R_ν. From (16) we have

$$\text{(17)} \qquad \varphi_5(t) = \int_0^{\infty} e^{itv} p[\bar{F}(v)]^{p-1} f(v)\, dv = 1 + it \int_0^{\infty} e^{itv}[\bar{F}(v)]^p dv, \quad t \in \mathbb{R}.$$

First, let R_ν and $R_{k+\nu} - R_k$ be identically distributed. Then $\varphi_4(t) = \varphi_5(t)$ for $t \in \mathbb{R}$. Comparing (15) with (17) we obtain

$$\text{(18)} \qquad \begin{aligned} &\int_0^{\infty}\int_0^{\infty} e^{itv} f(u)[-\log \bar{F}(u)]^{k-1} \left[\frac{\bar{F}(u+v)}{\bar{F}(u)}\right]^p du\, dv = \\ &= (k-1)! \int_0^{\infty} e^{itv}[\bar{F}(v)]^p dv, \quad t \in \mathbb{R}. \end{aligned}$$

Since $(k-1)! = \int_0^{\infty} f(u)[-\log \bar{F}(u)]^{k-1} du$, it follows from (18) that

$$\text{(19)} \qquad \int_0^{\infty}\int_0^{\infty} e^{itv} f(u)[-\log \bar{F}(u)]^{k-1} \left\{\left[\frac{\bar{F}(u+v)}{\bar{F}(u)}\right]^p - [\bar{F}(v)]^p\right\} du\, dv = 0, \quad t \in \mathbb{R}.$$

Taking limits of both sides of (19) as t goes to 0, we have

$$\text{(20)} \qquad \int_0^{\infty}\int_0^{\infty} f(u)[-\log \bar{F}(u)]^{k-1} \left\{\left[\frac{\bar{F}(u+v)}{\bar{F}(u)}\right]^p - [\bar{F}(v)]^p\right\} du\, dv = 0.$$

Since $F \in NBU$, we obtain $\left[\frac{\bar{F}(u+v)}{\bar{F}(u)}\right]^p - [\bar{F}(v)]^p \leqslant 0$, for $u, v > 0$. Applying the assumptions of Theorem 4 and (20), we get the equation (9). Therefore F is of the form (1).

If X_1 has the distribution function (1), then by (14) we have $f_{R_{k+\nu} - R_k}(z) = \lambda p e^{-\lambda p z}$ for $z > 0$, and by virtue of (16) we get $f_{R_\nu}(z) = \lambda p e^{-\lambda p z}$ for $z > 0$. Consequently, the statistics R_ν and $R_{k+\nu} - R_k$ are identically distributed (for every $k \in \mathbb{N}$).

Remark 2. Theorem 4 is true if the condition "$F \in NBU$" is replaced by "$F \in NWU$".

Ahsanullah [1] has proved that under the assumption that $F \in IFR$ (or $F \in DFR$) the identical distributions of

$$p \sum_{j=1}^{v} X_j \quad \text{and} \quad (n-r+1)(X_{r:n} - X_{r-1:n})$$

for some fixed r and n, where $1 \leqslant r \leqslant n$, $n \geqslant 2$, $X_{0:n} = 0$, characterize the exponential distribution. ($X_{1:n} \leqslant \dots \leqslant X_{n:n}$ are the order statistices of a random sample $(X_1, \dots, X_n)$.)

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Received on 29.12.1993.