F A S C I C U L I M A T H E M A T I C I

Nr 25 1995

DUŠAN KNEŽO, VINCENT ŠOLTÉS

EXISTENCE AND PROPERTIES OF NONOSCILLATORY SOLUTIONS OF THIRD ORDER DIFFERENTIAL EQUATION

In the paper the differential equation of the third order of the form

$$(r_2(t)(r_1(t)\varphi(y'))')'+f(t,y)=0,$$

is investigated. Nonoscillatory solutions of this equation are divided into types according to the values of the limits of the quasiderivatives. Conditions are found that guarantee the existence of the solutions of the formed types.

Key words: nonlinear differential equation with quasiderivatives, nonoscillatory solutions.

This paper is concerned with third order nonlinear differential equation of the form

$$(r_2(t)(r_1(t)\varphi(y'))')'+f(t,y)=0, \tag{1}$$

where

$$\left\{\begin{array}{l} \text{a) } r_i:[0,\infty)\to(0,\infty) \text{ are continuous for } i=1,2; \\ \text{b) } \varphi:R\to R \text{ is continuous, increasing and such that} \\ \quad \operatorname{sgn}\varphi(u)=\operatorname{sgn}u \text{ and } \varphi(R)=R; \\ \text{c) } f:[0,\infty)\times R\to R \text{ is continuous, increasing and such that} \\ \quad \operatorname{sgn}f(t,v)=\operatorname{sgn}v \text{ for every } t\ge 0. \end{array}\right. \tag{2}$$

Our main object is to investigate the existence and properties of nonoscillatory solutions of equation (1), where by solution of equation (1) we mean only proper solution of equation (1), i.e. such function $y:[T_y,\infty)\to R$, $T_y\ge 0$, which satisfies equation (1) and

$$\sup\{|y(t)|:t\ge T\}>0$$

for any $T\ge T_y$. By a nonoscillatory solution we mean a solution $y(t)$ of equation (1) if there exists $T\ge T_y$ such that $y(t)\ne 0$ for any $t\ge T$. Otherwies the solution is called oscillatory.

Throughout the paper we shall assume that

(3) $$f(t,y) \text{ is nondecreasing in } y,$$

(4) $$\int_0^\infty \frac{1}{r_2(t)}\,dt = \infty$$

and

(5) $$\int_0^\infty \left|\varphi^{-1}\left(\frac{k}{r_1(t)}\right)\right| dt = \infty$$

for every constant $k \neq 0$, where $\varphi^{-1}: R \to R$ denotes the inverse function of φ. We shall use the following notation:

(6) $$\begin{cases} \Phi_{k,T}(r_1,r_2;t) = \int_T^t \varphi^{-1}\left(\frac{1}{r_1(s)}\int_T^s \frac{k}{r_2(\tau)}\,d\tau\right) ds, & t \geq T, \\ \Phi_k(r_1,r_2;t) = \Phi_{k,0}(r_1,r_2;t), & t \geq 0. \end{cases}$$

Since (2a), (4) and (5) hold, it is obvious that

(7) $$\begin{cases} \lim_{t\to\infty} |\Phi_{k,T}(r_1,r_2;t)| = \infty \quad \text{for every } k \neq 0, \\ |\Phi_{k,T}(r_1,r_2;t)| > |\Phi_{l,T}(r_1,r_2;t)| \quad \text{pre } |k| > |l|,\ kl > 0, \\ \Phi_{k,T}(r_1,r_2;T) = 0. \end{cases}$$

Lemma 1. *Any nonoscillatory solution* $y(t)$ *of equation* (1) *is of one of the following types:*

I. $\lim_{t\to\infty} r_2(t)(r_1(t)\varphi(y'(t)))' = c_1 \neq 0,$

$\lim_{t\to\infty} |r_1(t)\varphi(y'(t))| = \infty,$

$\lim_{t\to\infty} |y(t)| = \infty,$

$\operatorname{sgn} y(t) = \operatorname{sgn} y'(t)$ *for every sufficiently large* t;

II. $\lim_{t\to\infty} r_2(t)(r_1(t)\varphi(y'(t)))' = 0,$

$\lim_{t\to\infty} |r_1(t)\varphi(y'(t))| > 0,$

$\lim_{t\to\infty} |y(t)| = \infty,$

$\operatorname{sgn} y(t) = \operatorname{sgn} y'(t)$ *for every sufficiently large* t;

III. $\lim_{t\to\infty} r_2(t)(r_1(t)\varphi(y'(t)))' = 0,$

$\lim_{t\to\infty} r_1(t)\varphi(y'(t)) = 0,$

$\lim_{t\to\infty} y(t) = c_2,$

$\operatorname{sgn} y(t) = -\operatorname{sgn} y'(t)$ *for every sufficiently large* t;

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1). Suppose $y(t) > 0$ for every $t \geq t_0 \geq T_y$ (the proof for $y(t) < 0$ is analogous). From equation (1) we have

$$r_2(t)(r_1(t)\varphi(y'(t)))')' < 0 \text{ for every } t \geq t_0,$$

and so function $r_2(t)(r_1(t)\varphi(y'(t)))'$ is decreasing on $[t_0, \infty)$. If for some $t^* \in [t_0, \infty)$ $r_2(t^*)(r_1(t^*)\varphi(y'(t^*)))' \leq 0$, then from (2b), (4) and (5) we have a contradiction to the fact, that $y(t) > 0$ for every $t \geq t_0$. It means that

$$r_2(t)(r_1(t)\varphi(y'(t)))' > 0 \text{ for every } t \geq t_0$$

and so

$$\lim_{t\to\infty} r_2(t)(r_1(t)\varphi(y'(t)))' = c_1 \geq 0$$

If $\lim_{t\to\infty} r_2(t)(r_1(t)\varphi(y'(t)))' = c_1 > 0$, then $r_2(t)(r_1(t)\varphi(y'(t)))' \geq c_1$ for every $t \geq t_0$, hence

$$r_1(t)\varphi(y'(t)) \geq r_1(t_0)\varphi(y'(t_0)) + c_1 \int_{t_0}^{t} \frac{1}{r_2(s)}\, ds$$

for every $t \geq t_0$. With regard to relation (4) we get $\lim_{t\to\infty} r_1(t)\varphi(y'(t)) = \infty$ and from this

$$y(t) > y(t_1) + \int_{t_1}^{t} \varphi^{-1}\left(\frac{k_1}{r_1(s)}\right) ds,$$

for some $k_1 > 0$ and for every $t \geq t_1 \geq t_0$, so with respect to (5) we have $\lim_{t\to\infty} y(t) = \infty$. We proved that the solution $y(t)$ is of the type I.

If $\lim_{t\to\infty} r_2(t)(r_1(t)\varphi(y'(t)))' = 0$, then with regard to the increasing function $r_1(t)\varphi(y'(t))$ two cases are possible: either there exists a $t_1 \geq t_0$ such that $r_1(t)\varphi(y'(t)) > 0$ for every $t \geq t_1$ or $r_1(t)\varphi(y'(t)) < 0$ for every $t \geq t_0$.

In the first case we have for every $t \geq t_2 \geq t_1$

$$r_1(t)\,\varphi(y'(t)) \geq r_1(t_2)\,\varphi(y'(t_2)) > 0,$$

so we obtain $\lim_{t\to\infty} r_1(t)\,\varphi(y'(t)) > 0$ and

$$y'(t) > \varphi^{-1}\left(\frac{r_1(t)\,\varphi(y'(t_2))}{r_1(t)}\right) \text{ for every } t \geq t_2.$$

From the last inequality using (5) we have $\lim_{t\to\infty} y(t) = \infty$, i.e. the solution $y(t)$ is of the type II.

In the second case $y(t)$ is a positive and decreasing function with a finite limit. It is not difficult to show that $\lim_{t\to\infty} r_1(t)\,\varphi(y'(t)) = 0$. It follows that $y(t)$ is of the type III. This completes the proof.

Remark 1. *From the proof of Lemma* 1 *it is obvious that*

$$\lim_{t\to\infty} r_2(t)(r_1(t)\,\varphi(y'(t)))' > 0$$

for positive nonoscillatory solutions of equation (1) *of the type* I *and*

$$\lim_{t\to\infty} r_2(t)(r_1(t)\,\varphi(y'(t)))' < 0$$

for negative nonoscillatory solutions of the type I.

Theorem 1. *Let equation* (1) *has a nonoscillatory solution of the type* I. *Then*

$$\int_0^\infty |f(t, c\Phi_k(r_1, r_2; t))|\,dt < \infty \tag{8}$$

for some constants $k \neq 0$ *a* $c > 0$.

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1) of the type I. We may suppose that $y(t) > 0$ for $t \geq t_0 \geq T_y$ (the proof for $y(t) < 0$ is similar). Since

$$\lim_{t\to\infty} r_2(t)(r_1(t)\,\varphi(y'(t)))' = c_1 > 0,$$

and moreover, $r_2(t)(r_1(t)\varphi(y'(t)))'$ is decreasing on $[t_0,\infty)$ we have

$$r_2(t)(r_1(t)\varphi(y'(t)))' \geq c_1 \text{ for every } t \geq t_0.$$

We can easily find out that $y(t) \geq \Phi_{c_1,t_0}(r_1,r_2;t)$ so there exists a positive constant c_2 such that

$$y(t) \geq c_2 \Phi_{c_1}(r_1,r_2;t) \text{ for every } t \geq t_0. \tag{9}$$

Integrating (1) from t to ∞ we have

$$\int_t^\infty f(s,y(s))\,ds < r_2(t)(r_1(t)\varphi(y'(t)))' < \infty.$$

From the last inequality in view of (9) and (3) we obtain

$$\int_t^\infty f(s,c_2\Phi_{c_1}(r_1,r_2;s))\,ds < \infty,$$

which implies (8) for $c_1 > 0$ a $c_2 > 0$. This completes the proof.

Theorem 2. *Suppose that for each fixed $k \neq 0$ and $T \geq 0$*

$$\lim_{l\to 0, lk>0} \frac{\Phi_{l,T}(r_1,r_2;t)}{\Phi_{k,T}(r_1,r_2;t)} = 0 \tag{10}$$

uniformly on any interval of the form $[T',\infty)$, $T' > T$. Equation (1) *has a nonoscillatory solution of the type I if* (8) *holds for some $k' \neq 0$ and $c > 0$.*

Proof. Suppose that (8) holds for some $c > 0$ and $k' > 0$. As (10) holds, there exists $l_1, k > l_1 > 0$, such that

$$\Phi_{l_1}(r_1,r_2;t) < c\Phi_{k'}(r_1,r_2;t).$$

Let $T > 0, l > 0$ be such that $2l < l_1 < k$ and

$$\int_T^\infty f(t,\Phi_{2l}(r_1,r_2;t))\,dt < l. \tag{11}$$

Let $C[T,\infty)$ be the space of all continuous functions which are defined on $[T,\infty)$ with topology of uniform convergence on compact subintervals of $[T,\infty)$. We define the set

$$Y = \{y \in C[T,\infty) : \Phi_{l,T}(r_1,r_2;t) \leq y(t) \leq \Phi_{2l,T}(r_1,r_2;t),\ t \geq T\} \tag{12}$$

and the mapping $\mathcal{T}:Y\to C[T,\infty)$

$$\mathcal{T}y(t)=\int_T^t \varphi^{-1}\left(\frac{1}{r_1(s)}\int_T^s \frac{1}{r_2(\tau)}\left(l+\int_T^\infty f(\xi,y(\xi))\,d\xi\right)d\tau\right)ds,$$

for every $t\geq T$. We will prove that the mapping $\mathcal{T}$ fulfils the assumptions of the Schauder-Tychonoff fixed point theorem.

(i) *$\mathcal{T}$ maps Y into iteself, because* if $y\in Y$, then for every $\tau\geq T$ we have

$$l\leq l+\int_\tau^\infty f(\xi,y(\xi))\,d\xi\leq\int_T^\infty f(\xi,\Phi_{2l}(r_1,r_2;\xi))\,d\xi\leq 2l,$$

$$\int_T^t\varphi^{-1}\left(\frac{1}{r_1(s)}\int_T^s\frac{l}{r_2(\tau)}\,d\tau\right)ds\leq\mathcal{T}y(t)\leq\int_T^t\varphi^{-1}\left(\frac{1}{r_1(s)}\int_T^s\frac{2l}{r_2(\tau)}\,d\tau\right)ds.$$

The last relation implies that $\mathcal{T}y\in Y$.

(ii) *$\mathcal{T}$ is continuous.* Let y_n be a sequence of elements of *Y converging to y* in topology of $C[T,\infty)$. The Lebesgue dominated convergence theorem shows that

$$\int_T^\infty f(t,y_n(t))\,dt\to\int_T^\infty f(t,y(t))\,dt\ \text{ for } n\to\infty$$

and so

$$\int_t^\infty f(s,y_n(s))\,ds\to\int_t^\infty f(s,y(s))\,ds\ \text{ for } n\to\infty$$

uniformly on $[T,\infty)$. It follows that $\mathcal{T}y_n(t)\to\mathcal{T}y(t)$ uniformly on compact subintervals of $[T,\infty)$, which imples convergence $\mathcal{T}y_n(t)\to\mathcal{T}y(t)$ in $[T,\infty)$.

(iii) *$\mathcal{T}(Y)$ is relatively compact.* For any $y\in Y$ we get the relation

$$0\leq(\mathcal{T}y)'(t)=\varphi^{-1}\left(\frac{1}{r_1(t)}\int_T^t\frac{1}{r_2(s)}\left(l+\int_s^\infty f(\tau,y(\tau))\,d\tau)\right)d\tau\right)ds\right)\leq$$

$$\leq\varphi^{-1}\left(\frac{1}{r_1(t)}\int_T^t\frac{1}{r_2(s)}\left(l+\int_s^\infty f(\tau,\Phi_{2l}(r_1,r_2;\tau)\,d\tau\right)ds\right).$$

Therefore, applying the Schauder-Tychonoff fixed point theorem we see that there exists an element $y\in Y$ such that $y=\mathcal{T}y$. Consequently,

$$y(t)=\int_T^t\varphi^{-1}\left(\frac{1}{r_1(s)}\int_T^s\frac{1}{r_2(\tau)}\left(l+\int_T^\infty f(\xi,y(\xi))\,d\xi\right)d\tau\right)ds$$

Differentiating the last integral equation we obtain that $y(t)$ is a solution of equation (1) of the type I.

Theorem 3. *Let the assumption of Theorem* 2 *hold. Then equation* (1) *has a nonoscillatory solution of the type* I *if and only if* (8) *holds for some constants* $k' \neq 0$ *and* $c > 0$.

The proof of Theorem 3 follows Theorem 1 and Theorem 2.

Theorem 4. *Equation* (1) *has a nonoscillatory solution of the type* III *with a nonzero limit for* $t \to \infty$ *if and only if*

$$\int_0^\infty \left| \varphi^{-1}\left(\frac{-1}{r_1(t)} \int_t^\infty \frac{1}{r_2(s)} \int_s^\infty f(\tau, c)\, d\tau\, ds \right) \right| dt < \infty \tag{13}$$

for some constant $c \neq 0$.

Proof. Let $y(t)$ be a positive solution of equation (1) of the type III, with $\lim_{t\to\infty} y(t) = c > 0$. Form (1), with regard to the properties of $y(t)$, we have

$$(r_1(t)\, \varphi(y'(t)))' = \frac{1}{r_2(t)} \int_t^\infty f(s, y(s))\, ds,$$

hence

$$y(t) - y(t_0) = \int_{t_0}^t \varphi^{-1}\left(\frac{-1}{r_1(s)} \int_s^\infty \frac{1}{r_2(\tau)} \int_\tau^\infty f(\xi, y(\xi))\, d\xi d\tau \right) ds. \tag{14}$$

Since $c^* \geq y(t) \geq c$, then considering that $f(t, y)$ is monotonous we obtain

$$-f(t, c^*) \leq -f(t, y(t)) \leq -f(t, c). \tag{15}$$

From (14) it is obvious that

$$0 > \int_{t_0}^\infty \varphi^{-1}\left(\frac{-1}{r_1(t)} \int_t^\infty \frac{1}{r_2(s)} \int_s^\infty f(\tau, y(\tau))\, d\tau\, ds \right) dt > -\infty,$$

and according to (15) we also have

$$\int_{t_0}^\infty \varphi^{-1}\left(\frac{-1}{r_1(t)} \int_t^\infty \frac{1}{r_2(s)} \int_s^\infty f(\tau, c)\, d\tau\, ds \right) dt > -\infty.$$

We have proved that the relation (13) holds for c. It follows therefore that the sufficient condition is true.

Now suppose that (13) holds for some $c > 0$ (the proof for $c < 0$ is similar). Then there exists a $T > 0$ such that

$$\int_T^\infty \varphi^{-1}\left(\frac{-1}{r_1(t)}\int_t^\infty \frac{1}{r_2(s)}\int_s^\infty f(\tau, c)\,d\tau\,ds\right)dt > -\frac{c}{2}.$$

Define the set

$$Y = \{y \in C[T, \infty) : \frac{c}{2} \le y(t) \le c,\ t \ge T\}$$

where $C[T, \infty)$ is the topological space as in the proof of Theorem 2 and the mapping

$$\mathcal{T}y(t) = c + \int_T^t \varphi^{-1}\left(\frac{-1}{r_1(s)}\int_s^\infty \frac{1}{r_2(\tau)}\int_\tau^\infty f(\xi, y(\xi))\,d\xi\,d\tau\right)ds,$$

for every $t \ge T$. Proceeding as in the proof of Theorem 2 we get, that for the mapping $\mathcal{T}$ the assupmtions of the Schauder-Tychonoff fixed point theorem hold which implies the existence of a continuous function $y \in Y$ such that

$$y(t) = c + \int_T^t \varphi^{-1}\left(\frac{-1}{r_1(s)}\int_s^\infty \frac{1}{r_2(\tau)}\int_\tau^\infty f(\xi, y(\xi))\,d\xi\,d\tau\right)ds.$$

Consequently, differentiation shows that $y(t)$ is the solution of equation (1) of the type III with $\lim_{t\to\infty} y(t) \ne 0$. This completes the proof.

Theorem 5. *Let the assumption of Theorem* 2 *hold. If* (8) *holds for some constants* $k' \ne 0$, $c > 0$ *and*

$$\int_0^\infty \left|\varphi^{-1}\left(\frac{1}{r_1(t)}\int_0^t \frac{1}{r_2(s)} f(\tau, d)\,d\tau\,ds\right)\right| dt = \infty \tag{16}$$

for some nonzero constant d *such that* $k'd > 0$, *then the equation* (1) *has a nonoscillatory solution of the type* II.

Proof. Suppose $k' > 0$ (the proof is similar for $k' < 0$). According to (10) and (8) there exist $l > 0$ and $T > 0$ such that

$$d + \Phi_l(r_1, r_2; t) \le c\Phi_{k'}(r_1, r_2; t) \text{ for every } t \ge T$$

and

$$\int_T^\infty f(t, d + \Phi_l(r_1, r_2; t))\,dt \le l. \tag{17}$$

Consider the set

$$Y=\{y\in C[T,\infty): d\le y(t)\le d+\Phi_l(r_1,r_2;t),\ t\ge T\}.$$

where $C[T,\infty)$ is topological space as in the proof of Theorem 2 and the mapping

$$\mathscr{T}y(t)=d+\int_T^t\varphi^{-1}\left(\frac{-1}{r_1(s)}\int_T^s\frac{1}{r_2(\tau)}\int_\tau^\infty f(\xi,y(\xi))\,d\xi\,d\tau\right)ds, \tag{18}$$

for every $t\ge T$ and $y\in Y$. From $y(t)\le d+\Phi_l(r_1,r_2;t)$ and the condition (3) it follows

$$f(t,y(t))\le f(t,d+\Phi_l(r_1,r_2;t)).$$

Using (17) we have from (18)

$$d\le\mathscr{T}y(t)\le d+\int_T^t\varphi^{-1}\left(\frac{1}{r_1(s)}\int_T^s\frac{l}{r_2(\tau)}d\tau\right)ds\le d+\Phi_l(r_1,r_2;t),$$

and so $\mathscr{T}$ maps Y into itself. In the same way as in the proof of Theorem 2 we can verify the others assumptions of the Schauder-Tychonoff fixed point theorem and therefore there exists a continuous function $y\in Y$ such that

$$y(t)=d+\int_T^t\varphi^{-1}\left(\frac{-1}{r_1(s)}\int_T^s\frac{1}{r_2(\tau)}\int_\tau^\infty f(\xi,y(\xi))\,d\xi\,d\tau\right)ds. \tag{19}$$

Differentiating the integral equation (19) we get that $y(t)$ is a solution of equation (1). Thus $y(t)\ge d$ for every $t\ge T$, we obtain

$$y(t)\ge d+\int_T^t\varphi^{-1}\left(\frac{-1}{r_1(s)}\int_T^s\frac{1}{r_2(\tau)}\int_\tau^\infty f(\xi,d)\,d\xi\,d\tau\right)ds,$$

and according to (16) $\lim_{t\to\infty}y(t)=\infty$. We see, by differentiating (19), that

$$r_1(t)\,\varphi(y'(t))=\int_T^t\frac{1}{r_2(\tau)}\int_\tau^\infty f(\xi,y(\xi))\,d\xi\,d\tau,$$

hence

$$r_2(t)(r_1(t)\,\varphi(y'(t)))'=\int_t^\infty f(\xi,y(\xi))\,d\xi.$$

and so respecting (8) we conclude

$$\lim_{t\to\infty}r_2(t)(r_1(t)\,\varphi(y'(t)))'=0.$$

It means that $y(t)$ is a solution of the type II and the theorem is proved.

Definition 1. *We say that equation* (1) *is strongly superlinear if there exists a constant* $\gamma > 0$ *such that* $|v|^{-\gamma}|f(t,v)|$ *is nondecreasing in* $|v|$ *for each fixed t and*

$$(20) \qquad \begin{cases} \int\limits_{M}^{\infty} \dfrac{dv}{\varphi^{-1}(v^{\gamma})} < \infty \\ \int\limits_{-M}^{-\infty} \dfrac{dv}{\varphi^{-1}(v^{\gamma *})} < \infty \end{cases}$$

for any $M > 0$.
Denote

$$R_2(t_0, t) = \int\limits_{t_0}^{t} \frac{1}{r_2(s)} ds \quad \text{for } t \geq t_0 .$$

Theorem 6. *Let the equation* (1) *be strongly superlinear and*

$$(21) \qquad \varphi^{-1}(uv) \geq \varphi^{-1}(u)\,\varphi^{-1}(v)$$

for every u, v *such that* $uv > 0$. *If*

$$(22) \qquad \int\limits_{0}^{\infty} \left| \varphi^{-1}\left(\frac{R_2(0,t)}{r_1(t)} \int\limits_{t}^{\infty} f(s,c)\,ds \right) \right| dt = \infty$$

for some constant $c \neq 0$, *then each nonoscillatory solution of equation* (1) *is of the type* III.

Proof. Let $y(t)$ be a positive nonoscillatory solution of equation (1) that is not of the type III. In consequence of Lemma 1 we have that it is of the type I or II. So there exists $t \geq t_0$ such that $y(t) \geq c$ for each $t \geq t_1$. Integrating (1) from t *to* ∞ leads to

$$(r_1(t)\,\varphi(y'(t)))' \geq \frac{1}{r_2(s)} \int\limits_{t}^{\infty} f(s, y(s))\,ds ,$$

hence

$$r_1(t)\,\varphi(y'(t)) \geq \int\limits_{t_1}^{t} \frac{1}{r_2(s)} \int\limits_{s}^{\infty} f(\tau, y(\tau))\,d\tau\,ds .$$

From the last relation by interchanging the order of integration we obtain

$$r_1(t)\,\varphi(y'(t)) \geq R_2(t_1, t) \int\limits_{t}^{\infty} f(\tau, y(\tau))\,d\tau ,$$

and so

$$y'(t) \geq \varphi^{-1}\left(\frac{R_2(t_1,t)}{r_1(t)} \int_t^{\infty} f(\tau, y(\tau))\, d\tau\right)$$

for every $t \geq t_1$. Now we divide the last inequality by the function $\varphi^{-1}((y(t))^\gamma)$, where $\gamma > 0$ is the constant of Definition 1 and we use (21). We obtain

$$\frac{y'(t)}{\varphi^{-1}((y(t))^\gamma)} \geq \varphi^{-1}\left(\frac{R_2(t_1,t)}{r_1(t)} \int_t^{\infty} \frac{f(\tau, y(\tau))}{(y(t))^\gamma}\, d\tau\right).$$

Since $y(t)$ is an increasing function and equation (1) is superlinear

$$\frac{f(\tau, y(\tau))}{(y(t))^\gamma} \geq \frac{f(\tau, y(\tau))}{(y(t))^\gamma} \geq \frac{f(\tau, c)}{c^\gamma}$$

for all $t \geq t_1$, so that

$$\frac{y'(t)}{\varphi^{-1}((y(t))^\gamma)} \geq \varphi^{-1}(c^{-\gamma})\, \varphi^{-1}\left(\frac{R_2(t_1,t)}{r_1(t)} \int_t^{\infty} f(\tau, c)\, d\tau\right).$$

Integrating the last inequality from t_1 to t we have

$$\int_{y(t_1)}^{y(t)} \frac{dv}{\varphi^{-1}(v^\gamma)} \geq \varphi^{-1}(c^{-\gamma}) \int_{t_1}^{t} \varphi^{-1}\left(\frac{R_2(t_1,s)}{r_1(s)} \int_s^{\infty} f(\tau, c)\, d\tau\right) ds,$$

which, in view of (22), contradicts (20). This completes the proof in the case when $y(t) > 0$. The proof of the case when $y(t) < 0$ is analogous.

Theorem 7. Let

$$\lim_{t\to\infty}\left(R_2(0,t) \int_t^{\infty} |f(\tau, c)|\, d\tau + \int_0^{t} R_2(0,t)|f(\tau, c)|\, d\tau\right) = \infty \tag{23}$$

for every constant $c \neq 0$. Then for any bounded nonoscillatory solution $y(t)$ of (1), $\lim_{t\to\infty} y(t) = 0$.

Proof. Let $y(t)$ be a bounded nonoscillatory solution of (1) such that $y(t) > 0$ for all $t \geq t_0$ and $\lim_{t\to\infty} y(t) = c > 0$. According to Lemma 1 it is of the type III. Integrating (1) from t to ∞ we obtain

$$(r_1(t)\, \varphi(y'(t)))' = \frac{1}{r_2(t)} \int_t^{\infty} f(\tau, y(\tau))\, d\tau \geq \frac{1}{r_2(t)} \int_t^{\infty} f(\tau, c)\, d\tau,$$

and so

$$r_1(t)\,\varphi(y'(t)) - r_1(t_0)\,\varphi(y'(t_0)) \geq \int_{t_0}^{t} \frac{1}{r_2(s)} \int_s^{\infty} f(\tau, c)\, d\tau\, ds$$

for every $t \geq t_0$. It follows that

$$-r_1(t_0)\,\varphi(y'(t_0)) \geq \int_t^{\infty} R_2(t_0, t) f(\tau, c)\, d\tau + \int_{t_0}^{t} R_2(t_0, \tau) f(\tau, c)\, d\tau$$

for every $t \geq t_0$ and this is a contradiction to (23). This completes the proof in the case $y(t) > 0$. Similar arguments we can use in the case $y(t) < 0$.

Theorem 8. *Let*

$$\int_0^{\infty} \left| \frac{1}{r_2(t)} \int_t^{\infty} f(s, c)\, ds \right| dt = \infty$$

for every constant $c \neq 0$. *Then all nonoscillatory solutions of equation* (1) *are of the type* III *and converging to zero for* $t \to \infty$.

The proof is simple and therefore is omitted.

References

[1] Edwards R. E., *Functional Analysis: Theory and applications*, Holt, Rinehart and Winston, New York 1965.

[2] Elbert Á., *Asymptotic Behaviour of Autonomous Half-linear Differential Systems on the Plane*, Studia Sc. Math. Hungarica, 19, 1984, pp. 447–464.

[3] Elbert Á., *Oscillation and Nonoscillation Theorems for Some Nonlinear Ordinary Equations*, Ordinary and Partial Differential Equations, Lecture Notes in Math., No. 964, pp. 187–212, Springer, Berlin-Heidelberg-New York 1982.

[4] Elbert Á., Kusano T., *Oscillation and Non-oscillation Theorems for a Class of Second Order Quasilinear Differential Equations*, Acta. Math. Hung. 56 (3–4), 1990, pp. 325–336.

[5] Mirzov D. D., *Ob koleblemosti rešenij odnoj nelinejnoj differencial'noj sistemy tipa Emdena-Fowlera*, Diff. Uravnenija, Tom XVI, No 11, 1980 pp. 1980-1984.

[6] Mirzov D. D., *Nekotoryje asimptotičeskije svojstva rešenij odnoj sistemy tipa Emdena-Fowlera*, Diff. Uravnenija, 23, 1987, pp. 1519-1532.

(Department of Mathematics, Technical University, Košice, Slovak Republic)
Received on 14.2.1994.