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JADWIGA KORCZAK, MAŁGORZATA MIGDA

COMPARISON THEOREMS FOR DIFFERENCE EQUATIONS

In the paper some comparison theorems for nonlinear difference equations of higher orders are presented.

Key words: difference equation, initial conditions, comparison theorems.

The comparison theorems play a very important role in the theory of difference equations as well as in the theory of differential equations. For the case of 2-nd order difference equations the problem was considered by Hooker, Patula [2], Olver [3], Patual [4]. Comparison theorems for difference equations of higher orders were considered by Popenda [5]. See also to the recent monograph by Agarwal [1] and the papers quoted there.

In this paper we present the comparison theorems for some types of nonlinear difference equations of an arbitrary order.

Let N denote the set of positive integers, R the set of real numbers and R_+ the set of positive real numbers. For a given function $x: N \to R$ we define the difference operators Δ^i as follows

$$\Delta^o x_n = x_n,\ \Delta^k x_n = \Delta(\Delta^{k-1} x_n) = \Delta^{k-1} x_{n+1} - \Delta^{k-1} x_n, \quad k \geq 1$$

where $x_n = x(n)$.

Theorem 1. *Let y and z be respectively the solutions of equations*

$$\Delta^m y_n + f(n, y_n) = 0, \quad n \in N \tag{E1}$$

$$\Delta^m z_n + g(n, z_n) = 0, \quad n \in N \tag{E2}$$

where $f, g: N \times R \to R_+$. If $g(n, x) \geq f(n, x)$ for all $n \geq 1$ and $x \in R$, f or g is a nondecreasing function with respect to x for each $n \in N$, and if y and z satisfy the conditions

$$\Delta^\mu (y_1 - z_1) \geq 0, \qquad \mu = 0, 1, ..., m-1, \tag{1}$$

then

$$\Delta^{\mu}(y_{n+1}-z_{n+1}) \geq \Delta^{\mu}(y_n-z_n) \geq 0, \quad n \geq 1, \quad \mu = 0,1,...m-1. \tag{2}$$

Proof. In the proof we will apply the mathematical induction. Let $n = 1$. From (1) we have $\Delta^{\mu}(y_1-z_1) \geq 0$ for $\mu = 0,1,...,m-1$. Therefore

$$\Delta^{\mu-1}(y_2-z_2) \geq \Delta^{\mu-1}(y_1-z_1) \geq 0, \quad \mu = 1,...,m-1,$$

that is $\Delta^{\mu}(y_2-z_2) \geq \Delta^{\mu}(y_1-z_1) \geq 0, \quad \mu = 1,...,m-2.$
We will show that $\Delta^{m-1}(y_2-z_2) \geq \Delta^{m-1}(y_1-z_1) \geq 0$. Subtracting the members of equations (E1) and (E2) we get the equality

$$\Delta^{m}(y_n-z_n) = g(n,z_n)-f(n,y_n). \tag{3}$$

Therefore for $n = 1$ we have

$$\Delta^{m}(y_1-z_1) = g(1,z_1)-f(1,y_1).$$

But for $\mu = 0$ (1) implies that $y_1 \geq z_1$, the function f or g is nonnegative and nondecreasing, hence

$$\Delta^{m}(y_1-z_1) \geq g(1,z_1)-f(1,y_1) \geq 0$$

or

$$\Delta^{m}(y_1-z_1) \geq g(1,y_1)-f(1,y_1) \geq 0.$$

Therefore

$$\Delta^{m-1}(y_2-z_2) \geq \Delta^{m-1}(y_1-z_1) \geq 0.$$

Now, let us assume that the inequality (2) holds for $n = k$, i.e.

$$\Delta^{\mu}(y_{k+1}-z_{k+1}) \geq \Delta^{\mu}(y_k-z_k) \geq 0, \quad \mu = 0,1,...,m-1. \tag{4}$$

Then

$$\Delta^{\mu-1}(y_{k+2}-z_{k+2})-\Delta^{\mu-1}(y_{k+1}-z_{k+1}) \geq 0, \quad \mu = 1,...,m-1.$$

Hence

$$\Delta^{\mu}(y_{k+2}-z_{k+2}) \geq \Delta^{\mu}(y_{k+1}-z_{k+1}) \geq 0, \qquad \mu = 0,1,....,m-2.$$

For $n = k+1$ we get from (3)

$$\Delta^{m}(y_{k+1}-z_{k+1}) \geq g(k+1,z_{k+1})-f(k+1,y_{k+1}).$$

From assumption (4) for $\mu=0$ we obtain $y_{k+1}\geq z_{k+1}$.
Hence, by virtue of the theorem's assumptions we get

$$g(k+1,z_{k+1})-f(k+1,y_{k+1})\geq g(k+1,z_{k+1})-f(k+1,z_{k+1})\geq 0$$

or

$$g(k+1,z_{k+1})-f(k+1,y_{k+1})\geq g(k+1,y_{k+1})-f(k+1,y_{k+1})\geq 0.$$

Therefore

$$\Delta^{m-1}(y_{k+2}-z_{k+2})\geq \Delta^{m-1}(y_{k+1}-z_{k+1}).$$

Thus inequality (2) is true for all $n\geq 1$. This completes the proof of Theorem 1.

Remark. If we put $f\equiv 0$ in equation (E1) then the theorem given above can be used for estimation of solutions of equation (E2) by the polynomial functions.

Example. Let us consider the following equations

(e1) $$\Delta^3 y_n = 0$$

and

(e2) $$\Delta^3 z_n + n z_n^2 = 0.$$

Let $y_1=z_1=1$, $\Delta y_1=\Delta z_1=3$, $2=\Delta^2 y_1\geq \Delta^2 z_1=0$ be the initial conditions. It is easy to verify that the solution of equation (e1), satisfying the above initial conditions is a polynomial $y=n^2$. By virtue of Theorem 1 the solution of equation (e2), defined by the given initial conditions, has the estimation $z_n\leq n^2$ for all $n\in N$.

Theorem 2. *Let the assumptions of Theorem* 1 *hold. Then we have*

(5) $$\Delta^{\mu}(y_n-z_n)\geq \sum_{i=0}^{m-\mu-1}\binom{n-1}{i}\Delta^{i+\mu}(y_1-z_1),\quad n\geq 1,\quad \mu=0,1,\dots,m-1.$$

Proof. From Theorem 1 (for $\mu=0$) it follows the sequence $\{\Delta^{m-1}(y_n-z_n)\}_{n=1}^{\infty}$ is nondecreasing. Therefore

$$\Delta^{m-1}(y_n-z_n)\geq \Delta^{m-1}(y_1-z_1).$$

Summing these inequalities over n we get

$$\sum_{j=1}^{n-1} \Delta^{m-1}(y_j - z_j) \geq \sum_{j=1}^{n-1} \Delta^{m-1}(y_1 - z_1).$$

Hence

$$\Delta^{m-2}(y_n - z_n) \geq \Delta^{m-2}(y_1 - z_1) + (n-1)\Delta^{m-1}(y_1 - z_1).$$

Again by summation over n we obtain

$$\sum_{j=1}^{n-1} \Delta^{m-2}(y_j - z_j) \geq \sum_{j=1}^{n-1} \Delta^{m-2}(y_1 - z_1) + \sum_{j=1}^{n-1} (j-1)\Delta^{m-1}(y_1 - z_1).$$

Hence

$$\Delta^{m-3}(y_n - z_n) \geq \Delta^{m-3}(y_1 - z_1) + (n-1)\Delta^{m-2}(y_1 - z_1) + \left(\sum_{j=1}^{n-1} \binom{j-1}{1}\right)\Delta^{m-1}(y_1 - z_1)$$

.

Since $\sum_{j=1}^{n-1} \binom{j-1}{k} = \binom{n-1}{k+1}$ we have

$$\Delta^{m-3}(y_n - z_n) \geq \Delta^{m-3}(y_1 - z_1) + \binom{n-1}{1}\Delta^{m-2}(y_1 - z_1) + \binom{n-1}{2}\Delta^{m-1}(y_1 - z_1).$$

After k steps we obtain

$$\Delta^{m-k-1}(y_n - z_n) \geq \sum_{i=0}^{k} \binom{n-1}{i}\Delta^{m-k-1+i}(y_1 - z_1), \quad k = 0, 1, \ldots, m-1.$$

Hence if we denote $\mu = \mathrm{m} - \mathrm{k} - 1$, then we obtain (5) and this completes the proof of Theorem 2.

Corollary. Applying (1) *to inequalities* (5) *we obtain the estimations*

(6) $$\Delta^{\mu}(y_n - z_n) \geq \binom{n-1}{m-\mu-1}\Delta^{m-1}(y_1 - z_1), \quad \mu = 0, 1, \ldots, m-1.$$

Example. Let $f \equiv 0$. Then the equations (E1) and (E2) have the form

(e1) $$\Delta^m y_n = 0$$

and

(e2) $$\Delta^m z_n + g(n, z_n) = 0.$$

Let the initial conditions satisfy the relations $\Delta^i z_1 \leq (m-1)^{(i)}(m-1)^{(m-1-i)}$ for $i = 0,1,...,m-2$, $\Delta^{m-1} z_1 = 0$.
The solution of equation (e1) satisfying conditions
$\Delta^i y_1 = (m-1)^{(i)}(m-1)^{(m-1-i)}$ for $i = 0,1,...,m-1$ is a polynomial $y_n = (n+m-2)^{(m-1)}$. By virtue of Corollary (for $\mu = 0$) the solution of equation (e2) defined by the given initial conditions has the estimation $z_n \leq (n+m-2)^{(m-1)} - (n-1)^{(m-1)}$ for all $n \in N$.

Remark. If in equations (E1) and (E2) we will put $f \equiv g$ then from Theorem 2 it follows that two distinct solutions of equation $\Delta^m y_n + f(n, y_n) = 0$ diverge with velocity not smaller then Cn^{m-1} where C is a positive constant.

The theorem given below can be proved similarly as Theorem 1.

Theorem 3. *Let y and z are solutions of the equations*

$$\text{(E3)} \qquad \Delta(r_n^{m-1}\Delta(...\Delta(r_n^1 \Delta y_n)...)) + f(n, y_n) = 0, \quad n \in N,$$

$$\text{(E4)} \qquad \Delta(r_n^{m-1}\Delta(...\Delta(r_n^1 \Delta z_n)...)) + g(n, z_n) = 0, \quad n \in N,$$

respectively, where $r_n^i : N \to R_+$, $i = 1,2,...,m-1$ *are nondecreasing functions,* f, $g : N \times R \to R_+$. *Let* $g(n,x) \geq f(n,x)$ for all $n \geq 1$ *and* $x \in R$, f *or* g *is a nondecreasing function with respect to the last argument. If y and z satisfy the conditions*

$$\text{(7)} \qquad \Delta(r_1^{\mu}\Delta(...\Delta(r_1^1 \Delta(y_1 - z_1)...)) \geq 0, \quad \mu = 2,...,m-2$$

then

$$\text{(8)} \qquad \Delta(r_{n+1}^{\mu}\Delta(...\Delta(r_{n+1}^1 \Delta(y_{n+1} - z_{n+1})...)) \geq 0, \quad n \geq 1, \quad \mu = 2,...,m-2.$$

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(Poznań University of Technology, Institute of Mathematics, 60-695 Poznań, Poland)
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