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A NOTE ON M-PARABOLIC PROBLEM FOR THE STRIP WITH BOUNDARY CONDITIONS OF EVEN ORDER

In the present paper we construct an explicit solution of the initial-boundary value parabolic problem for the time-spatial strip with initial conditions of Cauchy type and boundary conditions of even order.

Key words: M-parabolic problem, Green functions, iterated Green potentials.

1. Introduction

Let us consider the equation

$$(1) \quad P^m u(x, t) = f(x, t), \quad P = D_x^2 - D_t, \quad P^m = P(P^{m-1}), \quad m \in \mathbb{N}$$

in the domain

$$D = \{(x, t) : |x| < a, \quad t \in (0, T)\},$$

with the initial conditions

$$(2) \quad D_t^i u(x, 0) = f_i(x), \quad i = 0, 1, \dots, m-1, \quad x \in (-a, a),$$

$$(3) \quad D_x^{2i} u(-a, t) = h_i(t), \quad i = 0, 1, \dots, m-1, \quad t \in (0, T),$$

$$(4) \quad D_x^{2i} u(a, t) = k_i(t), \quad i = 0, 1, \dots, m-1, \quad t \in (0, T),$$

where $f, f_i, h_i, k_i, \quad i = 0, 1, \dots, m-1$, are given functions and a, T are positive constants.

In the paper [6] the m-parabolic initial-boundary value problem (1)–(4) is treated. In [6] it is considered more complicated reduction of boundary conditions to homogeneous conditions.

In this paper to construct the solution of the problem (1)–(4) we shall apply another reduction of the boundary conditions to the homogeneous conditions more simple as in [6].

The similar polyparabolic equation $P^m u(x, t) = f(x, t)$ in the strip and in the quart-time-plane with boundary conditions of Lauricella type are considered in [1] and [2], and with boundary conditions of Riquier type in the three-dimensional cylinder and in spherical shall are considered in the papers [3] and [4].

2. Some denotations

Definition 1. Denote by (K_1) the class of all functions $u: D \rightarrow \mathbb{R}$, such that $D_x^j D_t^i u(x, t) \in C(D)$, $j = 0, \dots, 2m$, $i = 0, \dots, m$.

Definition 2. Denote by (K_2) the class of all functions $f: D \rightarrow \mathbb{R} \cup S_0 \cup S_1 \cup S_2$, $S_0 = \{(x, 0): x \in [-a, a]\}$, $S_1 = \{(-a, t): t \in [0, T]\}$, $S_2 = \{(a, t): t \in [0, T]\}$, such that $D_s^i f(y, s)$, $i = 0, 1, \dots, m-1$ are continuous and bounded in $D \cup S_0 \cup S_1 \cup S_2$ and $D_y^i f(\pm a, s) = 0$, $i = 0, 1, 2, \dots, m-1$ for $s \in [0, T]$, the function $D_y f(y, s) \in C(D \cup S_0 \cup S_1 \cup S_2)$ and $D_t^i f(x, 0) = 0$, $x \in [-a, a]$, $i = 0, 1, 2, \dots, m-1$.

Definition 3. Denote by (K_3) the class of all functions $f_i: [-a, a] \rightarrow \mathbb{R}$, such that $f_i \in C^{2m-2i}([-a, a])$, $i = 0, 1, \dots, m-1$.

Definition 4. Denote by (K_4) the class of all functions $h_i: [0, T] \rightarrow \mathbb{R}$, such that $h_i \in C^{m-1}([0, T])$, $i = 0, 1, \dots, m-1$.

3. Reduction of problem (1)–(4) to the problem with homogeneous boundary conditions

In the paper [6], pp. 211–216 a complicated method of reduction of boundary conditions to homogeneous conditions was considered. We shall apply another much simpler method of reduction of problem (1)–(4) to the problem with homogeneous boundary conditions.

Let us consider the new unknown function

$$w(x, t) = u(x, t) - r(x, t),$$

where

$$r(x, t) = \sum_{i=0}^{m-1} A_i [(x+a)^{2i+1}(x-a)^{2i}k_i(t) - (x+a)^{2i}(x-a)^{2i+1}h_i(t)],$$

where

$$A_i = \frac{1}{(2a)^{2i+1}(2i)!}, \quad i = 0, 1, \dots, m-1.$$

Lemma 1. If $f_i \in (K_3)$, $h_i, k_i \in (K_4)$, $i = 0, 1, \dots, m-1$, $m \in \mathbb{R}$, $f \in (K_2)$ and $u \in (K_1)$ is the solution of the problem (1)–(4), then the function w belongs to (K_1) and satisfies the conditions

$$(1a) \quad P^m w(x, t) = F(x, t), \quad F(x, t) = f(x, t) - P^m r(x, t), \quad (x, t) \in D, \quad m \in \mathbb{N},$$

$$(2a) \quad D_t^i w(x, 0) = F(x), \quad i = 0, 1, 2, \dots, m-1, \quad F_i(x) = f_i(x) - D_t^i r(x, 0),$$

$$i = 0, 1, 2, \dots, m-1, \quad x \in (-a, a),$$

$$(3a) \quad D_x^{2i} w(-a, t) = 0, \quad i = 0, 1, 2, \dots, m-1, \quad t \in (0, T),$$

$$(4a) \quad D_x^{2i} w(a, t) = 0, \quad i = 0, 1, 2, \dots, m-1, \quad t \in (0, T).$$

Conversely, if $f_i \in (K_3)$, $h_i, k_i \in (K_4)$, $i = 0, 1, 2, \dots, m-1$, $m \in \mathbb{N}$, $f \in (K_2)$ and $w \in (K_1)$ is the solution of the problem (1a)–(4a), then the function $u = (w+r) \in (K_1)$ is the solution of the problem (1)–(4).

We omit the simple proof.

4. Green function

By [5], p. 476 the Green function for the equation $Pg(x, t; y, s) = 0$ for the domain D and boundary conditions of Dirichlet type $g(-a, t; y, s) = g(a, t; y, s) = 0$ is of the form:

$$g(x, t; y, s) = U_0(x, t; y, s) + \sum_{n=1}^{\infty} (-1)^n [U_n^1(x, t; y, s) + U_n^2(x, t; y, s)],$$

where

$$U_0(x, t; y, s) = (t-s)^{-1/2} \exp B \left((t, s)(x-y)^2 \right),$$

$$U_n^1(x, t; y, s) = (t-s)^{-1/2} \exp B \left((t, s)(x_n^1 - y)^2 \right), \quad n = 1, 2, 3, \dots,$$

$$U_n^2(x, t; y, s) = (t-s)^{-1/2} \exp B \left((t, s)(x_n^2 - y)^2 \right), \quad n = 1, 2, 3, \dots,$$

$$B(t, s) = (-4(t-s))^{-1},$$

$$\begin{aligned}
 x &= x_0^1 = x_0^2, \\
 x_{2n}^1 &= x + 4na, \quad x_{2n+1}^1 = -x - 4na - 2a, \quad n = 0, 1, 2, \dots \\
 x_{2n}^2 &= x - 4na, \quad x_{2n+1}^2 = -x + 4na + 2a, \quad n = 0, 1, 2, \dots
 \end{aligned}$$

5. Construction of solution of problem (1a)–(4a)

Similarly to the [6] p. 216, let us consider the function

$$(5) \quad w(x, t) = \sum_{j=0}^{m-1} w_j(x, t) + w_m(x, t),$$

where

$$\begin{aligned}
 w_m(x, t) &= A \int_0^t \int_{-a}^a F(y, s) (t-s)^{m-1} g(x, t; y, s) dy ds, \\
 w_i(x, t) &= A \int_{-a}^a H_i(y) t^i g(x, t; y, 0) dy, \quad i = 0, 1, \dots, m-1,
 \end{aligned}$$

$$A = (2\sqrt{\pi})^{-1},$$

$$\begin{aligned}
 H_i(x) &= \frac{1}{i!} F_i(y) - \frac{1}{(i-1)! 1!} D_y^2 F_{i-1}(y) + \frac{1}{(i-2)! 2!} D_y^4 F_{i-2}(y) + \dots + \\
 &+ (-1)^k \frac{1}{(i-k)! k!} D_y^{2k} F_k(y) + \dots + (-1)^i \frac{1}{0! i!} D_y^{2i} F_0(y).
 \end{aligned}$$

We have

Lemma 2. If $F(y, s) \in (K_2)$, then

1° $P^m w_m(x, t) = F(x, t)$, $(x, t) \in D$,

2° $D_x^i w_m(x, t) = 0$ as $(x, t) \rightarrow (x_0, 0)$, $i = 0, 1, \dots, m-1$, $x_0 \in (-a, a)$,

3° $D_x^{2i} w_m(\pm a, t) = 0$, $i = 0, 1, \dots, m-1$, $t \in (0, T)$.

The proof is given in [6], pp. 217–221.

$$\text{Let } Z(x, t) = \sum_{i=1}^{m-1} w_i(x, t).$$

We have

Lemma 3. If $h_i, k_i \in (K_4)$, $i = 0, 1, \dots, m-1$ and if $F_i \in (K_3)$, $i = 0, 1, \dots, m-1$, then

$$1^\circ P^m Z(x, t) = 0, \quad (x, t) \in D,$$

$$2^\circ D_x^i Z(x, t) - F(x_0) \text{ as } (x, t) \rightarrow (x_0, 0), \quad x_0 \in (-a, a), \quad i = 0, 1, \dots, m-1,$$

$$3^\circ D_x^{2i} Z(\pm a, t) = 0, \quad i = 0, 1, \dots, m-1, \quad t \in (0, T).$$

The proof is given in [6], pp. 222–224.

6. Fundamental theorem

By Lemmas 1, 2, 3 we obtain

Theorem. If $f \in (K_2)$, $f_i \in (K_3)$, $h_i, k_i \in (K_4)$, then the function w defined by (5) is the solution of (1a)–(4a) problem, and the function

$$u(x, t) = w(x, t) + r(x, t) = \sum_{j=0}^{m-1} w_j(x, t) + w_m(x, t) + \sum_{j=0}^{m-1} A_j [(x+a)^{2j+1} (x-a)^{2j} k_j(t) - (x+a)^{2j} (x-a)^{2j+1} h_j(t)]$$

is the solution of (1)–(4) problem.

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