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**PARABOLIC EQUATIONS
IN GENERALIZED SOBOLEV SPACES $W^{k,p(x)}$**

There are given the conditions for the solvability of an initial and boundary value problem for parabolic equations with coefficients of the type of variable powers.

Key words: partial differential equations, parabolic equation, Orlicz-Sobolev spaces, function spaces.

The problems concerning applications of Rothe's method (method of discretization in time) in evolution differential equations are frequently published (See e.g. the references in the monography J. Kačur [1], where there is given large list of the papers on this subject).

Section 2.4 of this monography deals with solutions for a parabolic equation

$$(1) \quad \frac{du}{dt} + \sum_{|i| \leq k} (-1)^{|i|} D^i a_i(x, \delta_k u) = f(x, t),$$

in Orlicz-Sobolev spaces, where the functions $a_i(x, t)$ have some nonpolynomial growth in y .

In this papers we shall derive analogous results in the case, when the growth of a_i is of the type of variable powers. We use some results from the paper of O. Kováčik and J. Rákosník [2]. Throughout this paper the terms measure, measurable etc. will mean the Lebesgue measure, Lebesgue measurable etc.

Let Ω be a bounded domain in R^N with Lipschitz boundary $\delta\Omega$ and let $I = (0; T)$ be a time niterval with $T < \infty$. Denote by $Q = I \times \Omega$ the time - space domain and by $i = \{i_1, \dots, i_N\}$, $i \in \{N_0\}^N$, so called multi-index with lenght $|i| = i_1 + \dots + i_N$. We use the notation

$$D^i v = \frac{\delta^{|i|} v}{\delta x_1^{i_1} \dots \delta x_N^{i_N}}$$

for the generalized derivative of a function v of order $|i|$ over the space variable $x = \{x_1, \dots, x_N\}$ (in the sense of distributions) and $\frac{dv}{dt}$ for the usual derivative over the time variable. By $\delta_k v$ we denote the vector of all derivatives $D^i v$, $|i| \leq k$, for which the arrangement of their components will be fixed.

In the paper [2] the authors deal with existence and uniqueness of weak solutions for the elliptic problem

$$(2) \quad \sum_{|i| \leq k} (-1)^{|i|} D^i a_i(x, \delta_k u(x)) = f(x),$$

$$(3) \quad D^i u(x) = 0, \quad |i| \leq k-1, \quad x \in \delta\Omega \quad (\text{in the sense of traces}),$$

where the coefficients $a_i(x, y)$ have the growth of variable powers in y . For this investigation there is applied the basic theory of generalized Sobolev spaces $W^{k, p(x)}$.

By $\pi(\Omega)$ we denote the family of all measurable functions $p: \Omega \rightarrow [1; \infty]$. Suppose $p \in \pi(\Omega)$. Define the functionals σ_p and $\|\cdot\|_p$ on the set of all measurable functions on Ω by

$$(4) \quad \sigma_p(f) = \int_{\Omega - \Omega_\infty} |f(x)|^{p(x)} dx + \text{esssup}_{\Omega_\infty} |f(x)|,$$

and

$$(5) \quad \|f\|_p = \inf \{ \beta > 0 : \sigma_p(f/\beta) \leq 1 \},$$

where $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$.

The generalized Lebesgue space $L^{p(x)}(\Omega)$ is the class of all functions f such that

$$\sigma_p(a, f) < \infty \quad \text{for some } a = a(f) > 0$$

is a normed linear space with the Luxemburg norm (5).

Define the conjugate function $p'(x)$ for $p(x) \in \pi(\Omega)$ by

$$p'(x) = \begin{cases} 1 & \text{for } x \in \Omega_\infty, \\ \infty & \text{for } x \in \Omega : p(x) = 1 \\ p(x)/\{p(x)-1\} & \text{for other } x \in \Omega. \end{cases}$$

Then we obtain $p'(x) \in \pi(\Omega)$.

The generalized Sobolev space $W^{k,p(x)}(\Omega)$ is the class of all functions f measurable on Ω such that $D^i f \in L^{p(x)}(\Omega)$ for every multi-index i with $|i| \leq k$, endowed with the norm

$$(6) \quad \|f\|_{k,p} = \sum_{|i| \leq k} \|D^i f\|_p.$$

By $W_0^{k,p(x)}(\Omega)$ we denote the subspace of $W^{k,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm (6).

The following propositions are valid (proofs see in [2]).

Lemma 1. The space $L^{p(x)}(\Omega)$ is a Banach space and $L^{p'(x)}(\Omega)$ will be its dual space if and only if $p \in L^\infty(\Omega)$. The space $L^{p(x)}(\Omega)$ is reflexive if and only if

$$(7) \quad 1 < \operatorname{ess\,inf}_\Omega p(x) \leq \operatorname{ess\,sup}_\Omega p(x) < \infty.$$

Lemma 2. Let $p \in \pi(\Omega) \cap L^\infty(\Omega)$. Then the set $C(\Omega) \cap L^{p(x)}(\Omega)$ is dense in $L^{p(x)}(\Omega)$ and $L^{p(x)}(\Omega)$ is separable. If, moreover, Ω is open, then the set $C_0^\infty(\Omega)$ is dense in $L^{p(x)}(\Omega)$.

Lemma 3. The spaces $W^{k,p(x)}(\Omega)$ and $W_0^{k,p(x)}(\Omega)$ are Banach spaces. They will be separable if $p \in L^\infty(\Omega)$ and reflexive if and only if p satisfies (7).

From the nonotone operator theory we get the following part on the solvability of the problem (2) and (3).

First, we shall say that a function $h: \Omega \times R^m \rightarrow R$, $m \in N$, satisfies the Carathéodory conditions ($h \in \text{CAR}(\Omega, m)$), if the function $h(\cdot, y)$ is measurable on Ω for every $y \in R^m$ and the function $h(x, \cdot)$ is continuous on R^m for a.e. $x \in \Omega$.

Lemma 4. Let $p \in \pi(\Omega)$ satisfy (7). Let $h \in \text{CAR}(\Omega, m)$, $m = \sum_{|i| \leq k} 1$, and let $0 \leq g \in L^{p'(x)}(\Omega)$ and $c > 0$ be such that

$$(8) \quad |h(x, y)| \leq g(x) + c, \quad \sum_{|i| \leq k} |y_i|^{p(x)-1}$$

holds for every $y \in R^m$ and a.e. $x \in \Omega$. Let $i \in \{N_0\}^N$, $|i| \leq k$. Then the operator $T_i: W^{k,p(x)}(\Omega) \rightarrow (W^{k,p(x)}(\Omega))^*$ defined by $(T_i u)(v) = \int_\Omega h(x, \delta_k u(x)) \cdot D^i v(x) dx$ for $u, v \in W^{k,p(x)}(\Omega)$ is continuous and bounded. (Proof see in [2].)

Now we consider a differential operator A of order $2k$ of the form

$$(9) \quad Au(x) = \sum_{|i| \leq k} (-1)^{|i|} D^i a_i(x, \delta_k u(x)),$$

where the functions $a_i \in CAR(\Omega, m)$ fulfill the growth condition (8) with nonnegative $g \in L^{p(x)}(\Omega)$ and $c > 0$. Let V be a Banach space of functions on Ω equipped with a norm $\|\cdot\|_V$ and such that $C_0^\infty(\Omega)$ is dense in V and, moreover, $W_0^{k,p(x)}(\Omega) \rightarrow V$, i.e. $W_0^{k,p(x)}(\Omega)$ is continuously embedded in V . By Lemma 2 we can put $V = L^{p(x)}(\Omega)$. Let $f \in V^*$ and let us denote by $\langle \cdot, \cdot \rangle_V$ the duality on V . A function $u \in W_0^{k,p(x)}(\Omega)$ is a weak solution for the Dirichlet boundary value problem $(A, 0, f)$ for the equation

$$Au = f$$

with the boundary condition given by

$$(3') \quad u \in W_0^{k,p(x)}(\Omega)$$

and if the identity

$$(10) \quad \sum_{|i| \leq k} \int_{\Omega} a_i(x, \delta_k u(x)) D^i v(x) dx = \langle f, v \rangle$$

holds for every $v \in W_0^{k,p(x)}(\Omega)$.

From the Lemma 4 we obtain, that the operator

$$T: W_0^{k,p(x)}(\Omega) \rightarrow (W_0^{k,p(x)}(\Omega))^*,$$

defined by the left-hand side of (10), is continuous and bounded.

Theorem 1. Let $p \in \pi(\Omega)$ satisfy (7). Let the functions a_i satisfy (8) and let the conditions

$$(11) \quad \sum_{|i| \leq k} [a_i(x, y) - a_i(x, w)] (y_i - w_i) \geq 0,$$

$$(12) \quad \sum_{|i| \leq k} a_i(x, y) y_i \geq c_1 \sum_{|i| \leq k} |y|^{p(x)} - c_2$$

hold for every $y, w \in R^m$, for a.e. $x \in \Omega$ and for some constants $c_1, c_2 > 0$. Then the boundary value problem $(A, 0, f)$ has at least one weak solution $u \in W_0^{k,p(x)}(\Omega)$. If, moreover, the inequality (11) is strict for $y \neq w$, then the solution is unique. (See in [2].)

The part 2.4 of [1] deals with the solutions of the parabolic equation (1). With respect to the nonpolynomial growth of functions a_i the corresponding space, in which the problem is formulated, is the Orlicz-Sobolev space $W^k L_G$ (see e.g. [3]), which (in general) is nonreflexive. Moreover, the corresponding operator A defined by (9) is assumed to be not everywhere defined and (in general) noncoercive.

The conditions for the solvability of the equation (1) will be obtained as a consequence of the solvability conditions for an abstract parabolic problem

$$\frac{du}{dt} + Au = F, \quad u(0) = u_0, \quad a.e., \quad t \in I,$$

where A is a pseudomonotone operator with respect to a complementary system $\{Z, Z_0; Y, Y_0\}$. Denote by $D(A)$ the domain of the operator A , which maps $D(A) \subset Y$ into Z , where Y, Z are Banach spaces. In general, the operator A will be nonreflexive and not coercive and $D(A)$ will be not a dense set in Y . Suppose that Y and Z are in duality with respect to a continuous pairing $\langle z, y \rangle$ for $z \in Z$ and $y \in Y$. Let Y_0, Z_0 be subspaces of Y and Z , respectively. The system $\{Z, Z_0; Y, Y_0\}$ we call to be a complementary system if, by means of $\langle \cdot, \cdot \rangle$, Y_0^* can be identified (i.e. is linearly homeomorphic) to Z and Z_0^* to Y with usually defined norms on Y, Y_0, Z, Z_0 according to the duality.

A real valued function $G(s)$ is said to be an N-function if it satisfies: $G(s) > 0$ for $s > 0$, $G(s)s^{-1} \rightarrow \infty$ for $s \rightarrow \infty$, $G(s)s^{-1} \rightarrow 0$ for $s \rightarrow 0$, $G(s)$ is convex and even for $s \in \mathbb{R}$. Let us put L_G the linear hull of the set

$$\{u \in L^1(\Omega); \int_{\Omega} G(u(x)) dx < \infty\}.$$

It is a Banach space (Orlicz space) with respect to the Luxemburg norm

$$\|u\|_G = \inf \{r > 0; \int_{\Omega} G(u/r) dx \leq 1\}.$$

The closure in L_G of the set of all bounded measurable functions in Ω is denoted by E_G .

The function $G(s)$ is said to satisfy the Δ_2 -condition if there exists a $k > 0$ such that $G(2s) \leq k \cdot G(s)$ holds for $s \geq s_1$ for some $s_1 > 0$. The inclusion $E_G \subset L_G$ takes place; the equality $E_G = L_G$ holds if and only if $G(s)$ satisfies the Δ_2 -condition. By $G'(s)$ we denote the N-function conjugate to $G(s)$ with respect to the Young inequality.

Then the system $\{L_G, E_G; L_G, E_G\}$ is a complementary system with respect to the pairing

$$\langle u, v \rangle = \int_{\Omega} uv dx.$$

By $W^k L_G$ there will be denoted the set $\{u \in L^1(\Omega); D^i u \in L_G \text{ for all } |i| \leq k\}$ with an equivalent norm

$$\|u\|_{k,G} = \left(\sum_{|i| \leq k} \|D^i u\|_G^2 \right)^{1/2}.$$

Another equivalent norm has the form

$$\|u\|_{k,G} = \sum_{|i| \leq k} \|D^i u\|_G.$$

Denote by $W_0^k L_G$ the closure of $C_0^\infty(\Omega)$ in $W^k L_G$ and by $W_0^k E_G$ the intersection $W_0^k L_G \cap \left(\prod_{|i| \leq k} E_G \right)$. Let us take $Y = W_0^k L_G$ and $Y_0 = W_0^k E_G$. Then we obtain $Y_0^* = Z = (W_0^k E_G)^*$ and $Z_0 = (W_0^k L_G)^*$. From here we can conclude, that this system $\{Z, Z_0; Y, Y_0\}$ is a complementary system with respect to

$$\langle w, v \rangle = \sum_{|i| \leq k} \int_{\Omega} D^i v w_i dx \text{ for } v \in Y, w = \{w_i\} \in Z = W_0^k E_G^*.$$

Let M be the class of all continuous functions $g(u)$ in R satisfying: $g(u) \rightarrow \infty$ for $u \rightarrow \infty$, $g(u)$ is odd and $u \cdot g(u)$ is convex for $u \geq u^0 > 0$, where u^0 is sufficiently large. We can verify that for any $g \in M$ there exists an N-function $G(u)$ (not uniquely determined) such that $G(u) = u \cdot g(u)$ for all $u \geq u_0$ and all these N-functions are equivalent and generate the same Orlicz space L_G .

Using of the Rothe's method the next theorem is proved.

Theorem 2. Let the functions a_i satisfy the conditions:

(i) $a_i(x, y)$ for $|i| \leq k$ are real valued functions for $x \in \Omega, y \in R^m$ which are measurable in x for fixed y and continuous in y for fixed x ;

(ii) $|a_i(x, y)| \leq a(x) + c_1 \sum_{|j| \leq k} |g(c_2, y_j)|$ where $g(u)$ is some function from M , $a(x) \in E_G(\Omega), c_1 > 0, c_2 > 0$;

(iii) $\sum_{|i| \leq k} [a_i(x, y) - a_i(x, w)] (y_i - w_i) \geq 0$ for all $y, w \in R^m, x \in \Omega$;

(iv) $\sum_{|i| \leq k} [a_i(x, y) y_i \geq c_3 \sum_{|i| \leq k} g(y_i/r) y_i - c_4$ for some $r > 1, c_3, c_4 > 0$ and for all $x \in \Omega, y, z \in R^m$.

Let there be considered the initial and boundary conditions for the equation (1) in the form

$$(13) \quad u(x, 0) = u_0(x),$$

$$(14) \quad \frac{\delta^l u}{\delta n^l} = 0 \text{ on } \delta\Omega \times I \text{ for all } l = 0, 1, \dots, k-1,$$

where n is the outward normal to $\delta\Omega$.

Suppose $f, \frac{df}{dt} \in L^2(Q)$, $u_0 \in D(A) \cap L^2$ and $Au_0 \in L^2$. Then there exists a unique (weak) solution

$$u \in L^\infty(I, W_0^k L_G \cap L^2)$$

of the problem (1), (13), (14) in the following sense:

(a) $u: I \rightarrow L^2$ is Lipschitz continuous, $u(t) \in D(A)$ for all $t \in I$ and $u(x, 0) = u_0(x)$;

(b) $\frac{du}{dt} \in L^\infty(I, L^2)$, $Au \in L^\infty(I, L^2)$;

(c) the equality

$$(15) \quad \left(\frac{du}{dt}, v \right) + \sum_{|i| \leq k} \int_{\Omega} a_i(x, \delta_k u) D^i v \, dx = \langle f, v \rangle$$

holds for all $v \in W_0^k L_G \cap L^2$ and for a.e. $t \in I$.

(See Theorem 2.4.23 in [1].)

Remark. We now compare the conditions from the Theorem 2 with the conditions from the Theorem 1. Suppose $p \in \pi(\Omega)$ and the inequality (7). Denote $p^* = \text{esssup } p(x)$ on Ω . Then the function $G(u) = |u|^{p(x)}$ is a Young function satisfying the Δ_2 -condition and $W^k L_G = W^k E_G = W^{k, p(x)}$, $W_0^k L_G = W_0^{k, p(x)}$ and $(W_0^k L_G)^* = (W_0^k E_G)^* = (W_0^{k, p(x)})^*$ take places. Then we can easily verify that the system

$$\{(W_0^{k, p(x)})^*, (W_0^{k, p(x)})^*, W_0^{k, p(x)}, W_0^{k, p(x)}\}$$

is a complementary system in the sense mentioned above. The conditions (8), (11), (12) are equivalent to the conditions (ii), (iii), (iv) supposed their validity for all $x \in \Omega$. Moreover, the condition (iv) follows from (12). The boundary condition (14) is, evidently, equivalent to the condition (3') for fixed $t \in I$.

According to this Remark we obtain the next theorem,

Theorem 3. Let $p \in \pi(\Omega)$ satisfy (7). Let the functions a_i satisfy (8), (11), (12) for every $y, w \in R^m$ and for all $x \in \Omega$, where the inequality (11) is strict for $y \neq w$. Let the conditions (13) and (3') be considered. Suppose

$$f, \frac{df}{dt} \in L^2(\Omega), u_0 \in D(A) \cap L^2 \text{ and } Au_0 \in L^2.$$

Then there exists a unique solution $u \in L^\infty(I, W_0^{k,p(x)} \cap L^2)$ of the problem (1), (13), (3') in the sense of (a), (b), (c), where the equality (15) holds for all $v \in W_0^{k,p(x)} \cap L^2$ and for a.e. $t \in I$.

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Received on 5.2.1993 and, in 16.6.1993.