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ON CERTAIN SOLUTION OF THE MAXWELL EQUATION WITH A NON-CONSTANT VELOCITY OF THE ELECTROMAGNETIC IMPULSE

The work shows, that adaptation of the solution of Picard's problem to the hyperbolic equation leads to movement of waves at a velocity depending on time. In the second part of the work it has also been shown that certain solutions of diffusion problems have waves qualities.

Key words: electromagnetic waves transport, differential equations..

1. Introduction

It is known from literature that the Maxwell equations confirm that the electromagnetic waves move with constant velocity $v = c$.

In this paper we show that there are solutions for which the electromagnetic wave moves with a velocity depending on time t .

We write Maxwell's equations:

$$(1a) \quad \frac{\partial E}{\partial t} + \operatorname{rot} H = 0, \operatorname{div} E = 0, x \in R^3, t \in R^1, E \in C^{2,2}, H \in C^{2,2},$$

$$(1b) \quad \frac{\partial H}{\partial t} - \operatorname{rot} E = j, \operatorname{div} H = q, H, E, j: R^3 \times R^1 \rightarrow R^3.$$

For $j = 0, \operatorname{div} E = 0, \operatorname{div} H = 0$ we can transform the equations into the hyperbolic equations. Differentiating (1a) and (1b) with respect to t and using (1b) and (1a) we have:

$$(1c) \quad \frac{\partial^2 E}{\partial t^2} + \operatorname{rot}(\operatorname{rot} E) = 0, \quad \frac{\partial^2 H}{\partial t^2} + \operatorname{rot}(\operatorname{rot} H) = 0.$$

Because $\operatorname{rot}(\operatorname{rot} A) = -D A + \operatorname{grad}(\operatorname{div} A)$ therefore we have:

$$(2) \quad \frac{\partial^2 E}{\partial t^2} = \Delta E, \quad \frac{\partial^2 H}{\partial t^2} = \Delta H.$$

The proof that the velocity of an impulse is constant results from a solution of Cauchy problem for equations (2), where we have (for $x \in R^1, t \in R^1$): $H(x,t) = F(x-t) + F(x+t)$ and $H(x,0) = 2F(x)$ (the initial condition $H(x,0)$ moves with constant velocity such that $|v|=1$) [1].

2. Main result

a. We can prove the existence of solutions for which the impulse moves with velocity v depending on time t . We consider the equation (2) for $x \in R^1$ (see fig. 1.):

$$(3) \quad \frac{\partial^2 H}{\partial t^2} = \frac{\partial^2 H}{\partial x^2}, H: R^1 \times R^+ \rightarrow R^1,$$

with conditions:

$$(4) \quad H(x,0) = \varphi(x), x \in R^1,$$

$$(5) \quad H(\lambda_i(t), t) = 0, i = 1, 2, t > 0.$$

We assume that:

$$(6) \quad \left| \frac{d\lambda_i(t)}{dt} \right| \leq 1.$$

We consider the domain $D = \{(x,t): x \in R^1, t > 0\}$. We show that there exists the solution satisfying (3) - (6). Hence we show that the hyperbolic impulse has a non-constant velocity v , dependent on t . We transform variables x and t . Let $x = \frac{p+q}{2}, t = \frac{p-q}{2}$. The equation (4) takes the form:

$$(7) \quad \frac{\partial^2 \bar{H}(p,q)}{\partial p \partial q} = 0,$$

while boundary conditions take the form:

$$(8) \quad \bar{H}(q = p, \frac{p+q}{2}) = \bar{\alpha}(q),$$

$$(9.i) \quad \bar{H}(\bar{\lambda}_i(p), p) = 0, \quad i = 1, 2 \quad (\text{see fig.2.})$$

We consider the line $q = \text{const}$, which passes through the point $A(-q_0, q_0)$. We suppose that the value of the solution (7) along the line (which is the characteristic line) is $f(p)$. We can find the solution of the Picard problem [2] for the equation (7) with boundary conditions:

$$(10) \quad \bar{H}(\bar{\lambda}_1(p), p) = 0$$

$$(11) \quad \bar{H}(q = \text{const}, p) = f(p) \quad (\text{for example } \text{const} = -q_0).$$

The solution of the Picard problem (7), (10), (11) can be written as:

$$(12) \quad \bar{H}(p, q) = f(p) - f(\bar{\lambda}_1^{-1}(q)).$$

We assume that the solution of the Picard problem is the solution of the problem (7) - (9) for unknown f . We will find the unknown f from the condition (9.2). From (11), (12) we have:

$$(13) \quad \begin{aligned} f(p) &= f(\bar{\lambda}_1^{-1}(\bar{\lambda}_2(p))), \quad \bar{\lambda}_2(p_0) = q_0, \\ \bar{\lambda}_1(-q_0) &= -q_0, \quad p > -p_0 \end{aligned}$$

We obtain the functional equation for the function f , for $\bar{\lambda}_1(p)$ and $\bar{\lambda}_2(p)$. We construct the solution of equation (7) with the velocity of the impulse depending on time t for $\bar{\lambda}_2^{-1}(q) = \bar{\lambda}_1^{-1}(q) + L$ (for example $L = 2\pi$). Therefore from (9.2), (13) we have:

$$(14) \quad f(\bar{\lambda}_1^{-1}(q) + 2\pi) - f(\bar{\lambda}_1^{-1}(q)) = 0.$$

From (14) we can see that f can be a periodic function (for example sinus).

Hence the periodic function with the period $\frac{2\pi}{n}$, $n = 1, 2, \dots$ are also

solutions of the functional equation (13). Therefore we can write the initial boundary condition as a sum of these periodic functions, which is a Fourier's series.

This construction ends the proof of existence of the solution of hyperbolic equation with velocity of waves depending on t .

b. On the other hand, we will show that there exists a solution of parabolic diffusion equation which has properties of waves. We consider the diffusion problem :

$$(15) \quad \frac{\partial p}{\partial t} = \Delta p, \quad p: R^3 \times R^1 \rightarrow R^+$$

We consider $p \in C^{2,1}$ for $(x, t) \in \Omega \subset R^1 \times R^1$. We write boundary conditions:

$$(16) \quad p(x, 0) = \varphi(x), \quad x \in (0, 1)$$

$$(17) \quad p(\lambda_i(t), t) = \Psi_i(t), \quad i = 1, 2, \text{ where } \varphi(0) = \Psi_1(0), \\ \varphi(1) = \Psi_2(0), \quad \Psi_j \text{ and } \varphi \text{ we will write next:}$$

$$(18) \quad \int_{\Omega \cap \{t\}} p(x, t) dx = \text{const for } \Omega = \{(x, t): t < x < t + 1, t > 0\}.$$

Let $x - t = y$. From (15) we have:

$$(19) \quad -\frac{dp}{dy} = \frac{d^2 p}{dy^2}$$

Hence $p(x, t) = C_1 + C_2 e^{-(x-t)}$, $C_1, C_2 = \text{const} > 0$.

We consider $\lambda_1(t) = t$, $\lambda_2(t) = t + 1$, from where $\Psi_1(t) = C_1 + C_2$, $\Psi_2(t) = C_1 + C_2 e^{-1}$. Hence

$$\int_t^{t+1} p(x, t) dx = C_1 - (e^{-1} - 1)C_2 \quad \text{and} \quad \int_{\Omega \cap \{t\}} p(x, t) dx = \text{const}.$$

We see that solutions of the diffusion problem satisfy the conservation law (18) and they have properties of the solutions of the wave equations. Such problem have not been known so far in the theory of diffusion (however the form of the solution is known [3].) Similarly,

we can obtain another solution for $y = x + t$. We have then $p(x, t) = C_1 + C_2 e^{-(x+t)}$.

The paper considers physical aspects of the hyperbolic problem (1) - (2) which has quality of solution of the free boundary problem for the parabolic equation. As for the free boundary problem we get a boundary which (in general) is not a linear time function.

Conversely, it has been shown that the free boundary problem (15) - (18) for the parabolic equation may have the quality of solutions of hyperbolic equations which describes waves moving at the constant velocity.

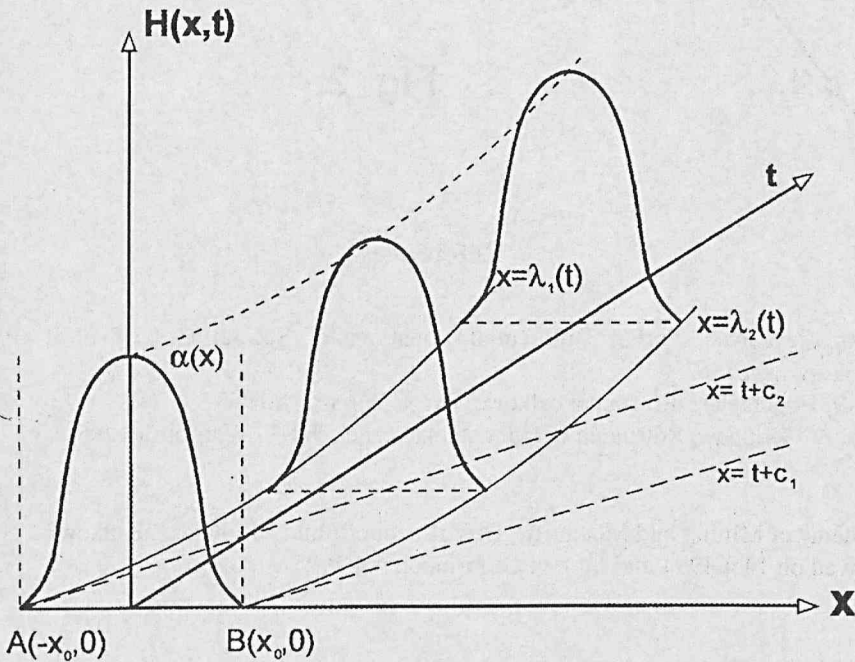


Fig. 1.

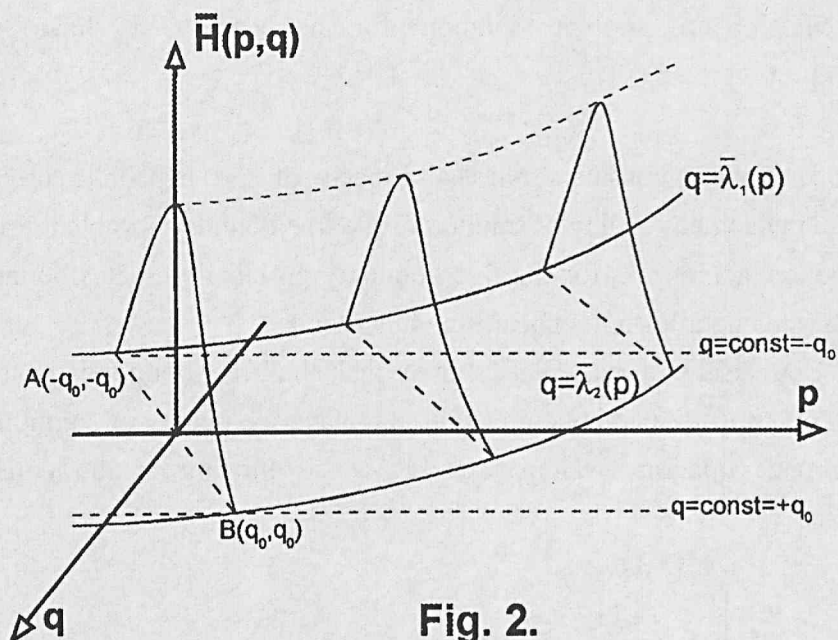


Fig. 2.

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 Received on 14.4.1994 and, in revised form, on 5.4.1995.