FASCICULI MATHEMATICI
Nr 26 1996

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OSCILLATION OF FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE WITH THE QUASI-DERIVATIVES

The purpose of this paper is to establish oscillation theorems for proper solutions of nonlinear differential equations with the quasi-derivatives.

Key words: neutral differential equation, oscillatory (nonoscillatory) solution, quasi-derivatives.

1. Introduction

In this paper we consider the n-th order, $n \geq 2$, neutral functional equations with the quasi-derivatives of the form

(E) $$L_n[x(t) - P(t)x(g(t))] + \delta Q(t) f(x(h(t))) = 0,$$

where $\delta = +1$ or -1 and L_n is disconjugate differential operator defined recursively by

$$L_0 x(t) = x(t),\ L_k x(t) = \frac{1}{a_k(t)}[L_{k-1}x(t)]',\ k = 1,2,\ldots,n,\ a_n = 1$$

The following conditions are assumed to hold throughout this paper:

(a) $a_i \in C[[t_0,\infty),(0,\infty)]$, $t_0 \geq 0$ and $\int_{t_0}^{\infty} a_i(t)dt = \infty$, $k = 1,2,\ldots,n-1$

(b) $P, Q, h, g \in C[[t_0,\infty),(0,\infty)]$, $\lim_{t\to\infty} g(t) = \infty$, $\lim_{t\to\infty} h(t) = \infty$ and $g(t)$ is strictly increasing;

(c) $f \in C[R,R]$ is nondecreasing, $xf(x) > 0$ for $x \neq 0$ and $xy(f(xy) \geq Kxyf(x)f(y)$ $(0 < K = \text{const})$.

By a solution of (E) we here mean a continuous function $x(t):[T_x,\infty) \to R$, $T_x \geq t_0$, such that $x(t) - P(t)x(g(t))$ has the continuous quasi-derivatives $L_i[x(t)-P(t)x(g(t))]$, $0 \le i \le n$ and $x(t)$ satisfies (E) for all sufficiently large $t \ge T_x$. Our attention is restricted to those solutions $x(t)$ of (E) which satisfy

$$\sup\{|x(t)| : t \ge T\} > 0, \quad \text{for any } T \ge T_x.$$

Such a solution is said to be a proper solution. We make the standing hypothesis that (E) possesses proper solutions. A proper solution of (E) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

The problem of oscillation of functional differential equations has received considerable attention in the last few years (see, for example, the papers [1-7]).

The purpose of this paper is to establish oscillation theorems for proper solutions of (E). The results from the papers [3] and [4] we extend for neutral nonlinear differential equations with the quasi-derivatives.

2. Classification of nonoscillatory solutions

We classify the possible nonoscillatory solutions of (E) in a similar way as in the paper [3].

Let $x(t)$ be a nonoscillatory solution of (E). From (E), (b) and (c) it follows that the function

(1) $$y(t) = x(t) - P(t)x(g(t))$$

has to be eventually constant sign, so that either

(2) $$x(t)y(t) > 0$$

or

(3) $$x(t)y(t) < 0$$

for all sufficiently large t. Assume first that (2) holds. Then the function $y(t)$ satisfies $\delta y(t) L_n y(t) < 0$ eventually and from the well known

Kiguradze's lemma it follows that there is an integer $l \in \{0,1,\ldots,n\}$, $(-1)^{n-l-1}\delta = 1$ and a $t_1 \geq t_0$ such that for every $t \geq t_1$ holds

$$(4)_l \qquad \begin{aligned} y(t)L_i y(t) > 0, \quad & 0 \leq i \leq l, \\ (-1)^{i-1} y(t)L_i y(t) > 0, \quad & l \leq i \leq n. \end{aligned}$$

A function $y(t)$ satisfying $(4)_l$ is said to be nonoscillatory function of degree l. The set of all solutions $x(t)$ of (E) satisfying (2) and $(4)_l$ will be denote by N_l^+. Now assume that (3) holds. Then $y(t)$ satisfies $(-\delta)y(t)L_n y(t) < 0$ for all large t and so it is a function of degree l for some $l \in \{0,1,\ldots,n\}$ with $(-1)^{n-l}\delta = 1$. The totality of nonoscillatory solutions $x(t)$ of (E) which satisfy (3) and $(4)_l$ will be denote by N_l^-. Consequently, if we denote by N the set of all possible nonoscillatory solutions of (E), then

$$(5) \qquad \begin{aligned} N &= N_1^+ \cup N_3^+ \cup \ldots \cup N_{n-1}^+ \cup N_0^- \cup N_2^- \cup \ldots \cup N_n^- && \text{for } \delta = 1 \text{ and } n \text{ even}, \\ N &= N_0^+ \cup N_2^+ \cup \ldots \cup N_{n-1}^+ \cup N_1^- \cup N_3^- \cup \ldots \cup N_n^- && \text{for } \delta = 1 \text{ and } n \text{ odd}, \\ N &= N_0^+ \cup N_2^+ \cup \ldots \cup N_n^+ \cup N_1^- \cup N_3^- \cup \ldots \cup N_{n-1}^- && \text{for } \delta = -1 \text{ and } n \text{ even}, \\ N &= N_1^+ \cup N_3^+ \cup \ldots \cup N_n^+ \cup N_0^- \cup N_2^- \cup \ldots \cup N_{n-1}^- && \text{for } \delta = -1 \text{ and } n \text{ odd}. \end{aligned}$$

It is now clear that the oscillation of all proper solutions of (E) is equivalent to the situation in which all classes N are empty.

3. Main results

Lemma 1. (i) *Let* $x(t)$ *be a nonoscillatory solution of* (E) *satisfying* (2). *Then*

$$(6) \qquad |x(t)| \geq |y(t)| \quad \textit{for all sufficiently large } t.$$

(ii) *Let* $x(t)$ *be a nonoscillatory solution of* (E) *satisfying* (3). *Then*

(7) $|x(t)| \geq |y(g^{-1}(t))| [P(g^{-1}(t))]^{-1}$ *for all sufficiently large t,*

where $g^{-1}(t)$ *denotes the inverse function of* g(t).

Proof of Lemma 1 is easy and will be omitted.

Let $x(t)$ be a nonoscillatory solution of (E). Lemma 1 and (c) implies that if (2) holds, then the function $y(t)$ satisfies the functional inequality

$$(\mathrm{I}^{+}, \delta) \qquad \{\delta L_n y(t) + Q(t) f(y(t)))\} \operatorname{sgn} y(t) \leq 0,$$

for all large t and that if (3) holds, then the functions $y(t)$ satisfies the functional differential inequality

$$(\mathrm{I}^{-}, \delta) \qquad \{\delta L_n y(t) - MQ(t) f([P(g^{-1}(h(t)))]^{-1}, \\ f(y(g^{-1}(h(t))))\} \operatorname{sgn} y(t) \geq 0,$$

for all large *t*, where $0 < -f(-1)K^2 = M (= \text{const})$.

Theorem 1. ([2, *Theorem* 3.1]) *Assume that the functional differential inequalities* (I^{-}, δ) *and* (I^{+}, δ) *have no nonoscillatory solutions. Then all proper solutions of* (E) *are oscillatory.*

In order to ensure the nonexistence of nonoscillatory solutions of the inequalities (I^{-}, δ) and (I^{+}, δ) we shall use the results (Lemmas 2-5) due to Kitamura [4] specialized to functional differential inequalities of the form

$$(\mathrm{I}, \delta) \qquad \{\delta L_n u(t) + q(t) f(u(r(t)))\} \operatorname{sgn} u(t) \leq 0,$$

where $n \geq 2$, $\delta = +1$ or -1, $r, q \in C[[t_0, \infty), (0, \infty)]$ and $r(t) \to \infty$ as $t \to \infty$.

We use the notation:

$$r_*(t) = \min\{r(t), t\};$$

$$A[r] = \{t \in [t_0, \infty) : r(t) > t\};$$
$$R[r] = \{t \in [t_0, \infty) : r(t) < t\};$$

$$I_0 = 1, \quad I_i(t,s;a_i,\ldots,a_1) = \int_s^t a_i(z) I_{i-1}(z,s;a_{i-1},\ldots,a_1)dz, \quad i = 1,2,\ldots,n;$$

$$H_l[r](t) = \int_{t_0}^{r_*(t)} I_{l-1}(r(t),s:a_1,\ldots,a_{l-1})a_l(s)I_{n-l-1}(t,s;a_{n-1},\ldots,a_{l+1})ds,$$

$$1 \le l \le n-1.$$

Lemma 2. *Let* $\delta = 1$ *and n be even. Assume that*

(8) $$\int^{\infty} \frac{dx}{f(x)} < \infty \quad \text{and} \quad \int^{-\infty} \frac{dx}{f(x)} < \infty,$$

$(9)_l$ $$\int^{\infty} H_l[r](t) q(t) dt = \infty$$

for $l = 1,3,\ldots,n-1$.
Then all proper solutions of $(I, +1)$ *are oscillatory.*

Lemma 3. *Let* $\delta = 1$ *and n be odd. Suppose that* (8), $(9)_l$ *for* $l = 2,4,\ldots,n-1$ *hold. Moreover, assume that*

(10) $$\int_{+0} \frac{dx}{f(x)} < \infty \quad \text{and} \quad \int_{-0} \frac{dx}{f(x)} < \infty,$$

(11) $$\int_{R[r]} I_{n-1}(t,r(t);a_{n-1},\ldots,a_1)q(t)dt = \infty.$$

Then all proper solutions of $(I, +1)$ *are oscillatory.*

Lemma 4. *Let* $\delta = -1$ *and* n *be even. Suppose that* (8), $(9)_l$ *for* $l = 2,4,\ldots,n-2$, (10) *and* (11) *hold. Moreover, assume that*

$$(12) \qquad \int_{A[r]} I_{n-1}(r(t),t;a_1,\dots,a_{n-1})q(t)dt = \infty .$$

Then all proper solutions of (I,−1) *are oscillatory.*

Lemma 5. *Let* $\delta = -1$ *and* n *be odd. Suppose that* (8), $(9)_l$ *for* $l = 1,3,\dots,n-2$, (12) *hold. Then all proper solutions of* (I,−1) *are oscillatory.*

Theorem 2. *Let* $\delta = 1$ *and* n *be even. Suppose that* (8) *and* (10) *hold. Moreover, assume that*

$$(13)_l \qquad \int^{\infty} H_l[h](t)Q(t)dt = \infty \quad for \quad l = 1,3,\dots,n-1;$$

$$(14)_l \qquad \int^{\infty} H_l[g^{-1}(h)h](t)Q(t) f([P(g^{-1}h(t))]^{-1})dt = \infty$$

$$for \quad l = 2,4,\dots,n-2;$$

$$(15) \qquad \int_{R[g^{-1}(h)]} I_{n-1}(t,g^{-1}(h(t));a_{n-1},\dots,a_1)Q(t)\times$$
$$\times f([P(g^{-1}(h(t)))]^{-1} dt = \infty;$$

$$(16) \qquad \int_{A[g^{-1}(h)]} I_{n-1}(t,g^{-1}(h(t)),t;a_1,\dots,a_{n-1})Q(t)\times$$
$$\times f([P(g^{-1}(h(t)))]^{-1} dt = \infty.$$

Then all proper solutions of (E) *are oscillatory.*

Proof. According to (5), N_l^+, $l \in \{1,3,\dots,n-1\}$ and N_l^-, $l \in \{0,2,\dots,n\}$ are the possible classes of nonoscillatory solutions of (E) with $\delta = 1$ and even n. We shall show that under the conditions of the theorem none of these solution classes has a member.

Suppose first that $N_l^+ \neq \varnothing$ for some $l \in \{1,3,\ldots,n-1\}$. Then the inequality $(I^+,+1)$ has a nonoscillatory solution. However, this is impossible, because from Lemma 2 applied to $(I^+,+1)$ it follows that (8) and $(13)_l$ for $l=1,3,\ldots,n-1$ prevent $(I^+,+1)$ from having a nonoscillatory solution. Thus we must have $N_l^+ \neq \varnothing$ for all $l \in \{1,3,\ldots,n-1\}$.

Suppose that $N_l^- \neq \varnothing$ for some $l \in \{0,2,\ldots,n\}$. Then the inequality $(I^-,+1)$ has a nonoscillatory solution. However, this is impossible, because from Lemma 4 applied to $(I^-,+1)$ it follows that (8), (10), $(14)_l$ for $l=0,2,\ldots,n-2$, (15) and (16) prevent $(I^-,+1)$ from having a nonoscillatory solution. Thus we must have $N_l^- \neq \varnothing$ for all $l \in \{0,2,\ldots,n-2\}$.

This completes the proof of Theorem 2.

Theorem 3. *Let* $\delta = 1$ *and* n *be odd. Suppose that* (8), (10), $(13)_l$ *for* $l=2,4,\ldots,n-1$, $(14)_l$ *for* $l=1,3,\ldots,n-2$ *and* (16) *hold. Moreover, assume that*

(17) $$\int_{R[h]} I_{n-1}(t,h(t);a_{n-1},\ldots,a_1)Q(t)dt = \infty .$$

Then all proper solutions of (E) *are oscillatory.*

Theorem 4. *Let* $\delta = -1$ *and* n *be even. Suppose that* (8), (10), $(13)_l$ *for* $l=2,4,\ldots,n-2$, $(14)_l$ *for* $l=1,3,\ldots,n-1$ *and* (17) *hold. Moreover, assume that*

(18) $$\int_{A[h]} I_{n-1}(h(t),t;a_1,\ldots,a_{n-1})Q(t)dt = \infty .$$

Then all proper solutions of (E) *are oscillatory.*

Theorem 5. *Let* $\delta = -1$ *and* n *be odd. Suppose that* (8), (10), $(13)_l$ *for* $l=1,3,\ldots,n-2$, $(14)_l$ *for* $l=2,4,\ldots,n-1$, (15) *and* (18) *hold. Then all proper solutions of* (E) *are oscillatory.*

Proofs of Theorems 3-5 are similar to proof of Theorem 1 and will be omitted.

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Received on 4.11.1994.