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# ON THE INDEFINITE INTEGRAL OF A STEPANOV'S ALMOST PERIODIC FUNCTION

In the paper the author proves that if the indefinite integral of an $S$-almost periodic function is bounded, then this integral is a $V$-almost periodic function.

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A set $E \subset (-\infty, \infty)$ is called relatively dense iff there is a positive number $l$ such that in each open interval $(\alpha, \alpha + l)$, $\alpha \in (-\infty, \infty)$, there is at least one element of the set $E$.

Let $x$ be a continuous function defined on the whole real axis and taking real values. If for $\varepsilon > 0$ there is

$$\sup_{-\infty < u < \infty} |x_\tau(u) - x(u)| \leq \varepsilon,$$

where $x_\tau(u) \equiv x(u + \tau)$, then the number $\tau$ is called $\varepsilon$ - almost period of $x$. Let us denote the set of $\varepsilon$ - almost periods of $x$ by $E\{\varepsilon; x\}$. The function $x$ is called uniformly almost periodic or a Bohr's almost periodic function ($B$-a.p.) iff for each $\varepsilon > 0$ the set $E\{\varepsilon; x\}$ is relatively dense. For example the function $x$ of the form

(1) $$x(u) = \sin u + \sin(\sqrt{2}\, u) \quad \text{for} \quad u \in (-\infty, \infty)$$

is uniformly almost periodic and $x$ is not periodic. ( See [1]).

Let us denote by $L^p$, where $p \geq 1$, the space of real functions $x$, measurable in the sense of Lebesgue for which

$$\int_a^b |x(s)|^p \, ds < \infty$$

for arbitrary $a, b \in (-\infty, \infty)$. For $x, y \in L^p$ let us put

$$D_{sp}(x,y) = \sup_{-\infty<u<\infty} \left\{ \int_u^{u+1} |x(s) - y(s)|^p \, ds \right\}^{\frac{1}{p}}.$$

If for $x \in L^p$ and for $\varepsilon > 0$ there is $D_{sp}(x_\tau, x) \leq \varepsilon$, then the number $\tau$ is called $S^p$, $\varepsilon$ - almost period of $x$. The function $x \in L^p$ is called Stepanov's almost periodic function ($S^p$ – a.p.) iff for each $\varepsilon > 0$ the set $E_{sp}\{\varepsilon; x\}$ of $S^p, \varepsilon$ - almost periods of $x$ relatively dense. If $p = 1$ we have an $S$-a.p. function. For example the continuous function

$$x(u) = \sin \frac{1}{2 + \cos u + \cos(\sqrt{2}u)}$$

for $u \in (-\infty, \infty)$ is $S$-a.p. and $x$ not $B$-a.p. (See[1]).

Let $X_0$ be the set of functions defined on the whole real axis taking finite real values. Let us denote for an arbitrary $t \in (-\infty, \infty)$ the Jordan variation of the function $x \in X_o$ on the interval $< t - 1, t + 1 >$ by $V(x; t)$. For $x \in X_0$ let us write

$$V(x) = \sup_{-\infty<t<\infty} \{|x(t) + V(x; t)\}.$$

We say that $x \in X_0$ satisfies the condition ($W$) iff for every $\alpha \in (-\infty, \infty)$ and for every $l > 0$ there exists $M > 0$ such that for every $t \in (\alpha, \alpha + l)$ we have $V(x; t) \leq M$. Let us put

$$\tilde{X}_o = \{x \in X_o : x \text{ is continuous and satisfies the condition } (W)\}.$$

The function $x \in \tilde{X}_o$ is called almost periodic in variation ($V$-a.p.) iff for $\varepsilon > 0$ the set $E_v\{\varepsilon; x\}$ of $V$ - almost periods of $x$, i.e. the set of numbers $\tau$ for which $V(x_\tau - x) \leq \varepsilon$, is relatively dense. Every $V$-a.p. function is a Bohr's a.p. function. For example the Bohr's a.p. function $x$ of the form (1) is $V$-a.p. Let us write $x(u) = x_1(u) + x_2(u)$ for $u \in (-\infty, \infty)$, where

$$x_1(u) = \begin{cases} 0 & \text{for } u = k \\ (u-k)\sin\dfrac{\pi}{u-k} & \text{for } u \in (k, k+1) \end{cases}, k = 0, \pm 1, \pm 2, \ldots ,$$

$x_2(u) = \sin(\sqrt{2}u)$ for $u \in (-\infty, \infty)$.

Then $x$ is a Bohr's a.p. function and $x$ is not $V$-a.p. (See[2]).

In [3] it was shown that if the indefinite integral of an $S$-a.p. function bounded and uniformly continuous, then this integral is $V$-a.p. The following theorem is true:

*Theorem* 1. *If $x$ is an S-a.p. function and the indefinite integral of $x$*

$$F(u) = \int_{u_o}^{u} x(s)ds + C \quad \textit{for} \quad u \in (-\infty, \infty)$$

*is bounded, then $F$ is V-a.p.*

*Proof.* Let $x$ be $S$-a.p. and $S_x(\cdot;h)$ be the Steklov function of $x$ of the form

$$S_x(u;h) = \frac{1}{2h}\int_{u-h}^{u+h} x(s)ds, \tag{2}$$

where $h > 0$, $u \in (-\infty, \infty)$. $S_x(\cdot;h)$ is $B$-a.p. Let us denote

$$F_x(w;h) = \int_{w_o}^{w} S_x(u;h)du + C \quad \text{for} \quad h > 0,$$

where $w \in (-\infty, \infty)$.

For $w \in (-\infty, \infty)$ we have

$$|F_x(w;h) - F(w)| \leq \frac{1}{2h}\int_{-h}^{h} \{ |\int_{w}^{w+s} x(u)du| + |\int_{w_o}^{w_o+s} x(u)du| \} ds.$$

Let us choose an $S, \varepsilon/2$-almost period $\tau$ of $x$ such that $\tau \in (-w, -w+l)$, where $l = l(\varepsilon) > 0$ is the number which characterizes the relative density of the set $E_{S^1}\{\varepsilon/2; x\}$. For $0 \leq s \leq 1$ we obtain

$$\int_{w}^{w+s} |x(u+\tau)|du \leq \int_{0}^{l+1} |x(u)|du < \infty,$$

because x is $S^1$-bounded. Hence there exists $\Delta = \Delta(\varepsilon) > 0$ such that for $0 \le s < \Delta$ we have

$$\int_w^{w+s} |x(u+\tau)|du < \frac{\varepsilon}{2}$$

and

$$|\int_w^{w+s} x(u)du| < \int_w^{w+s} |x(u+\tau) - x(u)|du + \frac{\varepsilon}{2} \le \varepsilon.$$

Therefore we obtain the following estimation

$$|\int_w^{w+s} x(u)du| < \varepsilon \qquad \text{for } |s| < \Delta$$

uniformly with respect to $w \in (-\infty, \infty)$, and so the sequence $(F_x(w;h_n))$, where $h_n \to 0$, is convergent to $F(w)$ uniformly with respect to $w \in (-\infty, \infty)$. The Steklov function $S_x(\cdot;h)$ of $x$ satisfies the following inequality

$$|S_{x_\tau}(u;h) - S_x(u;h)| \le \frac{1}{2h} \int_{u-h}^{u+h} |x(s+\tau) - x(s)|ds \quad \text{for every } u \in (-\infty, \infty).$$

Hence $S_x(\cdot;h)$ is $B$-a.p. Because for every $h > 0$ and every $w \in (-\infty, \infty)$

$$F_x(w;h) = \frac{1}{2h} \int_{-h}^{h} [F(w+s) - F(w_o + s)]ds + C$$

and $F$ is bounded, $F_x(\cdot;h)$ is also bounded on $(-\infty, \infty)$. Therefore $F_x(\cdot;h)$ is $B$-a.p. The limit $F$ of the sequence $(F_x(\cdot;h_n))$, where $h_n \to 0$, which is uniformly convergent, is $B$-a.p.

Because $x$ is $S^1$-bounded, for every $t \in (-\infty, \infty)$ we have

$$V(F;t) \le \int_{t-1}^{t+1} |x(s)|ds \le M,$$

where $M$ is a constant, and so we see that $F \in \tilde{X}_o$. For an arbitrary $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that for $n_0 > N$ we have

$$|F_x(w;h_{n_o}) - F(w)| \leq \frac{\varepsilon}{15}$$

uniformly with respect to $w \in (-\infty, \infty)$. It is known (see[1], p.29) that there exists $\varepsilon' = \varepsilon'(\varepsilon) > 0$ such that $\varepsilon' < \varepsilon/15$ and every $\varepsilon'$ - almost period of $S_x(\cdot;h_{n_o})$ is an $\varepsilon/3$-almost period of $F_x(\cdot;h_{n_o})$. Hence for $\tau \in E_{S^1}\{2h_{n0}\varepsilon';x\}$, where $h_{n0} \leq 1$, we obtain

$$V(F_\tau - F) \leq \sup_{-\infty<t<\infty} |F(t+\tau) - F_x(t+\tau;h_{n0})| +$$

$$+ \sup_{-\infty<t<\infty} |F_x(t+\tau;h_{n0}) - F_x(t;h_{n0})| + \sup_{-\infty<t<\infty} |F_x(t;h_{n0}) - F_x(t)| +$$

$$+ \sup_{-\infty<t<\infty} V(F_\tau - F;t) \leq \frac{7}{15}\varepsilon + \sup_{-\infty<t<\infty} \int_{t-1}^{t+1} |x(s+\tau) - x(s)| ds \leq \varepsilon,$$

and so $F$ is $V$-a.p.

*Theorem* 2. *Let us assume that* x *an S-a.p. function.*

*a) If the indefinite integral of x*

$$F(u) = \int_{u_o}^{u} x(s)ds + C \quad for \quad u \in (-\infty, \infty)$$

*satisfies the following condition*

$$\sup_{-\infty<t<\infty} \left| \int_t^{t+1} F(u)du \right| < \infty,$$

*then the function G of the form*

$$G(w) = \int_w^{w+1} F(u)du + C \qquad for \qquad w \in (-\infty, \infty)$$

*is B-a.p.*

*b) If the indefinite integral of x is $S^1$- bounded, then G is V-a.p.*

*Proof.* a) Let $x$ be $S$-a.p. and let $S_x(\cdot;h)$ be the Steklov function of $x$ of the form (2). Let us write for $w \in (-\infty, \infty)$

$$F_x(w;h) = \int_{w_o}^{w} S_x(u;h)du + C.$$

Similarly as in the proof of Theorem 1 we obtain that the sequence $(F_x(w;h_n))$, where $h_n \to 0$, is convergent to $F(w)$ uniformly with respect to $w \in (-\infty, \infty)$.

Because $x$ is $S$-a.p., for every fixed $s \in (-\infty, \infty)$ and every fixed $t \in (-\infty, \infty)$ the function $y_{st}$ of the form

$$y_{st}(u) = \int_{s+t+u}^{s+t+u+1} x(w)dw$$

is $B$-a.p. By the assumption it follows that there exists a constant $M > 0$ such that for every $t \in (-\infty, \infty)$ we have

$$\left| \int_{t}^{t+1} F(r)dr \right| \leq M.$$

Hence for every $r \in (-\infty, \infty)$ we obtain

$$|G_{st}(r)| = \left| \int_{r_o}^{r} y_{st}(u)du + C \right| \leq 2M + |C|,$$

and so $G_{st}$ $B$-a.p. It is known (see[1],p.29) that for an arbitrary $\varepsilon > 0$ there exists $0 < \varepsilon' = \varepsilon'(\varepsilon) < \varepsilon$ such that every $\varepsilon'$ - almost period of $y_{st}$ is an $\varepsilon$ - almost period of $G_{st}$. Hence for $\tau \in E_{S^1}\{\varepsilon'; x\}$ we have $\tau \in E\{\varepsilon; G_{st}\}$ and

$$\left| \int_{0}^{\tau} y_{st}(u)du \right| \leq \varepsilon$$

for every $s$ and every $t$. Therefore for every $t \in (-\infty, \infty)$ and for $\tau \in E_{S^1}\{\varepsilon'; x\}$ we obtain

$$\left| \int_{t+\tau}^{t+\tau+1} F_x(w;h)dw - \int_{t}^{t+1} F_x(w;h)dw \right| \leq \frac{1}{2h} \int_{-h}^{h} \left| \int_{0}^{\tau} y_{st}(u)du \right| ds \leq \varepsilon,$$

i.e. $E_{S^1}\{\varepsilon';x\} \subset E\{\varepsilon;z_h\}$, where

$$(3) \qquad z_h(t) = \int_t^{t+1} F_x(w;h)dw,$$

and hence $z_h$ is $B$-a.p. The sequence $(z_{h_n}(t))$ is convergent to $G(t)$ for every $h_n \to 0$ uniformly with respect to $t \in (-\infty,\infty)$, and so $G$ is $B$-a.p.

b) Because $F$ is $S^1$ - bounded, for every $t \in (-\infty,\infty)$ we have

$$V(G;t) \leq \int_{t-1}^{t+1} |F(u)|du + \int_{t}^{t+2} |F(u)|du \leq M,$$

where $M$ is a constant, and so $G \in \tilde{X}_o$.

For an arbitrary $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that for $n_o > N$ we obtain

$$\left| z_{h_{n_o}}(t) - G(t) \right| \leq \frac{\varepsilon}{9}$$

uniformly with respect to $t \in (-\infty,\infty)$, where $z_{h_{n_o}}$ is of the form (3). It is known that there exists $0 < \varepsilon' = \varepsilon'(\varepsilon) < \varepsilon/9$ such that $E_{S^1}\{\varepsilon';x\} \subset E\left\{\varepsilon/3; z_{h_{n_o}}\right\}$. Hence for every $t \in (-\infty,\infty)$ and every $\tau \in E_{S^1}\{\varepsilon';x\}$ we have

$$|G(t+\tau) - G(t)| \leq \frac{5}{9}\varepsilon$$

and

$$V(G_\tau - G) \leq \sup_{-\infty<t<\infty} |G(t+\tau) - G(t)| + 4 \sup_{-\infty<t<\infty} \int_t^{t+1} |x(s+\tau) - x(s)|ds < \varepsilon,$$

i.e. $G$ is $V$-a.p.

## References

[1] B.M. Levitan, Almost Periodic Functions, Moscow 1953 (in Russian).

[2] S. Stoiński, Real-valued Functions Almost Periodic in Variation, *Functiones et Approximatio* XXII (1993), 141-148.

[3] S. Stoiński, Some Remarks on Bohr's Almost Periodic Functions and Stepanov's Almost Periodic Functions, *Functiones et Approximatio* (in press).

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